On perfect squares and primes in numerical semigroups

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Stamp problem : proposed by Ian Stewart

How hard can it be?

Deliver the solution to an everyday puzzle and you could win the biggest prize in mathematics, says **Ian Stewart**

EVER since a Babylonian scribe decided to teach his students arithmetic by setting them problems using the formula "I found a stone but did not weigh it..." mathematicians have celebrated the hidden depths of apparently everyday problems. They have found inspiration in slicing pies, tying knots and spinning coins. But even mathematicians have been surprised by the depth of the mystery that lurks behind an innocent question about postage stamps.

Suppose that your post office sells stamps with just two values: 2 cents and 5 cents. By combining these values, you can make up almost any whole number of cents. For example, to post a letter costing 9¢, you could stick one 5¢ stamp and two 2¢ stamps on the envelope. Two values that you cannot achieve are 1¢ and 3¢ – and in fact these are the only impossible amounts. You can produce any even amount using 2¢ stamps – given a big enough envelope – and any odd value from 5¢ upwards, using one 5¢ stamp and multiple 2¢ stamps.

This example is typical. Given an unlimited supply of stamps, there is always some key value above which any total can be achieved by sticking the right combination of stamps on the envelope. This is also true if you have more than two denominations of stamp available.

But the million-dollar question is this: with *n* denominations of stamps available, what is that key value? The first person to consider a simple version of this question was James Joseph Sylvester in 1883 (to be precise, he was dealing with coins, but for our purposes we'll stick with stamps). Sylvester came up with a simple formula for finding this key value when dealing with just two denominations (see "Pushing the envelope", page 48).

In its general form, the postage-stamp problem really could be a million-dollar question: the Clay Mathematics Institute in Cambridge, Massachusetts, is offering exactly that amount to anyone who can solve a problem that is logically equivalent to it. We now have tantalising new hints that the postage-stamp problem – and therefore, perhaps, the related million-dollar enigma – might not be as daunting as it appears. So considering how to pay for posting our mail might lead to a breakthrough in one of the most significant mathematical problems of the 21st century.

It will never compute

The issue centres around the cost of solving a problem – not in dollars and cents, but in computational effort. We measure the difficulty of a calculation by the number of basic computational steps needed to complete it: for a particular size of problem – often measured in terms of the number of digits in the number to be crunched – what is the "running time" of the algorithm concerned? If the problem concerns 50-digit numbers rather than 25-digit numbers, say, how much longer does the algorithm take to get the answer? What about 100-digit numbers, or any number of digits? It should be noted that this running time is an abstract notion, related but not equivalent to the actual time taken by any given computer.

In broad terms, a computational method is practical – "efficient" or "easy", if you prefer to look at it that way – if the running time grows in step with some fixed power of the number of digits required to pose the question. For example, an algorithm for testing a number *n* to see whether it is prime may have a running time linked to the sixth power of the number of digits of *n*.

Such algorithms are said to be "class P", where the "P" stands for "polynomial". Algorithms that run in polynomial time are relatively stable: they do not get wildly slower with small increases in the size of the input. In contrast, non-P algorithms are generally impractical – "inefficient" or "hard" – and become unmanageable with relatively small increases in input size. It's not quite that straightforward, because some non-P algorithms are pretty efficient until the input size gets very big indeed, while some P algorithms depend on a parameter which is so large that they couldn't actually run within a human lifetime. Nevertheless, the distinction between P and non-P seems to be the most basic and important distinction in problems about the efficiency of algorithms – a way to formalise the intuitive ideas of "easy to compute" versus "hard to compute".

Are there any such things as truly hard problems? Yes, several kinds. The obvious ones are hard for a simple reason, such as printing out the answer takes too long. A good example is "print all ways to rearrange this list of symbols". With the 52 symbols in a pack of cards, the list would contain 80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000 arrangements, and you'd have to print the lot. These types of problem have to be excluded, which we do by introducing another class of algorithm, confusingly called NP, which run in "nondeterministic polynomial" time. A problem is NP if any proposed solution can be checked to determine whether it is right or wrong, in polynomial time – that is, in reasonable time. A rough analogy is solving a jigsaw puzzle. However long it takes to work out how to fit the pieces together – the nondeterministic aspect – a brief glance at the result usually reveals whether it is correct.

All this classification has led to a rather fundamental question, and whoever cracks it will take the Clay prize: is NP really any different from P? To put it plainly, if it is easy to check the accuracy of any proposed ▸

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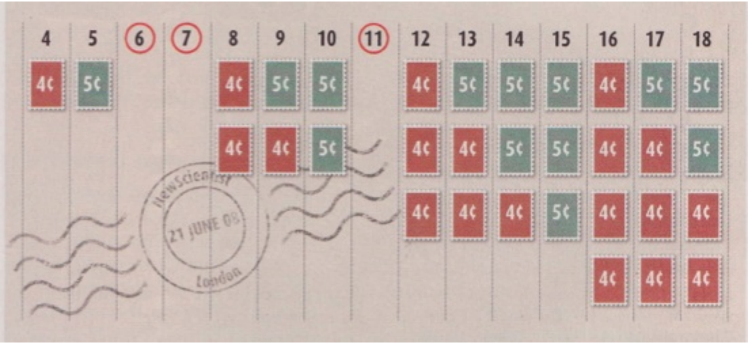

Frobenius variant

Let $a_1, \ldots, a_n$ be relatively prime positive integers and let

$$S = \langle a_1, \ldots, a_n \rangle = \left\{ \sum_{i=1}^{n} x_i a_i \mid x_i \in \mathbb{N} \right\}$$

semigroup generated by $a_1, \ldots, a_n$.

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k-power

P-type function

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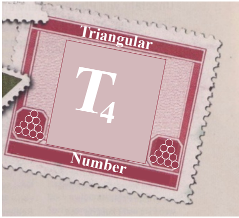
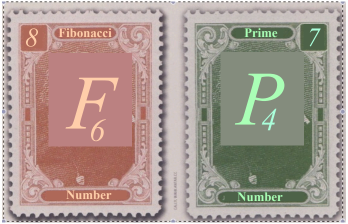
P : Triangular Fibonacci Prime





Perfect square

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- Upper bounds for ${}^2r(S_A)$ where S_A is arithmetic progression, i.e.,

$$S_A = \langle a, a+d, a+2d, \ldots, a+kd \rangle$$

- Some exact values for $S = \langle a, b \rangle$

Arithmetic progression

Lemma Let M be a non-negative integer and let x and y be the unique integers such that $M = ax + dy$, with $0 \leq y \leq a - 1$. Then,

$$M \in S_A \text{ if and only if } y \leq kx \text{ (with } x \geq 0).$$

Arithmetic progression

Lemma Let M be a non-negative integer and let x and y be the unique integers such that $M = ax + dy$, with $0 \leq y \leq a-1$. Then,

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Proposition (key) Let i be an integer. Let λ_i be the unique integer in $\{0, \ldots, d-1\}$ such that $\lambda_i a + i^2 \equiv 0 \pmod{d}$. Then,

$$(a-i)^2 \in S_A \text{ if and only if } (i+kd)^2 \leq \left(\left(\left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor + k\right) d - \lambda_i\right)(a+kd).$$

Arithmetic progression

Theorem (Chappelon+R.A. 2022) Let $d \geq 3$ and $a + kd \geq 4kd^3$.
Let $\{\alpha_1 < \cdots < \alpha_n\} \subseteq \{0, \ldots, d-1\}$ such that
$\lambda_{\alpha_j} = \lambda^* = \max\limits_{0 \leq i \leq d-1} \{\lambda_i\}$.

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$$^2r(S_A) \leq (a - ((\mu - k)d + \alpha_{j+1}))^2 = h(a, d, k)$$

where $(\mu d + \alpha_j)^2 \leq (kd - \lambda^*)(a + kd) < (\mu d + \alpha_{j+1})^2$.

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Corollay ($k = 1$) Let $d \geq 3$ and $a + d \geq 4d^3$. Then, $^2r(a, a + d) = h(a, d, 1)$.

For $\langle a, a+3 \rangle$

Theorem (Chappelon+R.A. 2022) Let $a \geq 2$ be an integer not divisible by 3. Then,

$$
{}^2r(a, a+3) = \begin{cases} (a-(3b-1))^2 & \text{if either } (3b+1)^2 \leq a+3 < (3b+2)^2, \quad a \equiv 1 \bmod 3, \\ & \text{or } (3b+1)^2 \leq 2(a+3) < (3b+2)^2, a \equiv 2 \bmod 3, \\ (a-(3b+1))^2 & \text{if either } (3b+2)^2 \leq a+3 < (3b+4)^2, \quad a \equiv 1 \bmod 3, \\ & \text{or } (3b+2)^2 \leq 2(a+3) < (3b+4)^2, a \equiv 2 \bmod 3. \end{cases}
$$

For $\langle a, a+4 \rangle$

Theorem (Chappelon+R.A. 2022) Let $a \geq 3$ be an odd integer not divisible by 4. Then,

$$
{}^2r(a, a{+}4) = \begin{cases} (a-(4b-1))^2 & \text{if either } (4b+1)^2 \leq a+4 < (4b+3)^2, \quad a \equiv 1 \bmod 4, \\ & \text{or } (4b+1)^2 \leq 3(a+4) < (4b+3)^2, a \equiv 3 \bmod 4, \\ & \\ (a-(4b+1))^2 & \text{if either } (4b+3)^2 \leq a+4 < (4b+5)^2, \quad a \equiv 1 \bmod 4, \\ & \text{or } (4b+3)^2 \leq 3(a+4) < (4b+5)^2, a \equiv 3 \bmod 4. \end{cases}
$$

For $\langle a, a+5\rangle$

Theorem (Chappelon+R.A. 2022) Let $a \geq 2$ be an integer not divisible by 5. Then,

$$
{}^2r(a, a{+}5) = \begin{cases}
1 & \text{if } a = 2 \text{ or } 4,\\
10^2 & \text{if } a = 13,\\
(a-6)^2 & \text{if } a = 27 \text{ or } 32,\\
(a-(5b-2))^2 & \text{if either } (5b+2)^2 \leq a+5 < (5b+3)^2, \quad a \equiv 4 \bmod 5,\\
 & \quad \text{or } \ (5b+2)^2 \leq 2(a+5) < (5b+3)^2, a \equiv 2 \bmod 5,\\
(a-(5b-1))^2 & \text{if either } (5b+1)^2 \leq a+5 < (5b+4)^2, \quad a \equiv 1 \bmod 5,\\
 & \text{or } \ (5b+1)^2 \leq 2(a+5) < (5b+4)^2, a \equiv 3 \bmod 5, a \neq 13,\\
(a-(5b+1))^2 & \text{if either } (5b+4)^2 \leq a+5 < (5b+6)^2, \quad a \equiv 1 \bmod 5,\\
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 & \text{or } \ (5b+3)^2 \leq 2(a+5) < (5b+7)^2, a \equiv 2 \bmod 5, a \neq 2, 27, 32.
\end{cases}
$$

For $\langle a, a+1 \rangle$ and $\langle a, a+2 \rangle$

Theorem (Chappelon+R.A. 2022) Let a be a positive integer such that $b^2 < a < a+1 < (b+1)^2$ for some integer $b \geq 1$. Then,

$$^{2}r(a, a+1) = (a-b)^2.$$

Theorem (Chappelon+R.A. 2022) Let $a \geq 3$ be an odd integer such that $(2b+1)^2 < a < a+2 < (2b+3)^2$ for some integer $b \geq 1$. Then,

$$^{2}r(a, a+2) = (a-(2b+1))^2.$$

Unexpected connection with $\sqrt{2}$

Let $(u_n)_{n\geq 1}$ be the recursive sequence : $u_1 = 1, u_2 = 2, u_3 = 3,$
$u_{2n} = u_{2n-1} + u_{2n-2}$ and $u_{2n+1} = u_{2n} + u_{2n-2}$ for all $n \geq 2.$

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This sequence appears in different contexts

Example :

This sequence corresponds to the denominators of Farey fraction approximations to $\sqrt{2}$ where the fractions are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{17}{12}, \frac{24}{17} \ldots$$

Conjecture (Chappelon+R.A. 2022) If $a=b^2$ for some integer $b\geq 1$ then

$$^2r(a,a+1)=\begin{cases}\left(a-\left\lfloor b\sqrt{2}\right\rfloor\right)^2 & \text{if } b\notin \bigcup\limits_{n\geq 0}\{u_{4n+1},u_{4n+2}\},\\ \left(a-\left\lfloor b\sqrt{3}\right\rfloor\right)^2 & \text{if } b\in \bigcup\limits_{n\geq 0}\{u_{4n+1},u_{4n+2}\}.\end{cases}$$

If $a+1=b^2$ for some integer $b\geq 1$ then

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- Verified by computer for all integers $a \geq 2$ up to 10^6.
- Recently D.S. Binner has made a surprising connections between this conjecture and Pell equations of the form $x^2 - 2y = 1$.

Natural related question

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A first (natural) easier step would be the following

Question : Is there an integer verifying property P belonging to S smaller than $g(S)$?

Prime numbers

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Question : Is there always a prime number p belonging to S with $p < g(S)$?

Primes in $\langle a, b\rangle$

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Theorem (R.A. + Skałba 2020) Let $3 \leq a < b$ be two relatively prime integers. Then, for any fixed $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\pi_{\langle a,b\rangle} > C(\varepsilon)\frac{g(a,b)}{\log(g(a,b))^{2+\varepsilon}}$$

for ab sufficiently large.

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• Prime number theorem : $\pi(x) \sim \frac{x}{\log x}$

• Siegel-Walfisz theorem : to approximate the number of primes p such that $p \equiv b \pmod{a}$.

$$\frac{1}{\varphi(a)} \int_{b}^{g(a,b)} \frac{du}{\log u} + R$$

where φ is the Euler totient function and $|R| < D'(\varepsilon)\frac{g(a,b)}{(\log(g(a,b)))^{2+2\varepsilon}}$ uniformly in $[a, \ldots, g(a, b)]$.

Inocent conjecture

Example

$\langle 5, 7 \rangle = \{0,$ 

$,$ 

$, 10, 12, 14, 15,$ 

$,$ 

$, 20, 21, 22, 24, \rightarrow\}$

$\uparrow$

$g(\langle 5, 7 \rangle) = 23$

$\pi(23) = 8$

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A number of computer experiments lead us to the following.

Conjecture (R.A. + Skałba 2020) Let $2 \leq a < b$ be two relatively prime integers. Then,

$$\pi_{\langle a,b\rangle} > 0$$

and

$$\pi_{\langle a,b\rangle} \sim \frac{\pi(g(\langle a,b\rangle))}{2} \quad \text{for } a \rightarrow \infty.$$


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Thank you
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attention
Jorge

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