On Kneser transversals and matroids

J. L. Ramírez Alfonsín

IMAG, Université de Montpellier

Universidade Federal do Rio Grande do Sul

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A little bit of convexity

Theorem (Helly) Let $\mathcal{A}$ be a finite family of at least $d+1$ convexes sets in $\mathbb{R}^d$. If every $d+1$ members of $\mathcal{A}$ have a common point then there is a common point to all members of $\mathcal{A}$.

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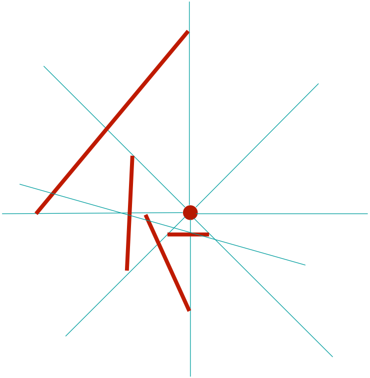
That is, does there exist an integer m such that if all membres of a finite family $\mathcal{A}$ of sets in $\mathbb{R}^2$ are intersected by a line then there is a line intersecting all members of $\mathcal{A}$?

Counterexample : avec $m = 5$, any subfamily fo 4 convexes have a transversal line but there is not a transversal line to all 5.



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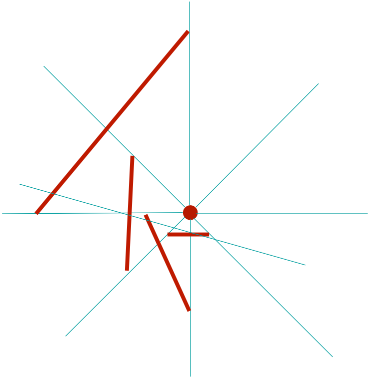
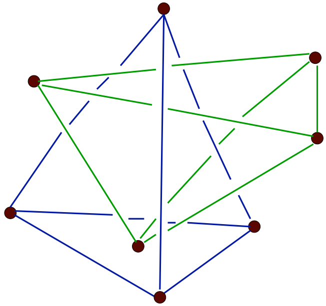


Theorem (Hadwiger) Let $\mathcal{A}$ be a finite family of convexe sets in $\mathbb{R}^2$ pairwise disjoints. If there exists a linear order of $\mathcal{A}$ such that any 3 membres of $\mathcal{A}$ are intersected by a line in the given induced order, then $\mathcal{A}$ admit a transversal line.

Let 8 points in $\mathbb{R}^3$ in general position.



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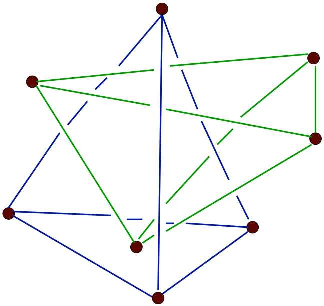
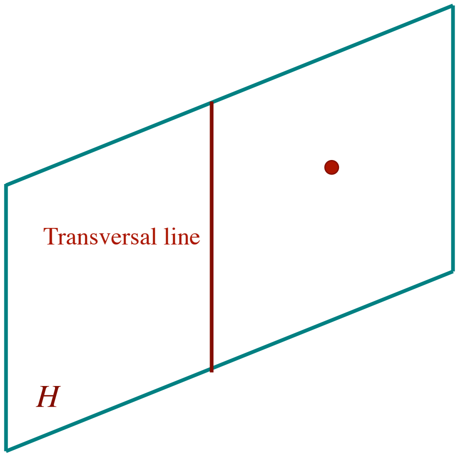


Question : Is there any transversal line to all the tetrahedra ?

NEVER



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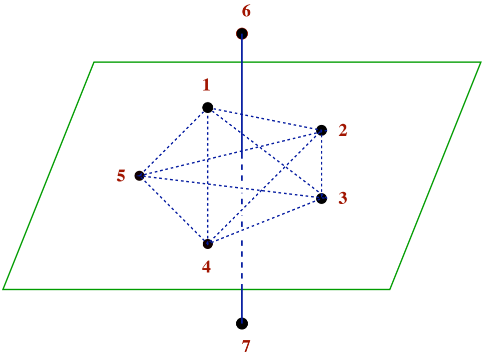
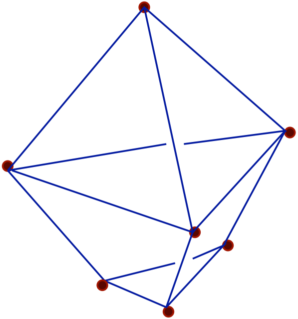
Some times YES and Some times NO





Definitions

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the largest integer n such that for any set of n points (no necessarely in general position) in $\mathbb{R}^d$, there is a $(d - \lambda)$-plane transversal to the convex hulls of all the k-set points.

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$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the smallest integer n such that for any set of $n$points in $\mathbb{R}^d$, do not admit a $(d - \lambda)$-plane transversal to the convex hulls of all the k-set points.

- $m(k,d,\lambda) < M(k,d,\lambda)$.

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Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$M(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

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The Kneser hypergraph $K^{\lambda+1}(n,k)$ is the hypergraph $(V, \mathcal{H})$ where V is the collection of all k-sets of n and
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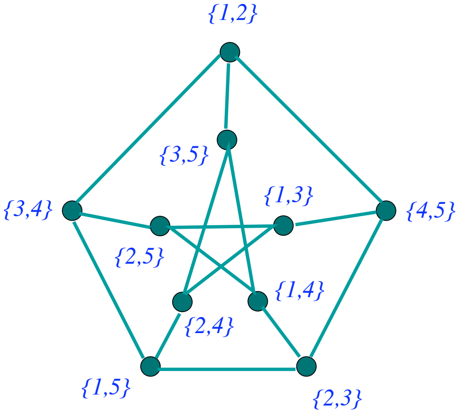
Remark : Kneser graphs are obtained when $\lambda = 1$.

Kneser graph with $n = 5$, $k = 2$ and $\lambda = 1$ (the well-known Petersen graph)



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A collection of vertices $\{S_1, \ldots, S_\rho\}$ of $K^{\lambda+1}(n, k)$ is in the same colour class if and only if either

a) $\rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho \neq \emptyset$ or

b) $\rho > \lambda + 1$ and any $(\lambda + 1)$-sub-family $\{S_{i_1}, \ldots, S_{i_{\lambda+1}}\}$ of $\{S_1, \ldots, S_\rho\}$ is such that $S_{i_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \emptyset$

(that is, they verify the λ-Helly property).

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- $d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq m(k,d,\lambda)$.
- $\chi(K^{\lambda+1}(n,k)) > \begin{cases} n - 2k + \lambda & \text{si } k \geq \lambda, \\ n - 2k & \text{si } k \leq \lambda. \end{cases}$

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Theorem (Lovász) $\chi(K^2(n,k)) = n-2k+2$.

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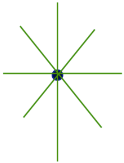

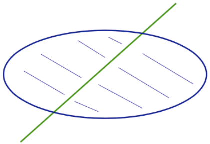

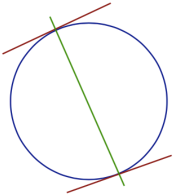

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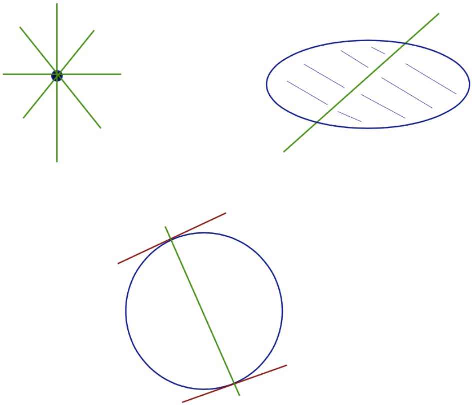

Fact : Two systems of lines in $\mathbb{R}^2$ coincide in one direction.

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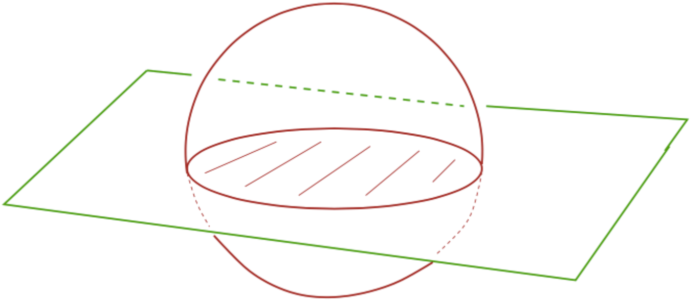

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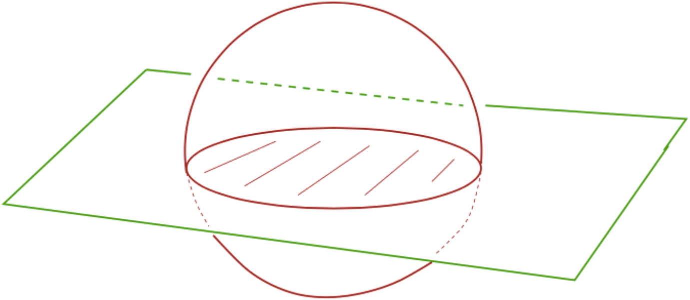

Fact : Three systems of planes in $\mathbb{R}^3$ coincide in one direction.

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Indeed, given ϕ it is enough to choose an hyperplane orthogonal to x going through $\phi(x)x$.

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That is, $\phi_i(x_0) = \phi_d(x_0)$ for all i.

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- Let L be a line passing through the origin.
- The projection of any red k-gon is a compact interval contained in L and any two of them intersect. Then, (by Helly) there is a point x common to all of them.

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• To each of the sides of Γ we have fewer than k points of V (otherwise Γ wouldn't intersect all k-gons).

• Therefore, there are at least $n-2(k-1)=n-2k+2$ points in Γ, contradicting the fact that $V \subset \mathbb{R}^{n-2k+1}$ is a set of points in general position.

Matroid : independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set ($E = \{1, \ldots, n\}$) and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Circuits

A subset $X \subseteq E$ is said to be minimal dependent if any proper subset of X is independent. A minimal dependent set of matroid M is called circuit of M.

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$(B2)$ *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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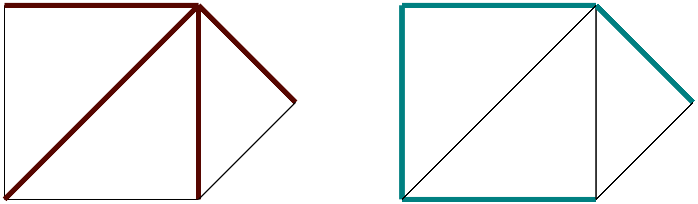
If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

Bases



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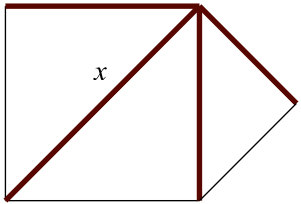

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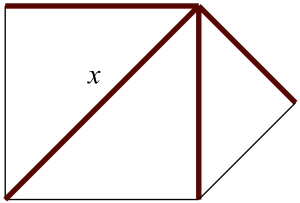

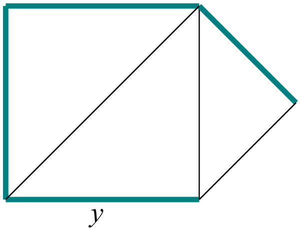

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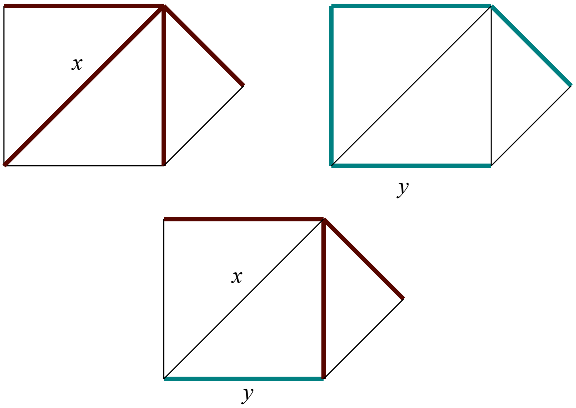

Matroid Kneser graph

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The classical Kneser graph is given by $KG(U_{r,n})$.

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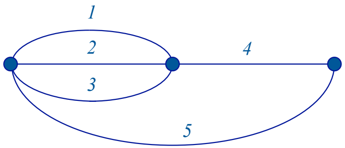
Theorem (Meunier, Montejano, R.A., 2018) Let M be a loopless matroid of rank $r \geq 1$ on $n \geq 2r$ elements. Then,

$$\chi(KG(M)) \leq \min\left\{\min_{C^* \in \mathcal{C}^*}\{|C^*|\}, n - 2r + 2\right\}$$

where $\mathcal{C}^*$ denotes the set of cocircuits of M.

Matroid Kneser graph

Let M_G be the graphic matroid associated to the graph G



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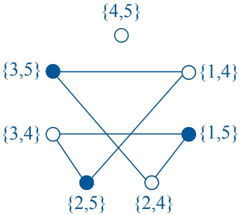

$KG(M_G)$

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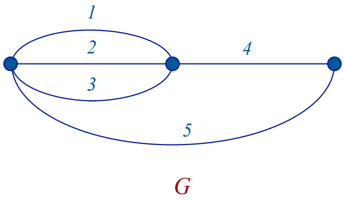

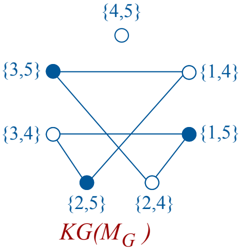

We have
$\mathcal{B}(M_G) = \{\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$ and
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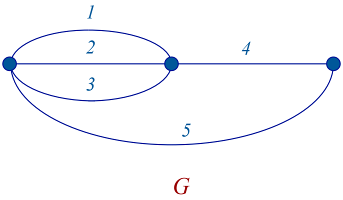

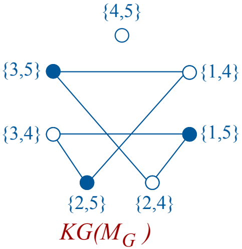

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Obtaining $\chi(KG(M_G)) \leq \min\{2, 5-2\cdot 2+2\} = \min\{2,3\} = 2$.

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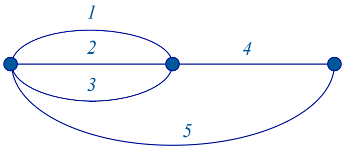

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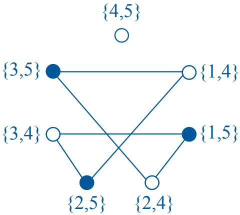

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We clearly have that $\chi(KG(M_G)) = 2$.