Oriented Matroids : introduction

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Notation

A signed set X is a set $\underline{X}$ divided in two parts (X^+, X^-), where X^+ is the set of the positive elements of X and X^- is the set of the negative elements. The set $\underline{X} = X^+ \cup X^-$ is called the support of X.

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The opposite of a signed set X, denoted by $-X$, is the signed set defined by $(-X)^+ = X^-$ and $(-X)^- = X^+$.
Given a signed set X and a set A we denote by $-_A X$ the signed set defined by $(-_A X)^+ = (X^+ \setminus A) \cup (X^- \cap A)$ and $(-_A X)^- = (X^- \setminus A) \cup (X^+ \cap A)$.

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We say that the signed set $-_A X$ is obtained by a reorientation of A.

Circuits

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(C0) $\emptyset \notin \mathcal{C}$,

(C1) (*symmetry*) $\mathcal{C} = -\mathcal{C}$,

(C2) (*incomparability*) for any $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$,

(C3) (*weak elimination*) for any $X, Y \in \mathcal{C}, X \neq -Y$, and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

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• Let $A \subseteq E$ and put $-_A\mathcal{C} = \{-_AX : X \in \mathcal{C}\}$. It is clear that $-_A\mathcal{C}$ is also the set of circuits of an oriented matroid, denoted by $-_AM$.

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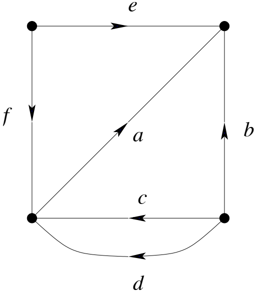
Notation For short, we write $X = a\overline{bc}de$ the signed set X defined by $X^+ = \{a, d, e\}$ and $X^- = \{b, c\}$.

Let D be the following oriented graph.



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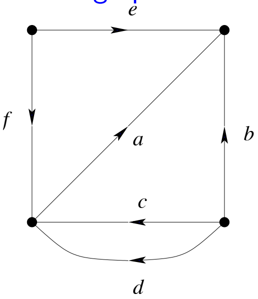

$$\begin{aligned}\mathcal{C}(D) \quad &= \{(a\bar{b}c), (a\bar{b}d), (a\bar{e}f), (c\bar{d}), (b\overline{ce}f), (b\overline{de}f),\\ &\quad (\bar{a}b\bar{c}), (\bar{a}b\bar{d}), (\bar{a}e\bar{f}), (\bar{c}d), (\bar{b}ce\bar{f}), (\bar{b}de\bar{f})\}.\end{aligned}$$

Configuration of vectors in the space

Let $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors generating a r-dimensional vector space over a ordered field, says $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^r$.

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We consider the minimal linear dependencies

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = 0 \text{ with } \lambda_i \in \mathbb{R}$$

We obtain an oriented matroid from E by considering the signed sets $X = (X^+, X^-)$ where

$$X^+ = \{i : \lambda_i > 0\} \text{ et } X^- = \{i : \lambda_i < 0\}$$

for all minimal dependencies among $\mathbf{v}_i$.

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This oriented matroid is called vectorial (or linear).

Configuration of points in the space

Any configuration of points $\{p_1, \ldots, p_n\}$ in the affine space induces an oriented matroid having as circuits the signed set from the coefficient of minimal affine dependencies, that is, linear combinations of the form

$$\sum_i \lambda_i p_i = 0 \text{ with } \sum_i \lambda_i = 0,\ \lambda_i \in \mathbb{R}.$$

Example

Let us consider the points in $\mathbb{R}^2$ given by the columns of matrix :

$$\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \overline{A} = \begin{pmatrix} -1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix} \end{array}$$

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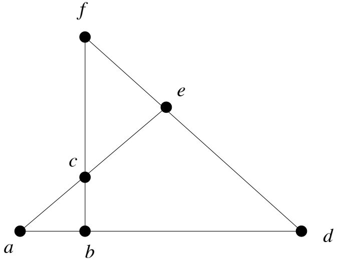
Matrix $\overline{A}$ correspond to points



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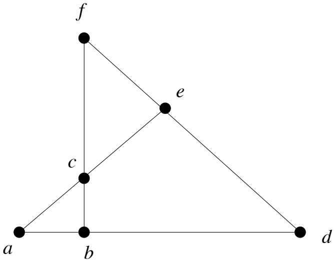

The set of circuits of the corresponding affine oriented matroid is

$$\begin{aligned}\mathcal{C}(\overline{A}) \quad &= \{(a\overline{b}d), (b\overline{c}f), (d\overline{e}f), (a\overline{c}e), (\overline{a}b\overline{e}f), (\overline{b}cd\overline{e}), (a\overline{c}df), \\ &\qquad (\overline{a}b\overline{d}), (\overline{b}c\overline{f}), (\overline{d}e\overline{f}), (\overline{a}c\overline{e}), (a\overline{b}e\overline{f}), (b\overline{cd}e), (\overline{a}c\overline{df})\}.\end{aligned}$$

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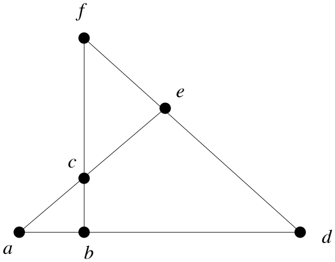

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For instance, $(a\bar{b}d)$ correspond to the affine dependecy $3(-1,0)^t - 4(0,0)^t + 1(3,0)^t = (0,0)^t$ with $3 - 4 + 1 = 0$.

Radon partitions

The circuits of an affine oriented matroid have a nice geometric interpretation. They can be thought as minimal Radon partitions.

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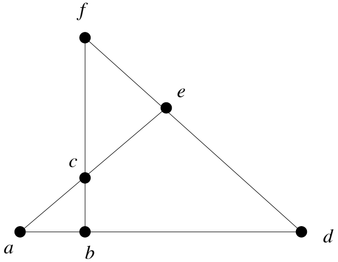
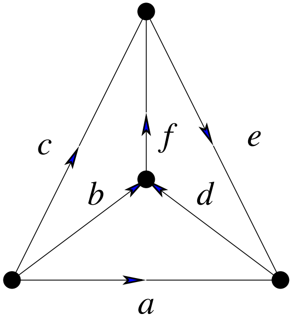
Given a circuit C, the convex hull of the positive elements of C intersect the convex hull of the negative elements of C.

Example



From the circuit $(a\bar{b}d)$ we see that the point b lies in the segment $[a, b]$ and from circuit $(\bar{a}b\bar{e}f)$ the segment $[a, e]$ intersect the segment $[b, f]$ (in the affine real espace).

We can check that the oriented matroid obtained form K_4 with the orientation illustrated below has the same set of circuits that $M(\overline{A})$



K_4 and $M(\overline{A})$ are isomorphic.

Let us consider the oriented matroid $-_d M(\overline{A})$ obtained by reorienting element d of $M(\overline{A})$. The set of circuits of $-_d M(\overline{A})$ is :

$$\begin{aligned}\mathcal{C} \quad = \{&(a\overline{b}\overline{d}), (b\overline{c}f), (\overline{d}\overline{e}f), (a\overline{c}e), (\overline{a}b\overline{e}f), (\overline{b}c\overline{d}\overline{e}), (a\overline{c}\overline{d}f),\\ &(\overline{a}bd), (\overline{b}c\overline{f}), (de\overline{f}), (\overline{a}c\overline{e}), (a\overline{b}e\overline{f}), (b\overline{c}de), (\overline{a}cd\overline{f})\}.\end{aligned}$$

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- $-_d M(\overline{A})$ is a graphic oriented matroid since it can be obtained by changing the orientation of the edge d.
- $-_d M(\overline{A})$ also correspond to the affine oriented matroid illustrated as before under the permutation $\sigma(a) = b, \sigma(b) = a, \sigma(c) = c, \sigma(d) = d, \sigma(e) = f, \sigma(f) = e$.

Minors

(Deletion) Let $M = (E, \mathcal{C})$ be an oriented matroid and let $F \subset E$. Then,

$$\mathcal{C}' = \{X \in \mathcal{C} : \underline{X} \subseteq F\}$$

the set of circuits in M contained in F, is the set of circuits of an oriented matroid in F.

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This oriented matroid is called a sub-matroid induced by F, and denoted by $M|_F$.

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(Contraction) Let $M = (E, \mathcal{C})$ be an oriented matroid and let $F \subset E$. Then,

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This oriented matroid is called a contraction of M over F, and it is denoted by M/F.

Duality

Two signed sets X et Y are said orthogonal, denoted by $X \perp Y$, if either $\underline{X} \cap \underline{Y} = \emptyset$ or if $X|_{X \cap Y}$ and $Y|_{X \cap Y}$ are neither opposite nor equal, that is, there exists $e, f \in \underline{X} \cap \underline{Y}$ such that $X(e)Y(e) = -X(f)Y(f)$.

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Let $M = (E, \mathcal{C})$ be an oriented matroid, then

(i) there exists a unique signature of $\mathcal{C}^*$ the cocircuits of $\underline{M}$ such that

$$(\perp) \qquad X \perp Y \;\text{ pour tout }\; X \in \mathcal{C} \;\text{ et }\; Y \in \mathcal{C}^*.$$

(ii) The collection $\mathcal{C}^*$ is the set of circuits of an oriented matroid over E, denoted by M^* and called dual (or orthogonal) of M.

$(iiii)$ We have $M^{**} = M$.

Geometric interpretation of cocircuits

Let E be a set of vectors generating $\mathbb{R}^d$ and let $M = (E, \mathcal{C})$ be the oriented matroid of rank r of linear dependencies of E.

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Let H be a hyperplane of $\underline{M}$, i.e., a closed set of E generating a hyperplane in $\mathbb{R}^d$. We recall that $D = E \setminus H$ is a cocircuit of $\underline{M}$.

Let h be the linear function in $\mathbb{R}^d$ such that $kernel(h)$ is H (unique up to scaling).

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The signature of D in M^* is given by

$$D^+ = \{e \in D : h(e) > 0\} \text{ and } D^- = \{e \in D : h(e) < 0\}.$$

Example

Let $V = \{a, b, c, e, f\}$ be the vectors given in the following matrix

$$\begin{array}{cc} & \begin{array}{ccccc} a & c & f & b & e \end{array} \\ A' = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

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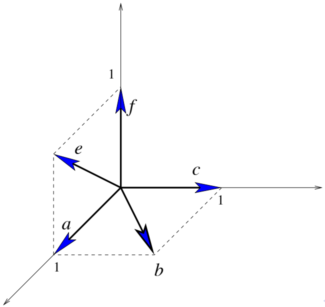
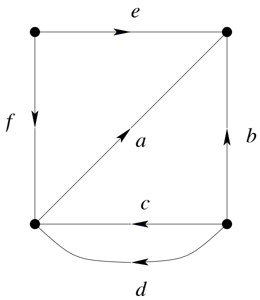
corresponding to vectors



The set of circuits of $M(A')$ is given by $M(D) \setminus d$ where D is the diagraph.



The vector configuration of the dual space V is given by the columns of

$$\begin{array}{cc} & \begin{array}{ccccc} a^{\perp} & c^{\perp} & f^{\perp} & b^{\perp} & e^{\perp} \end{array} \\ A'^{\perp} = & \begin{pmatrix} -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

We thus have that minimal dependencies among the columns of $A'^{\perp}$ are :

$$\begin{aligned}\mathcal{C}(A'^{\perp}) = \mathcal{C}^*(A') \quad &= \{a^\perp e^\perp b^\perp, a^\perp e^\perp \overline{c^\perp}, a^\perp \overline{f^\perp} b^\perp, a^\perp \overline{f^\perp c^\perp}, b^\perp c^\perp, e^\perp f^\perp, \\ &\overline{a^\perp e^\perp b^\perp}, \overline{a^\perp e^\perp} c^\perp, \overline{a^\perp} f^\perp \overline{b^\perp}, \overline{a^\perp}^{\perp} c^\perp, \overline{b^\perp c^\perp}, \overline{e^\perp f^\perp}\}.\end{aligned}$$

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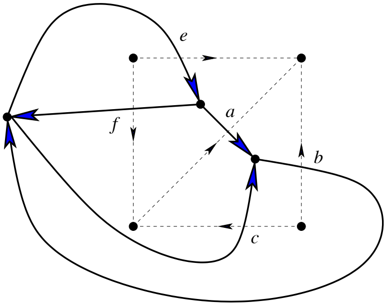
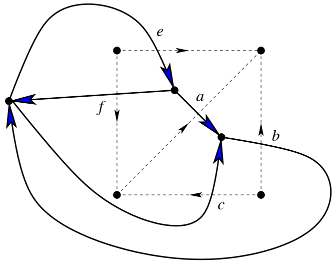
Recall that $M(A^{\perp})$ is isomorphic to $M(D')$ where D' is the oriented graph dual to the planar signed graph $D \setminus \{d\}$



Hyperplane-Cocircuits



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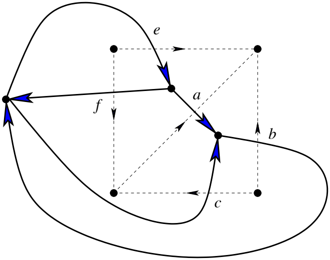
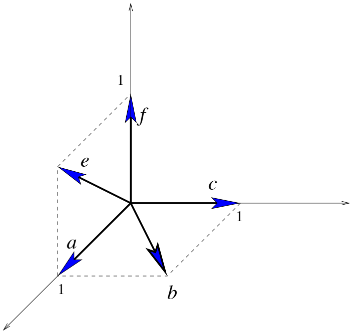


The set $\{e, f\}$ of D' is a minimal cut and thus a circuit of D' (or a cocircuit of $D \setminus \{d\}$). It corresponds to the hyperplane $E \setminus \{e, f\} = \{a, b, c\}$ of $D \setminus \{d\}$. The set $\{abc\}$ is a hyperplane since $r(\{abc\}) = 2$ and $cl(\{a, b, c\}) = \{a, b, c\}$.

Hyperplane-Cocircuits

Geometrically, the vectors $\{a, b, c\}$ generate a hyperplane but they do not form a base.



Geometric interpretation of cocircuits : affine case

Let E be a configuration of points in the $(d-1)$-affine space. Let D be a cocircuit of the oriented matroid of affine linear dependecies of E. The signature of D in M^* is

$$D^+ = D \cap H^+ \text{ et } D^- = D \cap H^-$$

where H^+ and H^- are the two open spaces in $\mathbb{R}^{d-1}$ determined by a hyperplan affine H containing $E \setminus D$.

Base orientations

A basis orientation of an oriented matroid M is an application from the set of ordered bases of M to $\{-1,+1\}$ verifying

$(B1)$ χ est alternating

(P) (pivoting property) if $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$ are two ordered bases of M with $e \neq f$ then,

$$\chi(f, x_2, \ldots x_r) = -C(e)C(f)\chi(e, x_2, \ldots, x_r)$$

where C is one of the two circuits of M in $(e, f, x_2, \ldots, x_r)$.

We notice that if χ is a basis orientation of $\mathcal{M}$ then $\mathcal{M}$ is determined only by $\underline{\mathcal{M}}$ and χ.

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Indeed, we can find the signs of the elements $C \in \mathcal{C}(\underline{M})$ from χ as follows : Choose $x_1, \ldots x_r, x_{r+1} \in M$ such that $C \subset \{x_1, \ldots, x_{r+1}\}$ and $\{x_1, \ldots, x_r\}$ is a base of $\underline{M}$. Then,

$$C(x_i) = (-1)^i \chi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+1}) \text{ for any } x_i \in C.$$

Dual version

We also have the dual version for the pivoting property (P) :

(P^*) (pivoting dual property) if $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$ are two ordered bases of M with $e \neq f$ then,

$$\chi(f, x_2, \ldots, x_r) = -D(e)D(f)\chi(e, x_2, \ldots, x_r)$$

where D is one of the two cocircuits of M complement to the hyperplane generated by $(x_2, \ldots, x_r)$ in M.

Chirotope

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($CH0$) $\chi \neq 0$,

($CH1$) χ is alternating, i.e., $\chi(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = sign(\sigma)\chi(x_1, \ldots, x_r)$ for any $x_1, \ldots, x_r \in E^r$ and any permutation σ.

($CH2$) for any $x_1, \ldots, x_r, y_1, \ldots, y_r \in E^r$ such that

$\chi(y_i, x_2, \ldots, x_r) \cdot \chi(y_1, \ldots, y_{i_1}, x_1, y_{i+1}, \ldots, y_r) \geq 0$ for any $i = 1, \ldots, r$

then

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Chirotope for linear matroids

If M is an oriented matroid of rank r of the linear dependencies of a set of vectors $E \subset \mathbb{R}^r$, then the corresponding chirotope χ is given by

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In this case the axiom $(CH2)$ is an abstraction of the Grassmann-Plücker relation for the determinant claiming that if $x_1, \ldots, x_r, y_1, \ldots, y_r \in \mathbb{R}^r$ then

$$det(x_1, \ldots, x_r) \cdot det(y_1, \ldots, y_r) = \sum_{i=1}^{r} det(y_i, x_2, \ldots, x_r) \cdot det(y_1, \ldots, y_{i_1}, x_1, y_{i+1}, \ldots, y_r)$$

Theorem Let $r \geq 1$ be an integer and let E be a finite set. An application

$$\chi : E^r \longrightarrow \{-1, 0, +1\}$$

is a basis orientation of an oriented matroid of rank r over E if and only if χ is a chirotope.

Contraction Let $A \subset E$. Recall that $\mathcal{C}/A = Min\{C \setminus A : C \in \mathcal{C}\}$.
Let $a_1, \ldots, a_{r-s}$ be a base of A in M. Then,

$$\begin{array}{rccc} \chi/A : & (E \setminus A)^s & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_s) & \longmapsto & \chi(x_1, \ldots, x_s, a_1, \ldots, a_{r-s}) \end{array}$$

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Deletion Let $A \subset E$ and suppose that $M \setminus A$ is of rank $s < r$. Recall that $\mathcal{C} \setminus A = \{C \in \mathcal{C} : C \cap A = \emptyset\}$. Let $a_1, \ldots, a_{r-s} \in A$ such that $E \setminus A \cup \{a_1, \ldots, a_{r-s}\}$ generate M. Then,

$$\begin{array}{rlcl} \chi \setminus A : & (E \setminus A)^s & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \ldots, x_s) & \longmapsto & \chi(x_1, \ldots, x_s, a_1, \ldots, a_{r-s}) \end{array}$$

Reorientation Let $A \subset E$ then the set of circuits of $-_A M$ is given by $-_A\mathcal{C} = \{-_A C : C \in \mathcal{C}\}$ where the signature of $-_A C$ is defined by $(-_A C)(x) = (-1)^{|A\cap\{x\}|} \cdot C(x)$. Then

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Duality Let $E = \{1, \dots, n\}$. Given a $(n-r)$-set $(x_1, \dots, x_{n-r})$, we write $(x'_1, \dots, x'_r)$ for one permutation of $E \setminus \{x_1, \dots, x_{n-r}\}$. In particular, $\{x_1, \dots, x_{n-r}, x'_1, \dots, x'_r\}$ is a permutation of $\{1, \dots, n\}$ where its sign, denoted by $sign\{x_1, \dots, x_{n-r}, x'_1, \dots, x'_r\}$, is given by the parity of the number of inversions of this set. Then,

$$\begin{array}{rccl} \chi^* : & E^{n-r} & \longrightarrow & \{-1, 0, +1\} \\ & (x_1, \dots, x_{n-r}) & \longmapsto & \chi(x'_1, \dots, x'_r) sign\{x_1, \dots, x_{n-r}, x'_1, \dots, x'_r\} \end{array}$$

Vectors and covectors

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$A \circ B = (A^+ \cup (B^+ \setminus \underline{A}),\ A^- \cup (B^- \setminus \underline{A}))$.

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$-$ $Y = (Y^+, Y^-)$ is a covector of M if and only if there is an affine hyperplane H (not necessarily generated by points of M) such that $Y^- = E \cap H^-$ and $Y^+ = E \cap H^+$ where H^- and H^+ are the open half-spaces induced by H.

Vectors and covectors

Theorem A collection V of signed subsets of a set E is the set of vectors of an oriented matroid if and only if the following properties are verified :

$(V0)$ $\emptyset \in V$,

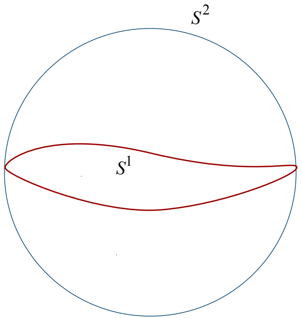
$(V1)$ (symmetry) $V = -V$,

$(V2)$ (composition) for all $X, Y \in V$ we have $X \circ Y \in V$,

$(V3)$ (vector strong elimination) for all $X, Y \in V, e \in X^+ \cap Y^-$ and $f \in (\underline{X} \setminus \underline{Y}) \cup (\underline{Y} \setminus \underline{X}) \cup (X^+ \cap Y^+) \cup (X^- \cap Y^-)$, there exists $Z \in V$ such that $Z^+ \subseteq (X^+ \cap Y^+) \setminus e$, $Z^- \subseteq (X^- \cap Y^-) \setminus e$ and $f \in \underline{Z}$.

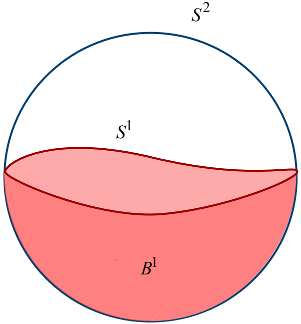
Topological representation

A sphere S of S^{d-1} is a pseudo-sphere if S is homeomorphic to S^{d-2} in a homeomorphism of S^{d-1}.



Topological representation

There are then two connected components in $S^{d-1} \setminus S$, each homeomorphic to a ball of dimension $d-1$ (called sides of S).



Topological representation

A finite collection $\{S_1, \ldots, S_n\}$ of pseudo-spheres in S^{d-1} is an arrangement of pseudo-spheres if

$(PS1)$ For all $A \subseteq E = \{1, \ldots, n\}$ the set $S_A = \cap_{e \in A} S_e$ is a topological sphere

$(PS2)$ If $S_A \not\subseteq S_e$ for $A \subseteq E, e \in E$ and S_e^+, S_e^- denote the two sides of S_e then $S_A \cap S_e$ is a pseudo-sphere of S_A having as sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

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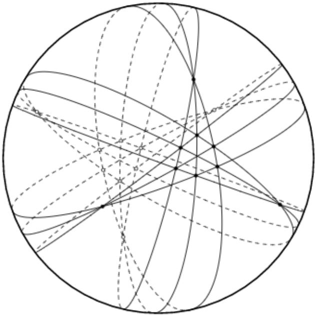
We say that the arrangement is signed if for each pseudo-sphere S_e, $e \in E$ it is chosen a positive and a negative side.

Topological representation

- Every essential arrangement of signed pseudo-sphere $\mathcal{S}$ partition the topological $(d-1)$-sphere in a complexe cellular $\Gamma(\mathcal{S})$. Each cell of $\Gamma(\mathcal{S})$ is uniquely determined by a sign vector in $\{-,0,+\}^E$ which is the codification of its relative position relative according to each pseudo-sphere S_i. Conversely $\Gamma(\mathcal{S})$ characterize $\mathcal{S}$



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Two arrangements (resp. signed arrangement) are equivalent if they are the same up to a homomorphism of S^{d-1} (resp. also the homeorphism preserve the signs). $\mathcal{S}$ is called realizable if there exists arrangement of sphere $\mathcal{S}'$ such that $\Gamma(\mathcal{S})$ is isomorphic to $\Gamma(\mathcal{S}')$.

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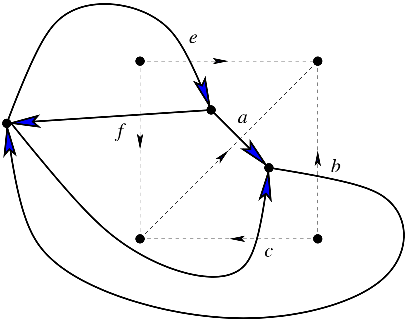
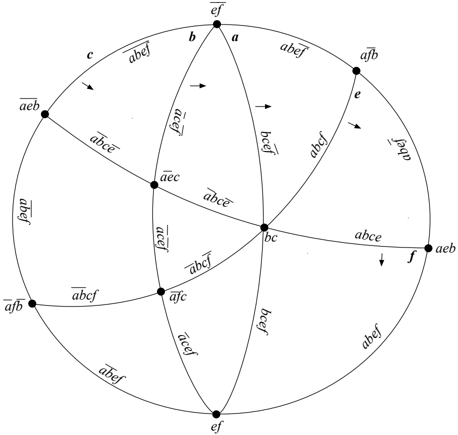
Theorem (Topological Representation) A loop-free oriented matroids of rank $d+1$ (up to isomorphism) are in one-to-one correspondence with arrangements of pseudospheres in S^d (up to topological equivalence) or equivalently to affine arrangements of pseudo-hyperplans in $\mathbb{R}^{d-1}$ (up to topological equivalence).

Topological representation





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• The number of subsets A of E such that $-_A M$ are totally cyclic is equals to $t(M; 0, 2)$.

Acyclic reorientations