MATROID BASE POLYTOPE DECOMPOSITION II : SEQUENCES OF HYPERPLANE SPLITS

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Abstract. This is a continuation of an early paper [Adv. Appl. Math. 47(2011), 158-172] about matroid base polytope decomposition. We will present sufficient conditions on a matroid M so its base polytope $P(M)$ has a *sequence* of *hyperplane* splits. These yield to decompositions of $P(M)$ with two or more pieces for infinitely many matroids M. We also present necessary conditions on the Euclidean representation of rank three matroids M for the existence of decompositions of $P(M)$ into 2 or 3 pieces. Finally, we prove that $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

Keywords: Matroid base polytope, polytope decomposition

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1. Introduction

This paper is a continuation of the paper [3] by the two present authors. For general background in matroid theory we refer the reader to [12, 15]. A *matroid* $M = (E, \mathcal{B})$ of *rank* $r = r(M)$ is a finite set $E = \{1, \dots, n\}$ together with a nonempty collection $\mathcal{B} = \mathcal{B}(M)$ of r-subsets of E (called the *bases* of M) satisfying the following *basis exchange axiom*:

if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

We denote by $\mathcal{I}(M)$ the family of *independent* sets of M (consisting of all subsets of bases of M). For a matroid $M = (E, \mathcal{B})$, the *matroid base polytope* $P(M)$ of M is defined as the convex hull of the incidence vectors of bases of M, that is,

$$P(M) := \text{conv}\left\{\sum_{i \in B} e_i : B \text{ a base of } M\right\},$$

where e_i is the i^{th} standard basis vector in $\mathbb{R}^n$. $P(M)$ is a polytope of dimension at most $n-1$.

A *matroid base polytope decomposition* of $P(M)$ is a decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

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where each $P(M_i)$ is a matroid base polytope for some matroid M_i and, for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is a face of both $P(M_i)$ and $P(M_j)$. It is known that nonempty faces of matroid base polytope are matroid base polytopes [5, Theorem 2]. So, the common face $P(M_i) \cap P(M_j)$ (whose vertices correspond to elements of $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$) must also be a matroid base polytope. $P(M)$ is said to be *decomposable* if it admits a matroid base polytope decomposition with $t \geq 2$ and *indecomposable* otherwise. A decomposition is called *hyperplane split* when $t = 2$.

Matroid base polytope decomposition were introduced by Lafforgue [9, 10] and have appeared in many different contexts : quasisymmetric functions [1, 2, 4, 11], compactification of the moduli space of hyperplane arrangements [6, 8], tropical linear spaces [13, 14], etc. In [3], we have studied the existence (and nonexistence) of such decompositions. Among other results, we presented sufficient conditions on a matroid M so $P(M)$ admits a hyperplane split. This yielded us to *different* hyperplane splits for infinitely many matroids. A natural question is the following one: given a matroid base polytope $P(M)$, is it possible to find a sequence of hyperplane splits providing a decomposition of $P(M)$? In other words, is there a hyperplane split of $P(M)$ such that one of the two obtained pieces has a hyperplane split such that, in turn, one of the two new obtained pieces has a hyperplane split, and so on, giving a decomposition of $P(M)$?

In [7, Section 1.3], Kapranov showed that all decompositions of a (appropriately parametrized) rank-2 matroid can be achieved by a sequence of hyperplane splits. However, this is not the case in general. Billera, Jia and Reiner [2] provided a decomposition into three indecomposable pieces of $P(W)$ where W is the rank three matroid on $\{1, \ldots, 6\}$ with $\mathcal{B}(W) = \binom{[6]}{3} \setminus \{\{1, 2, 3\}, \{1, 4, 5\}, \{3, 5, 6\}\}$. They proved that this decomposition cannot be obtained via hyperplane splits. However, we notice that $P(W)$ may admits other decompositions into three pieces that can be obtained via hyperplane splits; this is illustrated in Example 3.

A difficulty arising when we apply successive hyperplane splits is that the intersection $P(M_i) \cap P(M_j)$ also must be a matroid base polytope. For instance, consider a first hyperplane split $P(M) = P(M_1) \cup P(M_1')$ and suppose that $P(M_1')$ admits a hyperplane splits, say $P(M_1') = P(M_2) \cup P(M_2')$. This sequence of 2 hyperplane splits would give the decomposition $P(M) = P(M_1) \cup P(M_2) \cup P(M_2')$ if $P(M_1) \cap P(M_2), P(M_1) \cap P(M_2')$, and $P(M_2) \cap P(M_2')$ were matroid base polytopes. By definition of hyperplane split, $P(M_2) \cap P(M_2')$ is the base polytope of a matroid, however the other two intersections might not be matroid base polytopes. Recall that the intersection of two matroids is not necessarily a matroid (for instance, $\mathcal{B}(M_1) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ and $\mathcal{B}(M_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ are matroids while $\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1, 3\}, \{2, 3\}, \{2, 4\}\}$ is not).

In the next section, we give sufficient conditions on M so that $P(M)$ admits a sequence of $t \geq 2$ hyperplane splits. This allows us to provide decompositions of $P(M)$ with $t+1$ pieces for infinitely many matroids. We say that two decompositions $P(M) = \bigcup_{i=1}^{t} P(M_i)$ and $P(M) = \bigcup_{i=1}^{t} P(M'_i)$ are *equivalent* if there exists a permutation σ of $\{1, \ldots, t\}$ such that $P(M_i)$ is *combinatorially equivalent* to $P(M'_{\sigma(i)})$. They are *different* otherwise. We present a lower bound for the number of different decompositions of $P(U_{n,r})$ into t pieces. In Section 3, we present necessary geometric conditions (on the Euclidean representation) of rank three matroids M for the existence of decompositions of $P(M)$ into 2 or 3 pieces. Finally, in Section 4, we show that the *direct sum* $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

2. Sequence of hyperplane splits

Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $A \subseteq E$. We recall that the independent sets of the *restriction* of matroid M to A, denoted by $M|_A$, are given by $\mathcal{I}(M|_A) = \{I \subseteq A : I \in \mathcal{I}(M)\}$.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a t-partition of $E = \{1, \ldots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \ldots, t$. We say that $\bigcup_{i=1}^{t} E_i$ is a *good t-partition* if there exist integers $0 < a_i < r_i$ with the following properties :

$(P1)$ $r = \sum_{i=1}^{t} a_i$,

$(P2)$ (a) For any j with $1 \leq j \leq t-1$

if $X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq a_1$ and $Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})$ with $|Y| \leq a_2$, then $X \cup Y \in \mathcal{I}(M)$.

(b) For any pair j, k with $1 \leq j < k \leq t-1$

$$\begin{aligned} &\text{if } X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j}) && \text{with } |X| \leq \sum_{i=1}^{j} a_i, \\ &Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k}) && \text{with } |Y| \leq \sum_{i=j+1}^{k} a_i, \\ &Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t}) && \text{with } |Z| \leq \sum_{i=k+1}^{t} a_i, \\ &\text{then } X \cup Y \cup Z \in \mathcal{I}(M). \end{aligned}$$

Notice that the good 2-partitions provided by $(P2)$ case (a) with $t = 2$ are the *good partitions* defined in [3]. Good partitions were used to give sufficient conditions for the existence of hyperplane splits. The latter was a consequence of the following two results:

Lemma 1. [3, Lemma 1] *Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then,*

$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\}$ *and* $\mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_2| \leq a_2\}$ *are the collections of bases of matroids.*

Theorem 1. [3, Theorem 1] *Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then, $P(M) = P(M_1) \cup P(M_2)$ is a hyperplane split, where M_1 and M_2 are the matroids given by Lemma 1.*

We shall use these two results as the initial step in our construction of a sequence of $t \geq 2$ hyperplane splits.

Lemma 2. *Let $t \geq 2$ be an integer and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, i=1,...,t. Let*

$$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\}$$

and, for each $j = 1, \ldots, t$, let

$$\mathcal{B}(M_j) = \left\{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1, \ldots, |B \cap \bigcup_{i=1}^{j-1} E_i| \geq \sum_{i=1}^{j-1} a_i, \, |B \cap \bigcup_{i=1}^{j} E_i| \leq \sum_{i=1}^{j} a_i \right\}.$$

Then $\mathcal{B}(M_i)$ is the collection of bases of a matroid for each $i = 1, \ldots, t$.

Proof. By properties $(P1)$ and $(P2)$ we have that

if $X \in \mathcal{I}(M|_{E_1})$ with $|X| \leq a_1$ and $Y \in \mathcal{I}(M|_{E_2 \cup \cdots \cup E_t})$ with $|Y| \leq \sum_{i=2}^{t} a_i$,

then $X \cup Y \in \mathcal{I}(M)$. So, by Lemma 1, $\mathcal{B}(M_1)$ is the collection of bases of a matroid. Now, notice that $\mathcal{B}(\overline{M_1}) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1\}$ is also the collection of bases of a matroid on E. We claim that $P(\overline{M_1}) = P(M_2) \cup P(\overline{M_2})$ is a hyperplane split where

$$\mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \text{ and } |B \cap (E_1 \cup E_2)| \leq a_1 + a_2\}$$

and

$$\mathcal{B}(\overline{M_2}) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \text{ and } |B \cap (E_1 \cup E_2)| \geq a_1 + a_2\}.$$

Indeed, since $\mathcal{B}(\overline{M_1})$ is the collection of bases of a matroid on E, then, by properties $(P1)$ and $(P2)$ (a),

if $X \in \mathcal{I}(\overline{M}|_{E_1 \cup E_2})$ with $|X| \leq a_1 + a_2$ and $Y \in \mathcal{I}(\overline{M}|_{E_3 \cup \cdots \cup E_t})$ with $|Y| \leq \sum_{i=3}^{t} a_i$,

then $X \cup Y \in \mathcal{I}(\overline{M})$. So, by Lemma 1, $\mathcal{B}(M_2)$ is the collection of bases of a matroid (and thus $\mathcal{B}(\overline{M_2})$ also is). Inductively applying the above argument to $\overline{M}_j$, it can be easily checked that for all j $\mathcal{B}(M_j)$ is the collection of bases of a matroid. □

Theorem 2. *Let $t \geq 2$ be an integer and let $M = (E, \mathcal{B})$ be a matroid of rank r. Let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Then $P(M)$ has a sequence of t hyperplane splits yielding the decomposition*

$$P(M) = \bigcup_{i=1}^{t} P(M_i),$$

where M_i, $1 \leq i \leq t$, are the matroids defined in Lemma 2.

Proof. By Theorem 1, the result holds for $t = 2$. Moreover, by the inductive construction of Lemma 2, we clearly have that $P(M) = \bigcup_{i=1}^{t} P(M_i)$ with $\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$. We only need to show that $\mathcal{B}(M_j) \cap \mathcal{B}(M_k)$ is the collection of bases of a matroid for any $1 \leq j < k \leq t$. For, by definition of $\mathcal{B}(M_i)$, we have

$$\mathcal{B}(M_j) \cap \mathcal{B}(M_k) = \{B \in \mathcal{B}(M) : \text{the condition } C_h(B) \text{ is satisfied for all } 1 \leq h \leq k\}$$

where for $A \subseteq E$:

- $C_h(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{h} E_i| \geq \sum_{i=1}^{h} a_i$ and $1 \leq h \leq k$, $h \neq j, k$,
- $C_j(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i$,

  and
- $C_k(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i$.

We will check the exchange axiom for any $X, Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $X, Y \in \mathcal{B}(M)$ for any $e \in X \setminus Y$ there exists $f \in Y \setminus X$ such that $X - e + f \in \mathcal{B}(M)$. We will verify that $X - e + f \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. We distinguish three cases (depending which of the conditions $C_i(X - e)$ is satisfied).

Case 1. There exists $1 \leq l \leq j$ such that $C_l(X - e)$ is not satisfied. We suppose that l is minimal with this property. Since, by definition of $\mathcal{B}(M_j) \cap \mathcal{B}(M_k), l \leq j \leq k$, $C_l(X)$ is satisfied, and $C_l(X - e)$ is not satisfied, we obtain

(a) $\left| X \cap \bigcup_{i=1}^{l} E_i \right| = \sum_{i=1}^{l} a_i$,

(b) $e \in \bigcup_{i=1}^{l} E_i$,

(c) $|\underbrace{(X - e) \cap \bigcup_{i=1}^{l} E_i}_{I_1}| = \sum_{i=1}^{l} a_i - 1.$

Since $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$, then $\underbrace{|\, Y \cap \bigcup_{i=1}^{l} E_i \,|}_{I_2} \geq \sum_{i=1}^{l} a_i$.

Therefore, by using (c), $I_1, I_2 \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l}) \subseteq \mathcal{I}(M)$ with $|I_1| < |I_2|$. So, there exists $f \in I_2 \setminus I_1 \subset Y \setminus X$ with $I_1 \cup f \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l})$. Thus, $f \in \bigcup_{i=1}^{l} E_i$ and

$$|I_1 \cup f \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - 1. \tag{1}$$

Moreover, since X is a base, $|X| = r = \sum_{i=1}^{t} a_i$ and, by (a), we have

$$\underbrace{|\,(X - e + f) \cap \bigcup_{i=l+1}^{t} E_i\,|}_{I_3} \overset{(b)}{=} |X \cap \bigcup_{i=l+1}^{t} E_i| = \sum_{i=1}^{t} a_i - \sum_{i=1}^{l} a_i = \sum_{i=l+1}^{t} a_i.$$

We also have $I_3 \in \mathcal{I}(M|_{E_{l+1} \cup \cdots \cup E_t})$, thus, by $(P2)$ (b),

$I_1 \cup f \cup I_3 \in \mathcal{I}(M)$ with $|I_1 \cup f \cup I_3| = \sum_{i=1}^{l} a_i - 1 + 1 + \sum_{i=l+1}^{t} a_i = r$
and so $I_1 \cup f \cup I_3 = X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is $C_h(X - e + f)$ holds for each $1 \leq h \leq k$.

(i) $h < l$: Since l is the minimum for which $C_l(X - e)$ is not verified, $C_h(X - e)$ is satisfied for each $1 \leq h < l$ and thus $C_h(X - e + f)$ is also satisfied (we just added a new element).

(ii) $h = l$: By equation (1), $C_l(X - e + f)$ is satisfied.

(iii) $h > l$: Since $e, f \in \bigcup_{i=1}^{l} E_i$,

$$|X - e + f \cap \bigcup_{i=1}^{h} E_i| = |X \cap \bigcup_{i=1}^{h} E_i|,$$

thus $C_h(X - e + f)$ is satisfied if and only if $C_h(X)$ is satisfied, which is the case since $h > l$.

Case 2. $C_{l'}(X - e)$ is satisfied for all $1 \leq l' \leq j$ and there exists $j + 1 \leq l \leq k - 1$ such that $C_l(X - e)$ is not satisfied. We suppose that l is minimal with this property. Since $C_l(X)$ is satisfied and $C_l(X - e)$ is not,

(a) $\left| X \cap \bigcup_{i=1}^{l} E_i \right| = \sum_{i=1}^{l} a_i$,

(b) $e \in \bigcup_{i=j+1}^{l} E_i$ (since $C_j(X - e)$ is satisfied),

(c) $\underbrace{|\,(X-e)\cap\bigcup_{i=1}^{l}E_i\,|}_{I_1}=\sum_{i=1}^{l}a_i-1.$

Since $C_j(X-e)$ is satisfied,

$$\begin{aligned}\underbrace{|\,(X-e)\cap\bigcup_{i=j+1}^{l}E_i\,|}_{I_1} &= |(X-e)\cap\bigcup_{i=1}^{l}E_i|-|(X-e)\cap\bigcup_{i=1}^{j}E_i| \\ &\overset{(c)}{=}\sum_{i=1}^{l}a_i-1-\sum_{i=1}^{j}a_i=\sum_{i=j+1}^{l}a_i-1. \qquad (2)\end{aligned}$$

Let $Y\in\mathcal{B}(M_j)\cap\mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied,

$$\begin{aligned}\underbrace{|\,Y\cap\bigcup_{i=j+1}^{l}E_i\,|}_{I_2} &= |Y\cap\bigcup_{i=1}^{l}E_i|-|Y\cap\bigcup_{i=1}^{j}E_i| \\ &\geq\sum_{i=1}^{l}a_i-\sum_{i=1}^{j}a_i=\sum_{i=j+1}^{l}a_i.\end{aligned}$$

Since $|I_1|<|I_2|$, there exists $f\in I_2\setminus I_1$ such that $I_1+f\in\mathcal{I}(M|_{E_{j+1}\cup\cdots\cup E_l})$. So, $f\in\bigcup_{i=j+1}^{l}E_i$ and, by (b), we have

$$(X-e+f)\cap\bigcup_{i=1}^{j}E_i=X\cap\bigcup_{i=1}^{j}E_i.$$

Since X is a base, $X-e+f\cap\bigcup_{i=1}^{j}E_i\in\mathcal{I}(M|_{E_1\cup\cdots\cup E_j})$ (also notice that $(X-e+f)\cap\bigcup_{i=l+1}^{t}E_i\in\mathcal{I}(M|_{E_{l+1}\cup\cdots\cup E_t})$). Moreover, since $X\in\mathcal{B}_j\cap\mathcal{B}_k$, $C_j(X)$ is satisfied and thus

$$|(X-e+f)\cap\bigcup_{i=1}^{j}E_i|=\sum_{i=1}^{j}a_i \qquad (3)$$

and, by equation (2), we have

$$|(X-e+f)\cap\bigcup_{i=j+1}^{l}E_i|=\sum_{i=j+1}^{l}a_i \qquad (4)$$

obtaining that

$$|(X-e+f)\cap\bigcup_{i=l+1}^{t}E_i|=r-\sum_{i=1}^{j}a_i-\sum_{i=j+1}^{l}a_i=\sum_{i=l+1}^{t}a_i.$$

Now, by $(P2)$ (b), we have

$$\left((X-e+f)\cap\bigcup_{i=1}^{j}E_i\right)\cup\left((X-e+f)\cap\bigcup_{i=j+1}^{l}E_i\right)\cup\left((X-e+f)\cap\bigcup_{i=l+1}^{t}E_i\right)=X-e+f\in\mathcal{I}(M).$$

Since $|X-e+f|=r$, $X-e+f\in\mathcal{B}(M)$.

Finally we need to show that $X-e+f\in\mathcal{B}_j\cap\mathcal{B}_k$, that is, that $C_h(X-e+f)$ is verified for each $1\leq h\leq k$.

(i) $h<l$ and $h\neq j$: Since $C_h(X-e)$ is satisfied, by the minimality of l, $C_h(X-e+f)$ is also satisfied.

(ii) $h=j$: By equation (3), $C_j(X-e+f)$ is satisfied.

(iii) $h=l$: By equations (3) and (4), $C_l(X-e+f)$ is satisfied.

(iv) $h>l$: Since $e,f\in\bigcup\limits_{i=j+1}^{l}E_i$, $|X-e+f\cap\bigcup\limits_{i=1}^{h}E_i|=|X\cap\bigcup\limits_{i=1}^{h}E_i|$, thus $C_h(X-e+f)$ is satisfied if and only if $C_h(X)$ is satisfied, which is the case because $h>l$.

Case 3. $C_i(X-e)$ is satisfied for every $1\leq i\leq k$.

Subcase (a) $|(X-e)\cap\bigcup\limits_{i=1}^{k}E_i|=\sum\limits_{i=1}^{k}a_i$. We first notice that $e\in\bigcup\limits_{i=k+1}^{t}E_i$ (otherwise $|X-e\cap\bigcup\limits_{i=1}^{k}E_i|<|X\cap\bigcup\limits_{i=1}^{k}E_i|$ which is impossible since $C_k(X)$ holds). Now,

$$\underbrace{|\,(X-e)\cap\bigcup_{i=k+1}^{t}E_i\,|}_{I_1}=r-1-\sum_{i=1}^{k}a_i=\sum_{i=k+1}^{t}a_i-1. \tag{5}$$

Let $Y\in\mathcal{B}(M_j)\cap\mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied, $|Y\cap\bigcup\limits_{i=1}^{k}E_i|\leq\sum\limits_{i=1}^{k}a_i$, and so $\underbrace{|\,Y\cap\bigcup\limits_{i=k+1}^{t}E_i\,|}_{I_2}\geq\sum\limits_{i=k+1}^{t}a_i$.

Since $|I_1|<|I_2|$, there exists $f\in I_2\setminus I_1$ such that $I_1+f\in\mathcal{I}(M|_{E_{k+1}\cup\cdots\cup E_t})$. So, $f\in\bigcup\limits_{i=k+1}^{t}E_i$ and since $e\in\bigcup\limits_{i=k+1}^{t}E_i$,

$$(X-e+f)\cap\bigcup_{i=1}^{k}E_i=X\cap\bigcup_{i=1}^{k}E_i\in\mathcal{I}(M|_{E_1\cup\cdots\cup E_k}).$$

Also, since $(X-e+f)\cap\bigcup\limits_{i=k+1}^{t}E_i\in\mathcal{I}(M|_{E_{k+1}\cup\cdots\cup E_t})$, by $(P2)(b)$ we have

$$X-e+f=\left(X-e+f\cap\bigcup_{i=1}^{k}E_i\right)\cup\left(X-e+f\cap\bigcup_{i=k+1}^{t}E_i\right)\in\mathcal{I}(M).$$

Moreover, by using equation (5) and the fact that $f \in \bigcup_{i=k+1}^{t} E_i$ we obtain that

$$|(X-e+f)\cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=k+1}^{t} a_i.$$

Since $|(X-e)\cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i$,

$$|(X-e+f)\cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i.$$

Therefore,

$$|(X-e+f)\cap \bigcup_{i=1}^{t} E_i| = |(X-e+f)\cap \bigcup_{i=1}^{k} E_i| + |(X-e+f)\cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=1}^{t} a_i = r$$

and so $X-e+f \in \mathcal{B}(M)$.

Finally we need to show that $X-e+f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, that $C_h(X-e+f)$ is verified for each $1 \leq h \leq k$. Since $e, f \in \bigcup_{i=k+1}^{t} E_i$, $C_h(X-e+f)$ becomes $C_h(X)$ for all $1 \leq h \leq k$, which is satisfied.

Subcase (b) If $|(X-e)\cap \bigcup_{i=1}^{k} E_i| < \sum_{i=1}^{k} a_i$, then $e \in \bigcup_{i=j+1}^{t} E_i$ (otherwise $|(X-e)\cap \bigcup_{i=1}^{j} E_i| < |X \cap \bigcup_{i=1}^{j} E_i|$ which is impossible since $C_j(X)$ holds). Now, since $C_j(X-e)$ is satisfied,

$$|(X-e)\cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i,$$

and thus

$$\underbrace{|\,(X-e)\cap \bigcup_{i=j+1}^{t} E_i\,|}_{I_1} = \sum_{i=j+1}^{t} a_i - 1.$$

Let $Y \in \mathcal{B}(M_j)\cap \mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied,

$$|Y \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i,$$

and thus

$$\underbrace{|\,Y \cap \bigcup_{i=j+1}^{t} E_i\,|}_{I_2} = \sum_{i=j+1}^{t} a_i.$$

Since $|I_1| < |I_2|$, there exists $f \in I_2 \setminus I_1$ such that $I_1 + f \in \mathcal{I}(M|_{E_{j+1}\cup\cdots\cup E_t})$. So, $f \in \bigcup_{i=j+1}^{t} E_i$. Since $e \in \bigcup_{i=j+1}^{t} E_i$,

$$(X-e+f)\cap \bigcup_{i=1}^{j} E_i = X \cap \bigcup_{i=1}^{j} E_i \in \mathcal{I}(M|_{E_1\cup\cdots\cup E_j}) \qquad (6)$$

and, by $(P2)$ (b), we have

$$\left(X - e + f \cap \bigcup_{i=1}^{j} E_i\right) \cup \left(X - e + f \cap \bigcup_{i=j+1}^{t} E_i\right) \in \mathcal{I}(M)$$

Therefore, $X - e + f \in \mathcal{B}(M)$.

Finally, we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, $C_h(X - e + f)$ is verified for each $1 \leq h \leq k$.

(i) $h < j$: Since $C_h(X - e)$ is satisfied, $C_h(X - e + f)$ is also satisfied.

(ii) $h = j$: $C_j(X - e + f)$ is satisfied by equation (6).

(iii) $j + 1 \leq h \leq k - 1$: Since $C_h(X - e)$ is satisfied then $C_h(X - e + f)$ is also satisfied.

(iv) $h = k$: Since $|X - e \cap \bigcup\limits_{i=1}^{k} E_i| < \sum\limits_{i=1}^{k} a_i$ then $|X - e + f \cap \bigcup\limits_{i=1}^{k} E_i| \leq \sum\limits_{i=1}^{k} a_i$ and thus $C_h(X - e + f)$ is satisfied. □

2.1. **Uniform matroids.**

Corollary 1. *Let $n, r, t \geq 2$ be integers with $n \geq r + t$ and $r \geq t$. Let $p_t(n)$ be the number of different decompositions of the integer n of the form $n = \sum\limits_{i=1}^{t} p_i$ with $p_i \geq 2$ and let $h_t(U_{n,r})$ be the number of decompositions of $P(U_{n,r})$ into t pieces. Then,*

$$h_t(U_{n,r}) \geq p_t(n).$$

Proof. We consider the partition $E = \{1, \ldots, n\} = \bigcup\limits_{i=1}^{t} E_i$, where

$$\begin{aligned} E_1 &= \{1, \ldots, p_1\}, \\ E_2 &= \{p_1 + 1, \ldots, p_1 + p_2\}, \\ &\vdots \\ E_t &= \{\sum_{i=1}^{t-1} p_i + 1, \ldots, \sum_{i=1}^{t} p_i\}. \end{aligned}$$

We claim that $\bigcup\limits_{i=1}^{t} E_i$ is a good t-partition. For, we first notice that $M|_{E_i}$ is isomorphic to $U_{p_i, \min\{p_i, r\}}$ for each $i = 1, \ldots, t$. Let $r_i = r(M|_{E_i}) = \min\{p_i, r\}$. We now show that

$$\sum_{i=1}^{t} r_i \geq r + t. \tag{7}$$

For, we note that

$$\sum_{i=1}^{t} r_i = \sum_{i=1}^{t} r(M|_{E_i}) = \sum_{i \in T \subseteq \{1, \ldots, t\}} p_i + (t - |T|)r.$$

We distinguish three cases.

1) If $t = |T|$, then $\sum_{i=1}^{t} r_i = \sum_{i=1}^{t} p_i = n \geq r + t$.

2) If $t = |T| + 1$, then $\sum_{i=1}^{t} r_i = \sum_{i=1}^{t-1} p_i + r \geq 2(t-1) + r \geq t + t - 2 + r \geq t + r$.

3) If $t = |T| + k$, with $k \geq 2$, then $\sum_{i=1}^{t} r_i \geq kr \geq 2r \geq r + t$.

So, by equation (7), we can find integers $a'_i \geq 1$ such that $\sum_{i=1}^{t} r_i = r + \sum_{i=1}^{t} a'_i$. Therefore, there exist integers $a_i = r(M|_{E_i}) - a'_i$ with $0 < a_i < r(M|_{E_i})$ such that $r = \sum_{i=1}^{t} a_i$. Moreover, if $X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq \sum_{i=1}^{j} a_i$, $Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k})$ with $|Y| \leq \sum_{i=j+1}^{k} a_i$, and $Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t})$ with $|Z| \leq \sum_{i=k+1}^{t} a_i$ for $1 \leq j < k \leq t-1$, then $|X \cup Y \cup Z| \leq \sum_{i=1}^{t} a_i = r$ and so $X \cup Y \cup Z$ is always a subset of one of the bases of $U_{n,r}$. Thus, $X \cup Y \cup Z \in \mathcal{I}(U_{n,r})$ and $(P2)$ is also verified. □

Notice that there might be several choices for the values of a_i (each providing a good t-partition). However, it is not clear if these choices give different sequences of t hyperplane splits.

Example 1: Let us consider the uniform matroid $U_{8,4}$. We take the partition $E_1 = \{1,2\}$, $E_2 = \{3,4\}$, $E_3 = \{5,6\}$, and $E_4 = \{7,8\}$. Then $r(M|_{E_i}) = 2$, $i = 1, \ldots, 4$. It is easy to check that if we set $a_i = 1$ for each i then $E_1 \cup E_2 \cup E_3 \cup E_4$ is a good 4-partition and thus $P(U_{8,3}) = P(M_1) \cup P(M_2) \cup P(M_3) \cup P(M_4)$ is a decomposition where

$$
\begin{aligned}
\mathcal{B}(M_1) &= \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \leq 1\},\\
\mathcal{B}(M_2) &= \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \leq 1\},\\
\mathcal{B}(M_3) &= \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \geq 1,\ |B \cap \{5,6\}| \leq 1\},\\
\mathcal{B}(M_4) &= \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \geq 1,\ |B \cap \{5,6\}| \geq 1\}.
\end{aligned}
$$

2.2. **Relaxations.** Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $X \subset E$ be both a circuit and a hyperplane of M (recall that a *hyperplane* is a *flat*, that is $X = cl(X) = \{e \in E | r(X \cup e) = r(X)\}$, of rank $r-1$). It is known [12, Proposition 1.5.13] that $\mathcal{B}(M') = \mathcal{B}(M) \cup \{X\}$ is the collection of bases of a matroid M' (called, *relaxation* of M).

Corollary 2. *Let $M = (E, \mathcal{B})$ be a matroid and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition. Then, $P(M')$ has a sequence of t hyperplane splits where M' is a relaxation of M.*

Proof. It can be checked that the desired sequence of t hyperplane splits of $P(M')$ can be obtained by using the same given good t partition $E = \bigcup_{i=1}^{t} E_i$. □

We notice that the above result is not the only way to define a sequence of hyperplane splits for relaxations. Indeed it is proved in [3] that binary matroids (and thus graphic matroids) do not have hyperplane splits, however there is a sequence of hyperplane splits for relaxations of graphic matroids as it is shown in Example 3 below.

3. Rank-three matroids: geometric point of view

We recall that a matroid of rank three on n elements can be represented geometrically by placing n points on the plane such that if three elements form a circuit, then the corresponding points are collinear (in such diagram the lines need not be straight). Then the bases of M are all subsets of points of cardinal 3 which are not collinear in this diagram. Conversely, any diagram of points and lines in the plane in which a pair of lines meet in at most one point represents a unique matroid whose bases are those 3-subsets of points which are not collinear in this diagram.

The combinatorial conditions $(P1)$ and $(P2)$ can be translated into geometric conditions when M is of rank three. The latter is given by the following two corollaries.

Corollary 3. *Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2$ be a partition of the points of the geometric representation of M such that*
1) $r(M|_{E_1}) \geq 2$ and $r(M|_{E_2}) = 3$;
2) for each line l of M, if $|l \cap E_1| \neq \emptyset$, then $|l \cap E_2| \leq 1$.
Then, $E = E_1 \cup E_2$ is a 2-good partition.

Proof. $(P2)(a)$ can be easily checked with $a_1 = 1$ and $a_2 = 2$. □

Example 2. Let M be the rank-3 matroid arising from the configuration of points given in Figure 1. It can be easily checked that $E_1 = \{1, 2\}$ and $E_2 = \{3, 4, 5, 6\}$ verify the conditions of Corollary 3. Thus, $E_1 \cup E_2$ is a 2-good partition.

Corollary 4. *Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2 \cup E_3$ be a partition of the points of the geometric representation of M such that*
1) $r(M|_{E_i}) \geq 2$ for each $i = 1, 2, 3$,
2) for each line l with at least 3 points of M,
a) if $|l \cap E_1| \neq \emptyset$ then $|l \cap (E_2 \cup E_3)| \leq 1$,
b) if $|l \cap E_3| \neq \emptyset$ then $|l \cap (E_1 \cup E_2)| \leq 1$.
Then, $E = E_1 \cup E_2 \cup E_3$ is a 3-good partition.

Proof. $(P2)$ can be easily checked with $a_1 = a_2 = a_3 = 1$. □

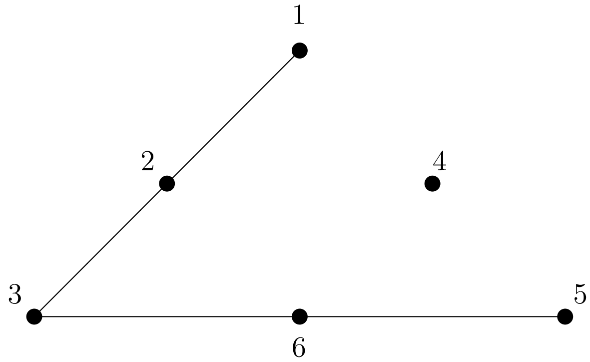

FIGURE 1. Set of points in the plane

Example 3. Let W^3 be the 3-whirl on $E = \{1, \ldots, 6\}$ shown in Figure 2. W^3 is the example given by Billera *et al.* [2] that we mentioned by the end of the introduction. W^3 is a relaxation of $M(K_4)$ (by relaxing circuit $\{2, 4, 6\}$) and it is not graphic.

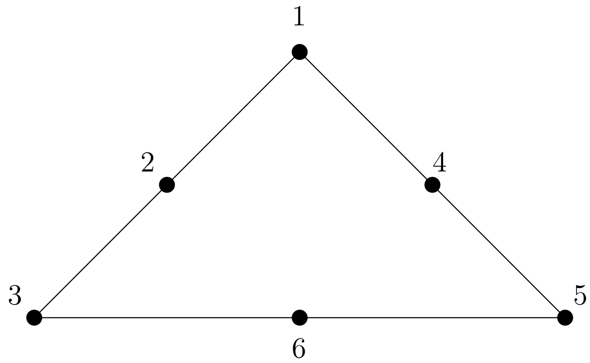

FIGURE 2. Euclidean representation of W^3

It can be checked that $E_1 = \{1, 6\}$, $E_2 = \{2, 5\}$, and $E_3 = \{1, 4\}$ verify the conditions of Corollary 4. Thus, $E_1 \cup E_2 \cup E_3$ is a good 3-partition.

We finally notice that given the 2-good partition $E_1 \cup E_2$ of the matroid M in Example 2, we can apply a hyperplane split to the matroid $M|_{E_2}$ induced by the set of points in $E_2 = \{3, 4, 5, 6\}$. Indeed, it can be checked that $E_2^1 = \{3, 4\}$ and $E_2^2 = \{5, 6\}$ verify conditions in Corollary 3 and thus it is a good 2-partition of $M|_{E_2}$. Moreover, it can be checked that $E_1 = \{1, 2\}$, $E_2^1 = \{3, 4\}$, and $E_2^2 = \{5, 6\}$ verify the conditions of Corollary 4. and thus $E_1 \cup E_2 \cup E_3$ is a good 3-partition for M.

4. Direct sum

Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. The *direct sum*, denoted by $M_1 \oplus M_2$, of matroids M_1 and M_2 has as ground set the disjoint union $E(M_1 \oplus M_2) = E(M_1) \cup E(M_2)$ and as set of bases $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 | B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$. Further, the rank of $M_1 \oplus M_2$ is $r_1 + r_2$.

In [3], we proved the following result.

Theorem 3. [3] *Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a hyperplane split.*

Our main result in this section is the following.

Theorem 4. *Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ admits a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ admits a sequence of hyperplane splits.*

Proof. Without loss of generality, we suppose that $P(M_1)$ has a sequence of hyperplane splits yielding to the decomposition $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$. For each $i = 1, \ldots, t$, we let

$$L_i = \{X \cup Y : X \in \mathcal{B}(N_i), Y \in \mathcal{B}(M_2)\}.$$

Since N_i and M_2 are matroids, L_i is also the matroid given by $N_i \oplus M_2$.

Now for all $1 \leq i, j \leq t$, $i \neq j$ we have

$$L_i \cap L_j = \{X \cup Y : X \in \mathcal{B}(N_i) \cap \mathcal{B}(N_j), Y \in \mathcal{B}(M_2)\}$$

Since $\mathcal{B}(N_i) \cap \mathcal{B}(N_j) = \mathcal{B}(N_i \cap N_j)$ and M_2 are matroids, $L_i \cap L_j$ is also a matroid given by $(N_i \cap N_j) \oplus M_2$. Moreover, $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ so $\mathcal{B}(M_1) = \bigcup_{i=1}^{t} \mathcal{B}(N_i)$ and thus

$$\begin{aligned} \bigcup_{i=1}^{t} L_i &= \{X \cup Y : X \in \bigcup_{i=1}^{t} \mathcal{B}(N_i), Y \in \mathcal{B}(M_2)\} \\ &= \{X \cup Y : X \in \mathcal{B}(M_1), Y \in \mathcal{B}(M_2)\} \\ &= \mathcal{B}(M_1 \oplus M_2). \end{aligned}$$

We now show that this matroid base decomposition induces a t-decomposition of $P(M_1 \oplus M_2)$. Indeed, we claim that $P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i)$. For, we proceed by induction on t. The case $t = 2$ is true since, in the proof of Theorem 3, was showed that $P(M_1 \oplus M_2) = P(L_1) \cup P(L_2)$. We suppose that the result is true for t and let

$$P(M_1) = \bigcup_{i=1}^{t-1} P(N_i) \cup P(N_t^1) \cup P(N_t^2), \tag{8}$$

where N_i, $i = 1, \dots t-1$, N_t^1, N_t^2 are matroids. Moreover, we suppose that throughout the sequence of hyperplane splits of $P(M_1)$ we had $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ and that the last hyperplane split was applied to $P(N_t)$ (obtaining $P(N_t) = P(N_t^1) \cup P(N_t^2)$) and yielding to equation (8).

Now, by the inductive hypothesis, the decomposition $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ implies the decomposition $P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i)$. But, by the case $t = 2$, $P(N_t) = P(N_t^1) \cup P(N_t^2)$ implying the decomposition $P(N_t \oplus M_2) = P(L_t^1) \cup P(L_t^2)$ where

$L_t^1 = \{X \cup Y : X \in \mathcal{B}(N_t^1), Y \in \mathcal{B}(M_2)\}$ and $L_t^2 = \{X \cup Y : X \in \mathcal{B}(N_t^2), Y \in \mathcal{B}(M_2)\}$

Therefore,

$$P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i) = \bigcup_{i=1}^{t-1} P(L_i) \cup P(L_t^1) \cup P(L_t^2).$$

□

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