Knots through combinatorics

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A challenging problem

Question : have you ever tried to untangle a rope, headphone cables, necklace or any other strand ?

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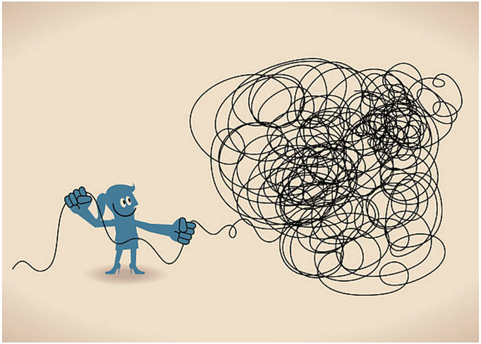

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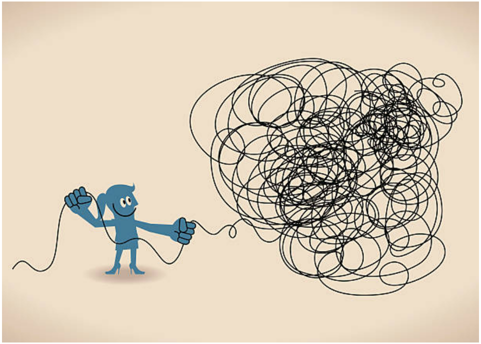

The longer a strand, the more likely it is to tangle.

Knot theory

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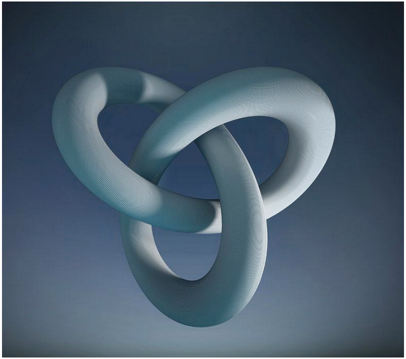

Knot theory : diagrams

Trivial knot
0_1

Trefoil knot
3_1

Figure-eight knot
4_1

Pentafoil knot
5_1

Trivial link
0^2_1

Hopf link
2^2_1

Solomon link
4^2_1

Borromean link
6^3_2

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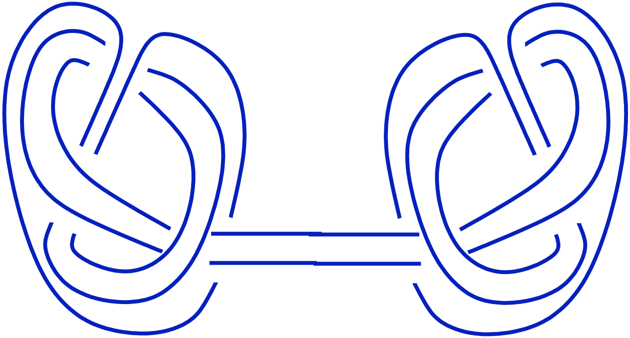

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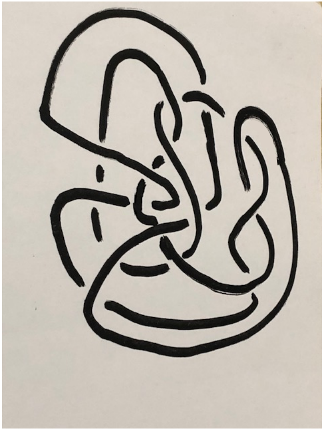
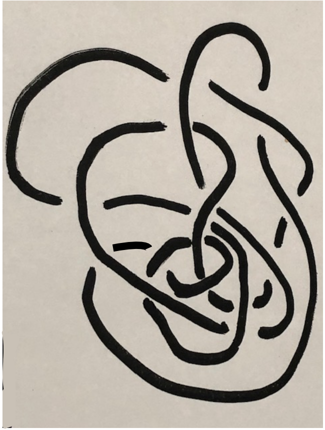
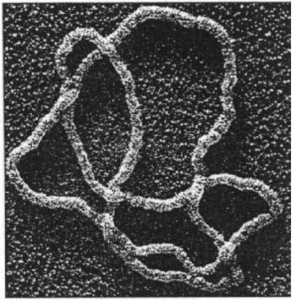




Application : DNA (Deoxyribonucleic Acid)

A fundamental application of this problem can be found in the study of DNA



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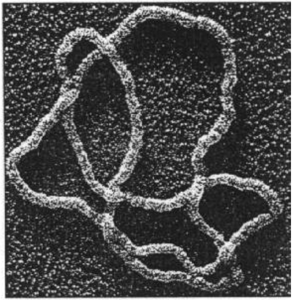

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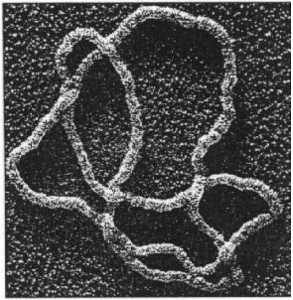
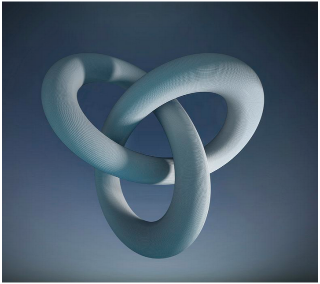


when an enzyme acts on the initial form of DNA, it gives new forms of DNA. We would like to know if they are still unraveled. Difficulty : DNA is very tangled inside the cell (equivalent to approximately 200 km of fishing line inside a football).

First step ...

Is the trefoil trivial ?



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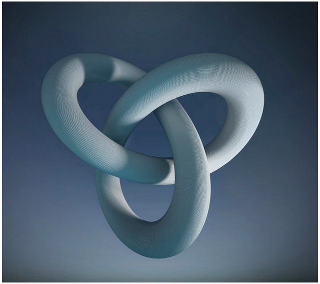

Theorem (Papakyrikopoulos, 1957) Un knot K is trivial if and only if the fundamental group of the complementary space of K is abelian.

3-coloring

(R. Fox) A knot diagram K is 3-colorable if one can color each arc of the diagram with **red**, **blue** and **green**, such that

- at least 2 colors are used,

- at each crossing we have either 3 different colors or only one color.

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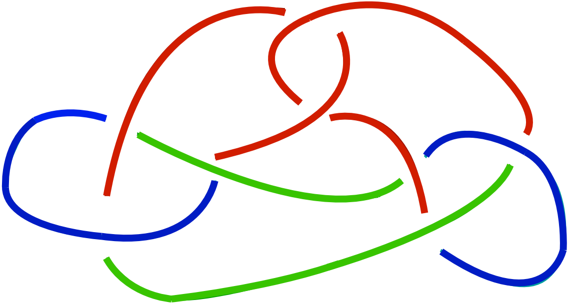

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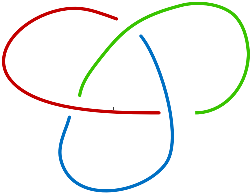

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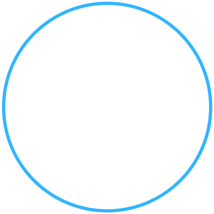

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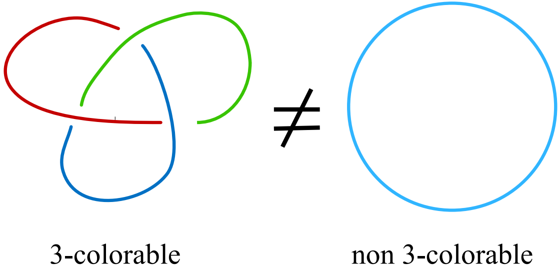

Colorability $\pmod p$

A knot diagram K is colorable $\pmod p$ if each arc of the diagram can be labeled with an integer in $\{1, \ldots, p-1\}$ such that

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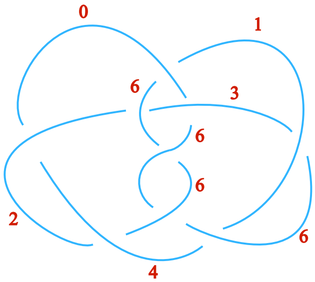

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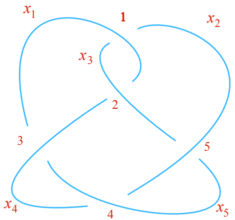

$$M = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 2 & -1 & 0 & -1 \end{pmatrix}$$

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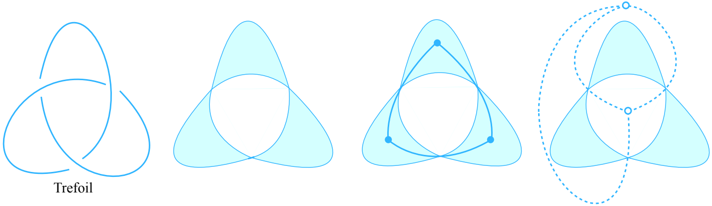
- $\det(K)$ is an invariant of K
- $\det(K) = |\triangle_K(-1)|$ where $\triangle_K(t)$ is the Alexander polynomial
- $\det(K) = |J_K(-1)|$ where $J_K(t)$ is the Jones polynomial

Tait graphs



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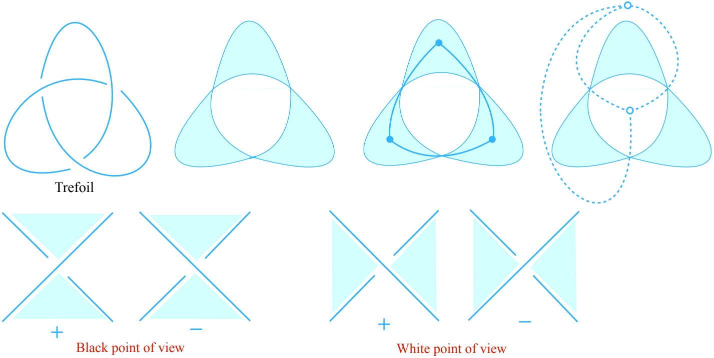

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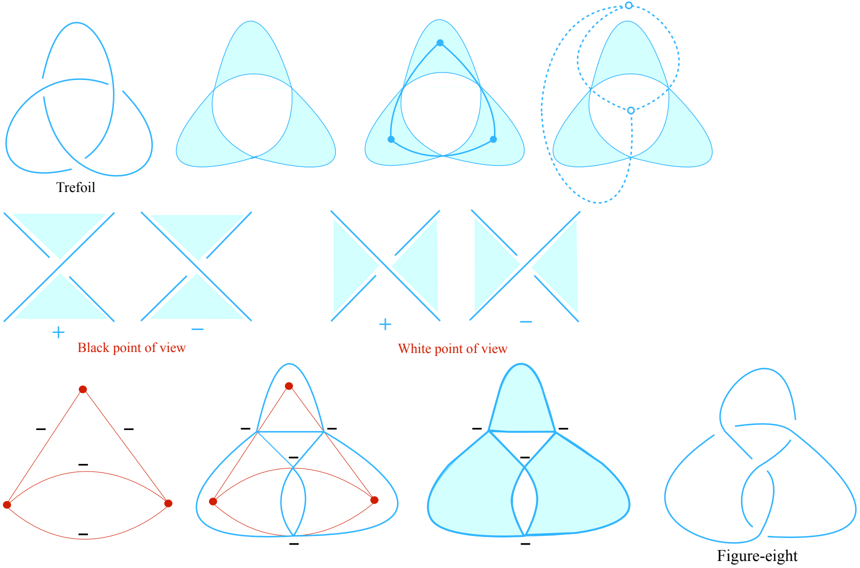
Trefoil
+
−
Black point of view
+
−
White point of view
−
−
−
−
−
−
−
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−
−
−
−
Figure-eight

Spanning trees

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Remark : If L_G is alternating then

$$det(L_G) = \# \text{ of spanning trees of } G$$

Fourier-Hadamard transforms

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function.
Let $\text{supp}(f) = \{\boldsymbol{x} \in \mathbb{F}_2^n \mid f(\boldsymbol{x}) \neq 0\}$ be its support.

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The Fourier-Hadamard transform of f is defined as

$$\widehat{f}(\boldsymbol{u}) = \sum_{\boldsymbol{x} \in \mathbb{F}_2^n} f(\boldsymbol{x})(-1)^{\boldsymbol{x} \cdot \boldsymbol{u}} = \sum_{\boldsymbol{x} \in \text{supp}(f)} (-1)^{\boldsymbol{x} \cdot \boldsymbol{u}}$$

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f can be represented by elements in the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$, called Numerical Normal Form (NNF). It can be written as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{F}_2^n} \lambda_{\boldsymbol{y}} \boldsymbol{x}^{\boldsymbol{y}}$$

where $\boldsymbol{x}^{\boldsymbol{y}} = \prod_{i=1}^{n} x_i^{y_i}$.

Fourier-Hadamard transforms and determinant

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We define $FH_G(x_1, \ldots, x_n)$, to be the NNF of the Fourier-Hadamard transform of f_G, that is,

$$FH_G(x_1, \ldots, x_n) = \widehat{f}_G(x_1, \ldots, x_n).$$

Formula for the determinant

Theorem (Gros, Pastor-Diaz, R.A. 2024) Let (G, χ_E) be an edge-signed connected planar graph and let L_G be the link arising from G. Then,

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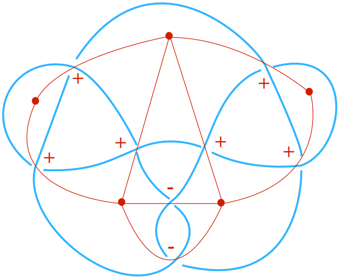
Our method yields to a straightforward procedure to answer this question.

Example



$\left|FH_G(1,1,0,0,0,0,0,0)\right| = 15 = \det(8_{21})$ (difference between the 24 negative-spanning trees and 9 positive-spanning trees.

Oriented matroids

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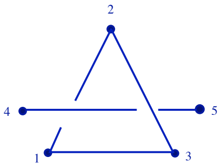
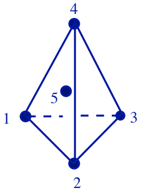
Example : $d = 3$.



(+,+,+,–,–)



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These are called Radon partitions

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Spatial representation of K_5

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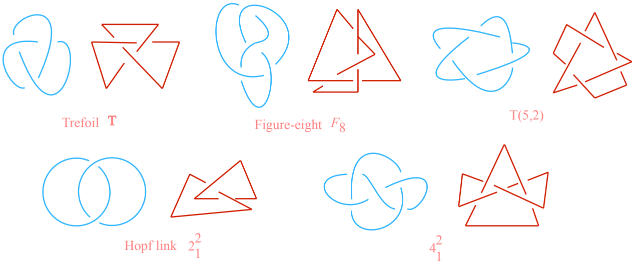

Some values of $m(L)$

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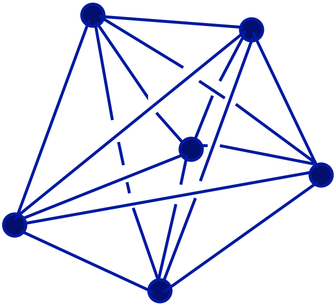

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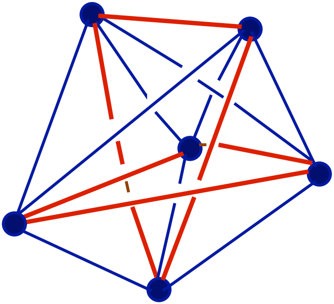

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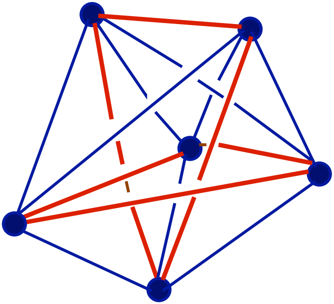

Theorem (R.A. 1998, 2000, 2009)
$m(T \text{ or } T^*) = 7,\ m(4_1^2) > 7,\ m(F_8) > 8,\ m(T(5,2)) > 8$

Las Vergnas' question

Let $X = (x_0, \ldots, x_{n-1})$ be a n-uple of points in $\mathbb{R}^3$ in general position. Let K_X be the polygonal knot defined by the segments $[x_i, x_{i+1}]$ (addition $\pmod{n}$)

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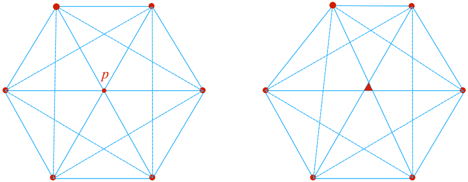
Question (M. Las Vergnas) Is it true that K_X only depends on the oriented matroid induced by $x_0, \ldots, x_{n-1}$?

In other words

Question Let X and Y be two sets of n points. Is it true that if there is a bijection $\varphi : X \rightarrow Y$ preserving Radon partitions then K_X is isotopic to K_Y ?

Strong geometry



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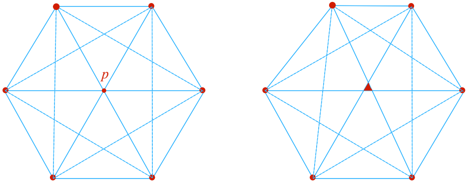

Two configurations of points having the same oriented matroid

We introduce a new oriented matroid $M_{\wedge}(X)$ arising from the set of lines spanned by X.

Strong geometry

Let X be a n-uple of points in the space.

We define strong geometry associated to X, denoted by $SGeom(X)$, as the structure composed by $M(X)$ and $M_{\wedge}(X)$.

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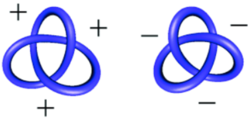
Theorem (Gros, R.A. 2025) K_X can be completely determined by $SGeom(X)$.

Proof. Combining the information of $SGeom(X)$ and Gauss diagrams.

Achirality



Trefoil and its mirror

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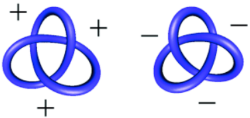

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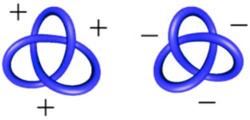

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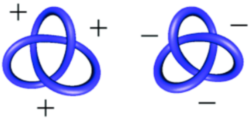

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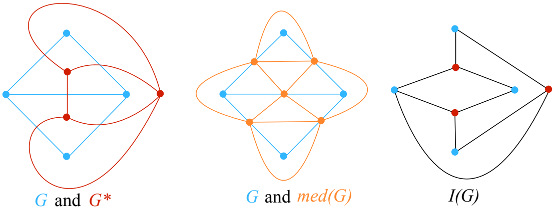
Remark : the Trefoil is not achiral while the Figure-eight is achiral.

Maps

G planar, G^* dual, $med(G)$ medial, $I(G)$ incident



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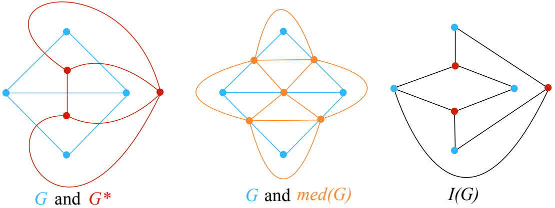

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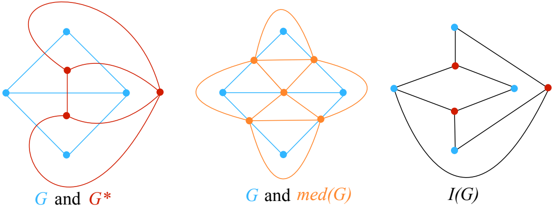

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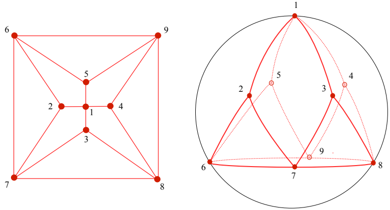

Maps

Any embedding of G into $\mathbb{S}^2$ partition the 2-sphere into simply connected regions of $\mathbb{S}^2 \setminus G$ called the faces of the embedding

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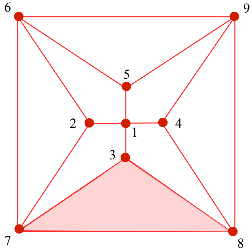
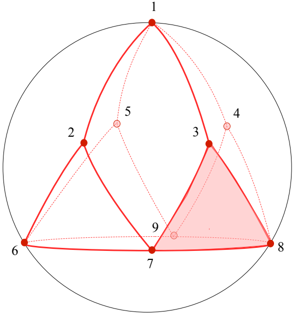
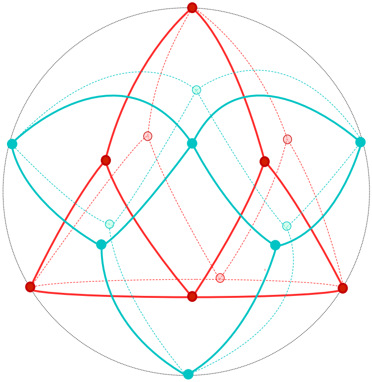




An embedding of G and its dual G^* in $\mathbb{S}^2$.



Antipodally self-dual maps

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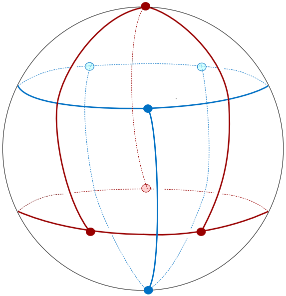

Maps of K_4 and K_4^*

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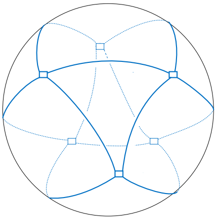

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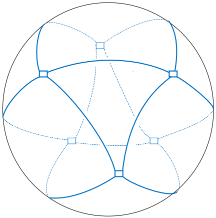

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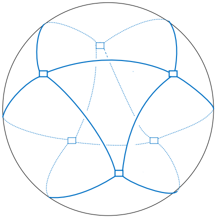

Theorem (Montejano, R.A., Rasskin, 2022) If G is antipodally self-dual then $med(G)$ is antipodally symmetric.

Question : Which graphs are antipodally self-dual ?

Antipodally self-dual : charaterization

Theorem (Montejano, R.A., Rasskin, 2022) G is antipodally self-dual if and only if $I(G)^{\square}$ admits an *involutive labeling without fixed vertex*.

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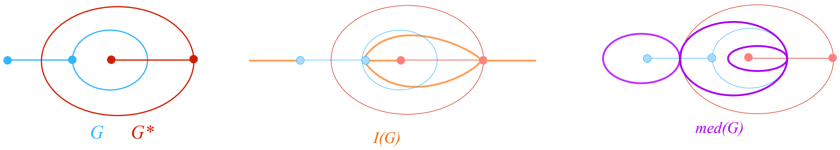

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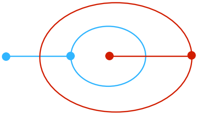
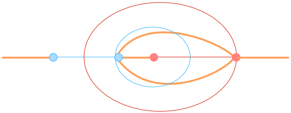
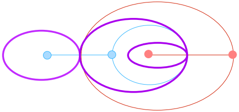


G G^*



I(G)



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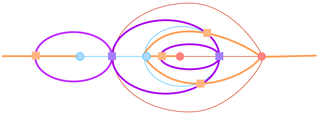
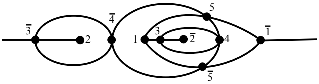
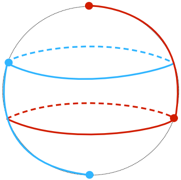


$I(G)^{\square}$

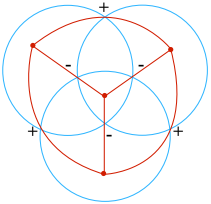


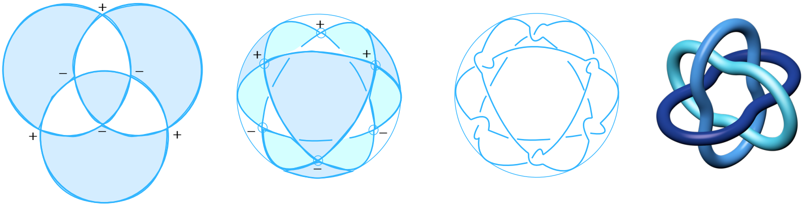
involutive labeling



antipodally embedding

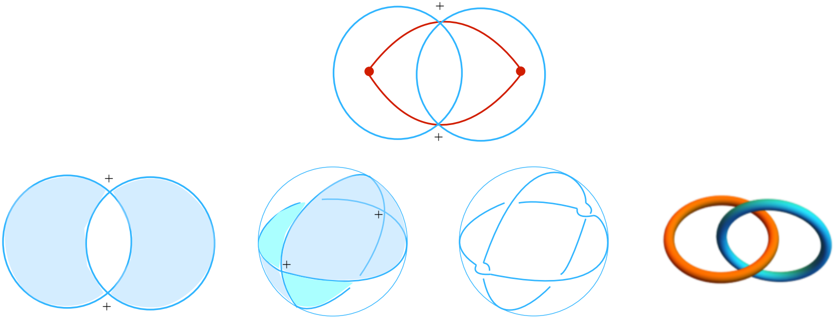
Maps and diagrams





Borromean rings

Maps and diagrams



Hopf link

Achiriality

Theorem (Montejano, R.A., Rasskin, 2022) Let (G, S_E) be antipodally self-dual edge-signed map ($med(G)$ is antipodally symmetric, realized by a map α). If either

(a) α is color-preserving and sign-reversing ; or

(b) α is color-reversing and sign-preserving,

then the link L obtained from (G, S_E) is achiral.

Self-dual pairing

$Aut(G)$: automorphism group of G (isomorphisms of G into G)

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Following ϕ the correspondance $*$ gives a permutation on $(V \cup E \cup F)$ (preserve incidences and reverse dimension of elements). All such permutation generate a group

$$Cor(G) = Aut(G) \cup Dual(G)$$

where $Aut(G)$ is a subgroup of $Cor(G)$ of index 2.

Isometries

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Theorem (Montejano, R.A., Rasskin, 2022) Let G be a self-dual map. If either

• there exists $\sigma \in Dual(G)$ such that the isometry $\tilde{\sigma}$ is oriented-preserving or

• there exists $\sigma \in Aut(G)$ such that the isometr $\tilde{\sigma}$ is not oriented-preserving

then the link $L(G, S)$ is achiral for every signature S


THANKS FOR
YOUR ATTENTION

Antipodally self-dual : necessary conditions

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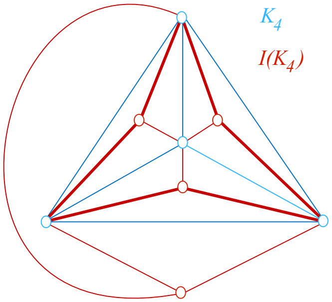
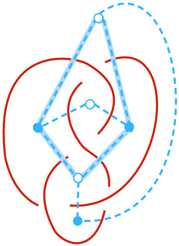
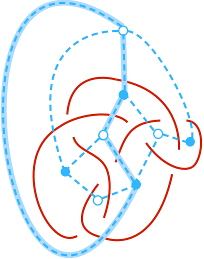


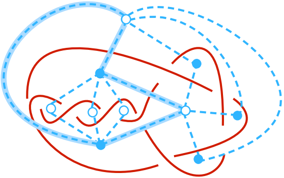
Some achiral knots



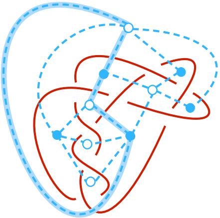
4_1 (figure-eight)



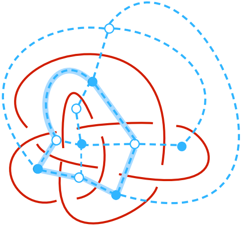
6_3



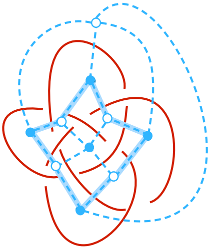
8_3



8_9



8_{12}



8_{18}

Isometries

Theorem (B. Servatius, H. Servatius, 1995) The self-dual pairings are classified in 24 classes :
$[2,q] \triangleright [q]$, $[2,q]^+ \triangleright [q]^+$, $[2^+,2q] \triangleright [2q]$, $[2,q^+] \triangleright [q]^+$, $[2^+,2q^+] \triangleright [2q]^+$, $[2] \triangleright [1]$, $[2] \triangleright [2]^+$, $[4] \triangleright [2]$, $[2]^+ \triangleright [1]^+$, $[4]^+ \triangleright [2]^+$, $[2,2] \triangleright [2,2]^+$, $[2,4] \triangleright [2^+,4]$, $[2,2] \triangleright [2,2^+]$, $[2,4] \triangleright [2,2]$, $[2,4]^+ \triangleright [2,2]^+$, $[2^+,4] \triangleright [2,2]^+$, $[2^+,4] \triangleright [2^+,4^+]$, $[2,4^+] \triangleright [2^+,4^+]$, $[2,2^+] \triangleright [2^+,2^+]$, $[2,4^+] \triangleright [2,2^+]$, $[2,2^+] \triangleright [1]$, $[3,4] \triangleright [3,3]$, $[3,4]^+ \triangleright [3,3]^+$ and $[3^+,4] \triangleright [3,3]^+$

Achiriality

Theorem (Montejano, R.A., Rasskin, 2022) Let (G, S_E) be an edge-signed self-dual map. If the self-dual pairing of the map G is other than $[2, q^+] \triangleright [q]^+, [2^+, 2q^+] \triangleright [2q]^+, [2] \triangleright [2]^+, [2, 2] \triangleright [2, 2]^+, [2^+, 4] \triangleright [2, 2]^+, [3^+, 4] \triangleright [3, 3]^+$ then $L(G, S_E)$ is achiral for every signature S_E.