Matroids Polytope and Ehrhart polynomial

J.L. Ramírez Alfonsín

IMAG, Université de Montpellier

São Paulo, Brasil
August 2017

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

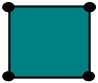
$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}.$

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}.$

Q_2 

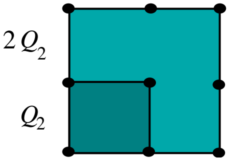
k	1
$L_{Q_2}(k)$	4

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}.$



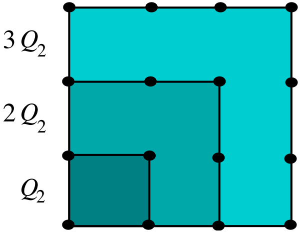
k	1	2
$L_{Q_2}(k)$	4	9

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}$.



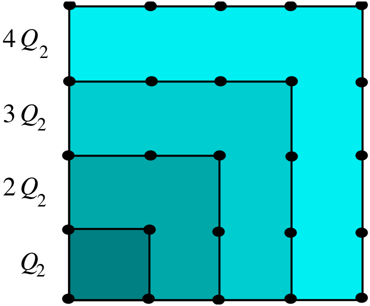
k	1	2	3
$L_{Q_2}(k)$	4	9	16

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}$.



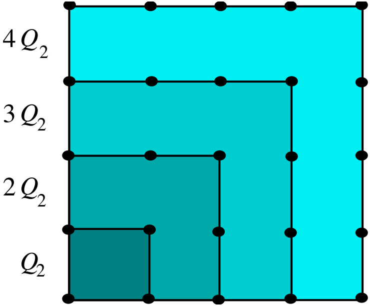
k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

Ehrhart theory

A Lattice polytope $P \subset \mathbb{R}^d$ is a convex hull of a finite set of points in $\mathbb{Z}^d$. For $k \in \mathbb{Z}_{>0}$ let $L_P(k) := \#(kP \cap \mathbb{Z}^d)$

Example

$Q_2 = conv\{(0,0),(1,0),(0,1),(1,1)\} = \{x, y \in \mathbb{R} : 0 \leq x, y \leq 1\}$.



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$\mathbf{L_{Q_2}(k) = (k+1)^2}$

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

$$L_{Q_d^\circ}(k) = (k-1)^d$$

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d$$

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

Theorem (Macdonald 1971)
$L_P(-k) = (-1)^{dim(P)} L_{P^\circ}(k)$ **(Reciprocity law)**.

Ehrhart polynomial

d-dimensional cube : $L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$

Theorem (Ehrhart 1962) For any lattice polytope P, $L_P(k)$ is a polynomial in k of degree $dim(P)$ with rational coefficients, with leading term $vol(P)$ and constant term 1.

Let P° denotes the interior of P.

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

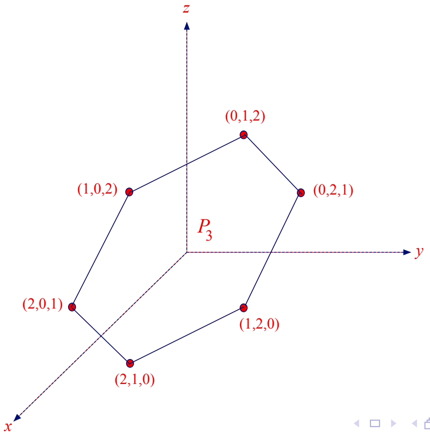
Theorem (Macdonald 1971)
$L_P(-k) = (-1)^{dim(P)} L_{P^\circ}(k)$ **(Reciprocity law)**.
Therefore, $(-1)^{dim(P)} L_P(-k)$ enumerates the interior lattice points in kP.

Permutahedron

The d-dimensional permutahedron P_d is defined as
$P_d := conv\{(\pi(1)-1, \pi(2)-1, \ldots, \pi(d)-1) : \pi \in \mathcal{S}_d\}$

Permutahedron

The d-dimensional permutahedron P_d is defined as
$P_d := conv\{(\pi(1)-1, \pi(2)-1, \ldots, \pi(d)-1) : \pi \in \mathcal{S}_d\}$



Permutahedron

Theorem

$$L_{P_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where f_i is the number of forests on $\{1, \ldots, d\}$ with i vertices.

Permutahedron

Theorem

$$L_{P_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where f_i is the number of forests on $\{1, \ldots, d\}$ with i vertices.

Remark f_d is the number of spanning trees on the complete graph K_d.

Permutahedron

Theorem

$$L_{P_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where f_i is the number of forests on $\{1, \ldots, d\}$ with i vertices.

Remark f_d is the number of spanning trees on the complete graph K_d.

$$f_d = d^{d-2}$$

Permutahedron

Theorem

$$L_{P_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where f_i is the number of forests on $\{1, \ldots, d\}$ with i vertices.

Remark f_d is the number of spanning trees on the complete graph K_d.

$$f_d = d^{d-2} = vol(P_d)$$

Cyclic polytope

Let $m(t) = (t, t^2, \ldots, t^d)$ be the moment curve in $\mathbb{R}^d$.

Cyclic polytope

Let $m(t) = (t, t^2, \ldots, t^d)$ be the moment curve in $\mathbb{R}^d$.

The d-dimensional Cyclic polytope $C_d = C_d(t_1, \ldots, t_n)$ is defined as

$$C_d := conv\{m(t_1), \ldots, m(t_n)\}$$

Cyclic polytope

Let $m(t) = (t, t^2, \ldots, t^d)$ be the moment curve in $\mathbb{R}^d$.

The d-dimensional Cyclic polytope $C_d = C_d(t_1, \ldots, t_n)$ is defined as

$$C_d := conv\{m(t_1), \ldots, m(t_n)\}$$

Theorem

$$L_{C_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where $f_i = vol(C_i(t_1, \ldots, t_n))$.

Ehrhart series

The standard d-simplex

$$\begin{aligned}\Delta &= \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\} \\ &= conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}\end{aligned}$$

$$L_\Delta(t) = \binom{t+d}{d}$$

Ehrhart series

The standard d-simplex

$$\begin{aligned} \Delta \quad &= \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\} \\ &= conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} \end{aligned}$$

$$L_\Delta(t) = \binom{t+d}{d}$$

$L_\Delta(t)$ comes with the friedly generating function

$$\sum_{t \geq 0} \binom{t+d}{d} z^t$$

Ehrhart series

The standard d-simplex

$$\begin{aligned}\Delta &= \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\} \\ &= conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}\end{aligned}$$

$$L_\Delta(t) = \binom{t+d}{d}$$

$L_\Delta(t)$ comes with the friedly generating function

$$\sum_{t\geq 0} \binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}}$$

Ehrhart series

The standard d-simplex

$$\Delta = \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\}$$
$$= conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$$

$$L_\Delta(t) = \binom{t+d}{d}$$

$L_\Delta(t)$ comes with the friedly generating function

$$\sum_{t\geq 0} \binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}}$$

This motivate to define the Ehrhart serie of the lattice polytope P as

$$Ehr_P(z) := 1 + \sum_{t\geq 1} L_P(t) z^t$$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

The h^*-vector are the coefficients of

$$h^*(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*$$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

The h^*-vector are the coefficients of

$$h^*(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*$$

- $L_P(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d}$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

The h^*-vector are the coefficients of

$$h^*(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*$$

- $L_P(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d}$
- $h^*(0) = 1$ and $h^*(1) = dim(P)!vol(P)$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

The h^*-vector are the coefficients of

$$h^*(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*$$

- $L_P(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d}$
- $h^*(0) = 1$ and $h^*(1) = dim(P)!vol(P)$
- $h_d^* = \#(P^\circ \cap \mathbb{Z}^d)$ and $h_1^* = \#(P \cap \mathbb{Z}^d) - d - 1$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

$$\frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

The h^*-vector are the coefficients of

$$h^*(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*$$

- $L_P(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d}$
- $h^*(0) = 1$ and $h^*(1) = dim(P)!vol(P)$
- $h_d^* = \#(P^\circ \cap \mathbb{Z}^d)$ and $h_1^* = \#(P \cap \mathbb{Z}^d) - d - 1$
- Theorem (Stanley 1980) $h_0^*, \ldots, h_d^*$ are nonnegative integers

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

(*I*1) $\emptyset \in \mathcal{I}$,

(*I*2) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

(*I*3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

$(I1)$ $\emptyset \in \mathcal{I}$,

$(I2)$ If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

$(I3)$ If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.
The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Bases

A base of a matroid is a maximal independent set.
We denote by $\mathcal{B}$ the set of all bases of a matroid.

Bases

A base of a matroid is a maximal independent set.
We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma All the bases of a matroid have the same cardinality r.

Bases

A base of a matroid is a maximal independent set.
We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma All the bases of a matroid have the same cardinality r.

The rank of a matroid M, denoted by $r(M)$, is the rank of one of its bases.

Bases

A base of a matroid is a maximal independent set.
We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma All the bases of a matroid have the same cardinality r.

The rank of a matroid M, denoted by $r(M)$, is the rank of one of its bases.

The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ (**exchange propety**) $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Bases

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

Bases

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

Two examples

- Let $U_{r,n} = \binom{[n]}{r}$ (i.e., the family of all r-sets of $\{1, \ldots, n\}$).

Bases

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

Two examples

• Let $U_{r,n} = \binom{[n]}{r}$ (i.e., the family of all r-sets of $\{1,\ldots,n\}$). $U_{r,n}$ is a matroid (called the uniform matroid of rank r on n elements).

Bases

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

Two examples

• Let $U_{r,n} = \binom{[n]}{r}$ (i.e., the family of all r-sets of $\{1, \ldots, n\}$). $U_{r,n}$ is a matroid (called the uniform matroid of rank r on n elements).

• Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Let $\mathcal{B}$ be the set of all maximal forest in G.

Bases

Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

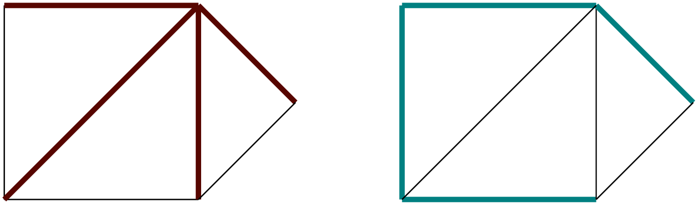
Two examples

• Let $U_{r,n} = \binom{[n]}{r}$ (i.e., the family of all r-sets of $\{1, \ldots, n\}$). $U_{r,n}$ is a matroid (called the uniform matroid of rank r on n elements).

• Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Let $\mathcal{B}$ be the set of all maximal forest in G.
Then, $M(G) = (\mathcal{B}, E)$ is a matroid with $r(M(G)) = n - c$ where c is the number of connected components of G.

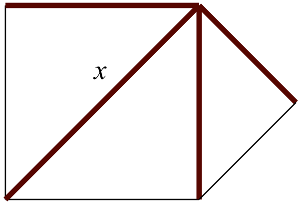
Bases

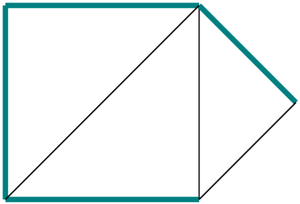
Example



Bases

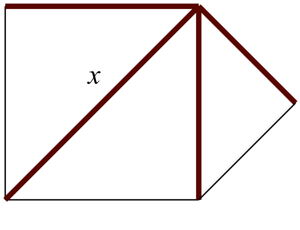
Example





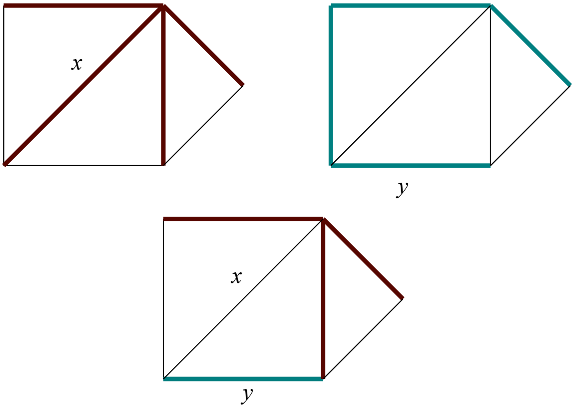
Bases

Example



Bases

Example



Matroid base polytope

Let $M = (\mathcal{B}, E)$ with $|E| = n$. For each base $B \in \mathcal{B}$, the incident vector $e_B \in \mathbb{R}^E$ is defined by

$$e_B = \sum_{i \in B} e_i$$

where e_i denotes i^{th} standard base vector in $\mathbb{R}^n$.

Matroid base polytope

Let $M = (\mathcal{B}, E)$ with $|E| = n$. For each base $B \in \mathcal{B}$, the incident vector $e_B \in \mathbb{R}^E$ is defined by

$$e_B = \sum_{i \in B} e_i$$

where e_i denotes i^{th} standard base vector in $\mathbb{R}^n$.

The base polytope of M, denoted by P_M, is defined as

$$P_M = conv\{e_B : B \in \mathcal{B}\}$$

Matroid base polytope

Let $M = (\mathcal{B}, E)$ with $|E| = n$. For each base $B \in \mathcal{B}$, the incident vector $e_B \in \mathbb{R}^E$ is defined by

$$e_B = \sum_{i \in B} e_i$$

where e_i denotes i^{th} standard base vector in $\mathbb{R}^n$.

The base polytope of M, denoted by P_M, is defined as

$$P_M = conv\,\{e_B : B \in \mathcal{B}\}$$

Let Δ_E be the simplexe in $\mathbb{R}^E$, i.e.,

$$\Delta_E = conv(e_i : i \in E) = \{x \in \mathbb{R}^E : \sum_{i \in E} x_i = 1,\ x_i \geq 0 \text{ for all } i \in E\}$$

Matroid base polytope

Theorem (Gel'fand, Goresky, MacPherson, Serganova 1987)
Let $P \subseteq \mathbb{R}^E$ be a convex polytope. Then, P is the base polytope of a matroid $M = (\mathcal{B}, E)$ if and only if

- $P_M \subseteq r\Delta_E$ where $r = r(M)$ (implying that $dim(P) \leq n-1$)
- the vertices of P belong to $\{0,1\}^E$ and
- each edge of P_M is a translation of $conv(e_i, e_j)$ for all $i, j \in E, i \neq j$.

Matroid base polytope

Theorem (Gel'fand, Goresky, MacPherson, Serganova 1987)
Let $P \subseteq \mathbb{R}^E$ be a convex polytope. Then, P is the base polytope of a matroid $M = (\mathcal{B}, E)$ if and only if

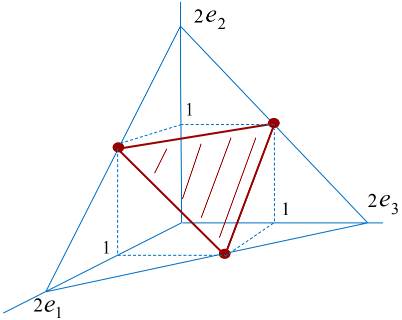
- $P_M \subseteq r\Delta_E$ where $r = r(M)$ (implying that $dim(P) \leq n - 1$)
- the vertices of P belong to $\{0, 1\}^E$ and
- each edge of P_M is a translation of $conv(e_i, e_j)$ for all $i, j \in E, i \neq j$.

Example

$$P_{U_{2,3}} = conv\{(1,1,0),(1,0,1),(0,1,1)\}$$

Example

$$P_{U_{2,3}} = conv\{(1,1,0),(1,0,1),(0,1,1)\}$$

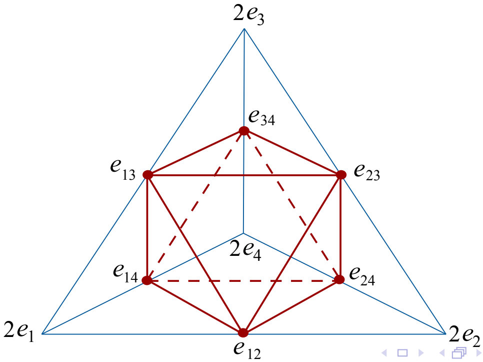


Example

$$P_{U_{2,4}} = conv \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

Example

$$P_{U_{2,4}} = conv \left\{ \begin{pmatrix} 1\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$



Proposition Let M be a matroid on E. Then,
$i \sim j \iff i = j$ or there exist bases A and B such that $B = (A \setminus i) \cup j$
is an equivalent relation on E.

Matroid base polytope

Proposition Let M be a matroid on E. Then,
$i \sim j \;\Leftrightarrow\; i = j$ or there exist bases A and B such that $B = (A \setminus i) \cup j$ is an equivalent relation on E.

The equivalent classes are the connected components of M. We denote by $c(M)$ the number of connected components of M and we say that M is connected if $c(M) = 1$.

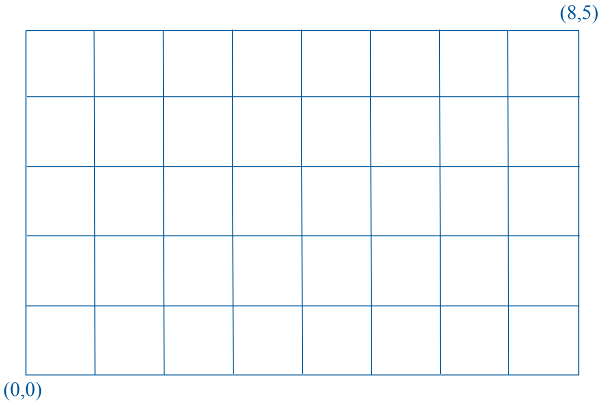
Matroid base polytope

Proposition Let M be a matroid on E. Then,
$i \sim j \ \Leftrightarrow \ i = j$ or there exist bases A and B such that $B = (A \setminus i) \cup j$ is an equivalent relation on E.

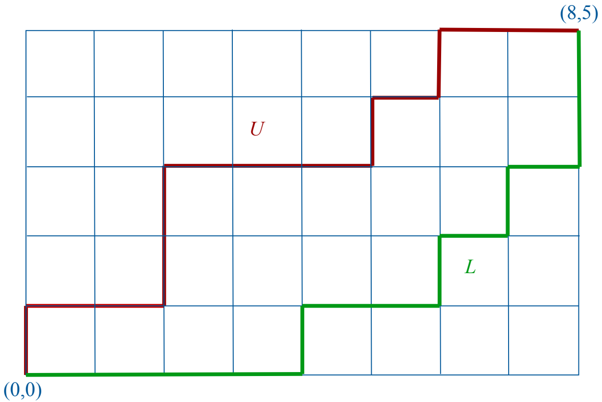
The equivalent classes are the connected components of M. We denote by $c(M)$ the number of connected components of M and we say that M is connected if $c(M) = 1$.

Theorem $dim(P_M) = n - c(M)$ where $n = |E|$.

Lattice path matroid



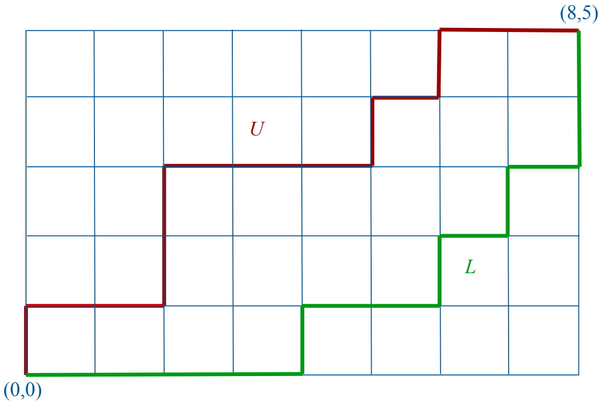
Lattice path matroid



U=NEENNEEENENENEE

L=EEEENEENENENENN

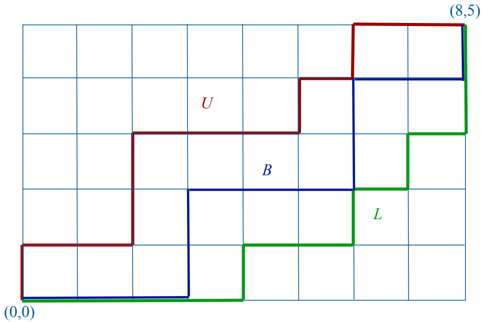
Lattice path matroid



U={1,4,5,9,11}
U=NEENNEEENENEE

L={5,8,10,12,13}
L=EEEENEENENENN

Lattice path matroid

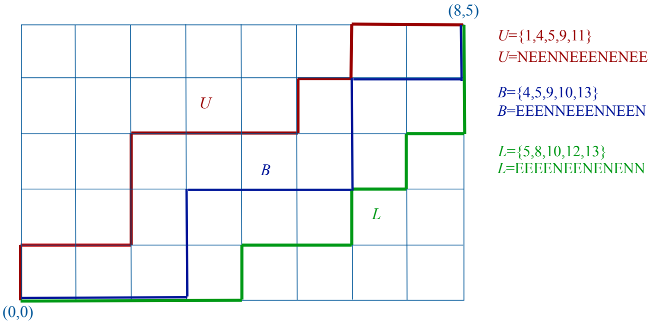


U={1,4,5,9,11}
U=NEENNEEENENEE

B={4,5,9,10,13}
B=EEENNEEENNEEN

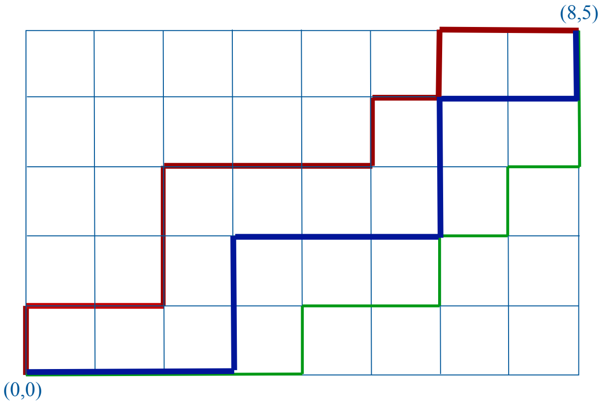
L={5,8,10,12,13}
L=EEEENEENENENN

Lattice path matroid



$M[U, L]$ lattice path matroid (LPM) of rank r (# rows) on $r + m$ (# rows + # columns) elements.

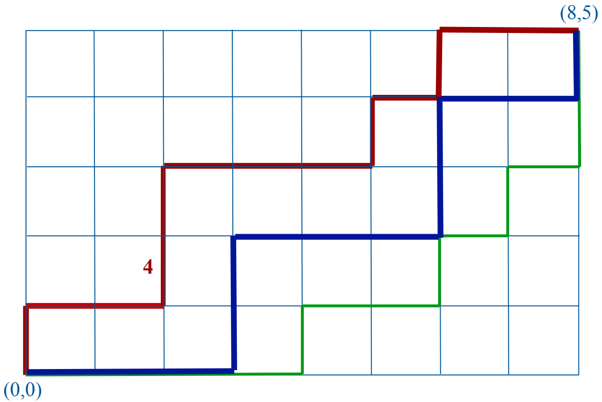
LPM base exchange base



U={1,4,5,9,11}
U=NEENNEEENENEE

B={4,5,9,10,13}
B=EEENNEEENNEEN

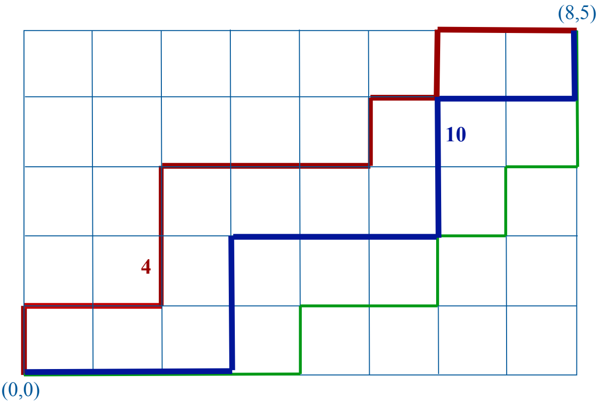
LPM base exchange base



U={1,**4**,5,9,11}
U=NEE**N**NEEENENEE

B={4,5,9,10,13}
B=EEENNEEENNEEN

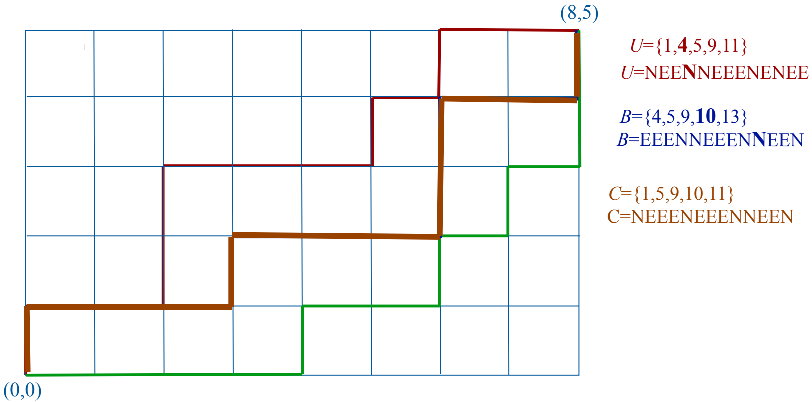
LPM base exchange base



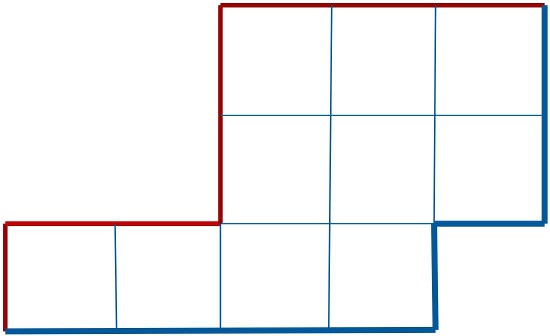
U={1,**4**,5,9,11}
U=NEE**N**NEEENENEE

B={4,5,9,**10**,13}
B=EEENNEEEN**N**EEN

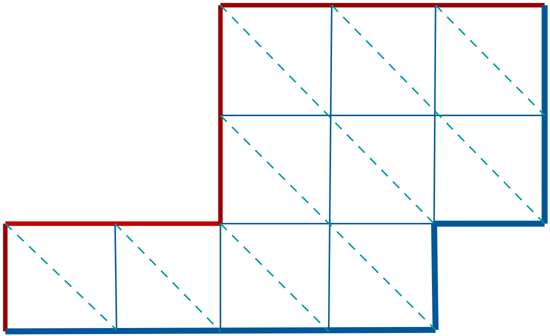
LPM base exchange base



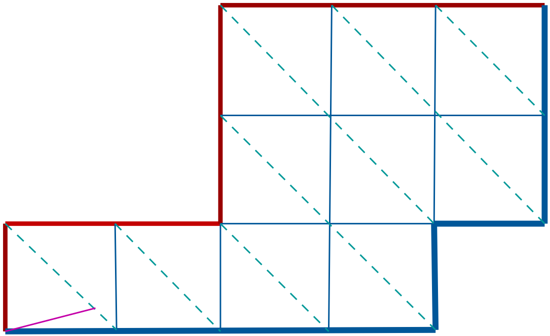
Generalized lattice path



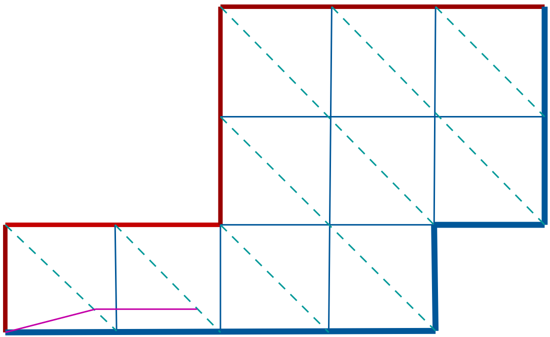
Generalized lattice path



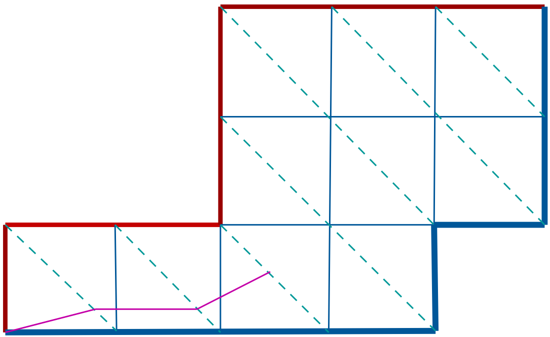
Generalized lattice path



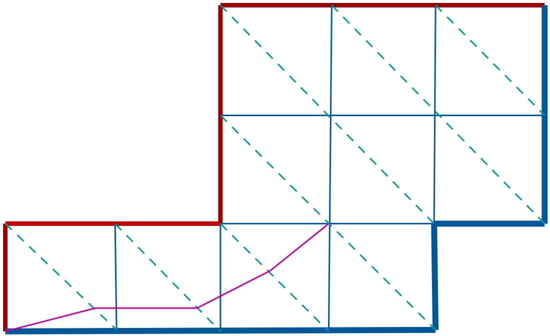
Generalized lattice path



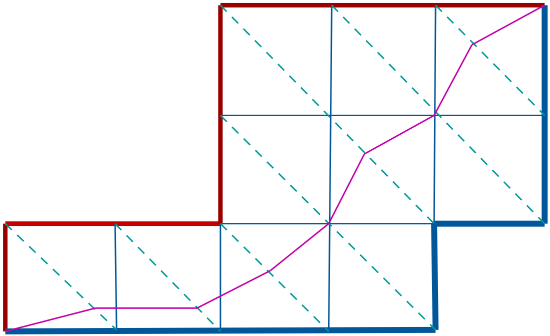
Generalized lattice path



Generalized lattice path

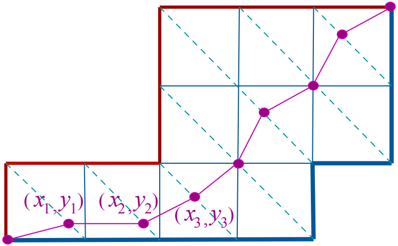


Generalized lattice path



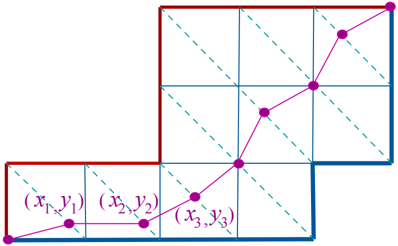
Generalized lattice path

A generalized path P starts at $(0,0)$ and ends at $(r, r+m)$ and it is monotonously increasing $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$.



Generalized lattice path

A generalized path P starts at $(0,0)$ and ends at $(r, r+m)$ and it is monotonously increasing $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$.



Let $st(P) = (p_1, \dots, p_{r+m})$ where $p_{i+1} = y_{i+1} - y_i$ for each i.

We call $st(P)$ step vector of P.

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r + m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r + m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.
Let $\mathcal{C}_M$ be the family of step vectors of all generalized lattice path in $M[U, L]$.

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r+m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.
Let $\mathcal{C}_M$ be the family of step vectors of all generalized lattice path in $M[U, L]$. Then,

$$\mathcal{C}_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1, \sum_{j=1}^{i} l_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} u_j \ \forall i \right\}$$

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U,L]$ be a LPM of rank r on $r+m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.
Let $\mathcal{C}_M$ be the family of step vectors of all generalized lattice path in $M[U,L]$. Then,

$$\mathcal{C}_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1, \sum_{j=1}^{i} l_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} u_j \ \forall i \right\}$$

- $0 \leq y_{i+1} - y_i \leq 1$

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r + m$ elements.
Let $st(L) = (l_1, \ldots, l_{r+m})$ and $st(U) = (u_1, \ldots, u_{r+m})$.
Let $\mathcal{C}_M$ be the family of step vectors of all generalized lattice path in $M[U, L]$. Then,

$$\mathcal{C}_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \leq p_i \leq 1, \sum_{j=1}^{i} l_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} u_j \ \forall i \right\}$$

- $0 \leq y_{i+1} - y_i \leq 1$
- Any generalized path stay between U and L.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope.Then, $P_M = \mathcal{C}_M$.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope. Then, $P_M = \mathcal{C}_M$.
Proof (idea).
$P_M = conv\{$ characteristic vectors of $\mathcal{B}(M)\} \subseteq conv\{\mathcal{C}_M\} = \mathcal{C}_M$.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope. Then, $P_M = \mathcal{C}_M$.

Proof (idea).
$P_M = conv\{$ characteristic vectors of $\mathcal{B}(M)\} \subseteq conv\{\mathcal{C}_M\} = \mathcal{C}_M$.

$P_M \supseteq \mathcal{C}_M$ by induction on the number of elements.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope. Then, $P_M = \mathcal{C}_M$.

Proof (idea).
$P_M = conv\{$ characteristic vectors of $\mathcal{B}(M)\} \subseteq conv\{\mathcal{C}_M\} = \mathcal{C}_M$.

$P_M \supseteq \mathcal{C}_M$ by induction on the number of elements.

Corollary (Knauer, Martinez-Sandoval, R.A., 2017)
Let $\mathcal{C}_M^k$ be the family of step vectors of all generalized paths P in $M = [U, L]$ such that each (x_i, y_i) in P satisfy $kx_i, ky_i \in \mathbb{Z}$.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope.Then, $P_M = \mathcal{C}_M$.

Proof (idea).

$P_M = conv\{$ characteristic vectors of $\mathcal{B}(M)\} \subseteq conv\{\mathcal{C}_M\} = \mathcal{C}_M$.

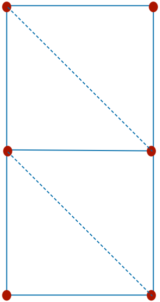
$P_M \supseteq \mathcal{C}_M$ by induction on the number of elements.

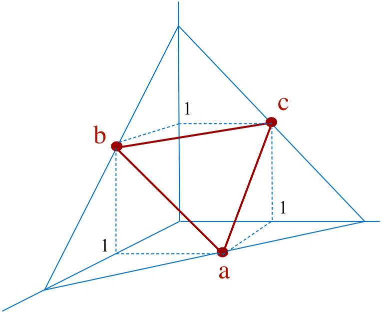
Corollary (Knauer, Martinez-Sandoval, R.A., 2017)
Let $\mathcal{C}_M^k$ be the family of step vectors of all generalized paths P in $M = [U, L]$ such that each (x_i, y_i) in P satisfy $kx_i, ky_i \in \mathbb{Z}$.Then,

$$kP_M \cap \mathbb{Z}^{r+m} = \mathcal{C}_M^k$$

Integer points in LPM polytopes

Example : Consider $P_{U_{2,3}}$

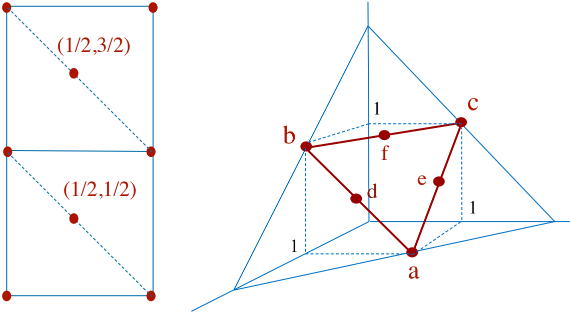




a=(1,1,0)
b=(1,0,1)
c=(0,1,1)

Integer points in LPM polytopes

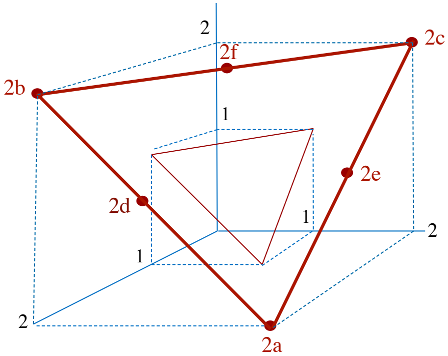
Example : Construct paths in $\mathcal{C}^2_{U_{2,3}}$



a=(1,1,0) d=(1,1/2,1/2)
b=(1,0,1) e=(1/2,1,1/2)
c=(0,1,1) f=(1/2,1/2,1)

Integer points in LPM polytopes

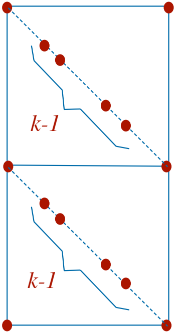
Example : $2P_{U_{2,3}}$



2a=(2,2,0) 2d=(2,1,1)
2b=(2,0,2) 2e=(1,2,1)
2c=(0,2,2) 2f=(1,1,2)

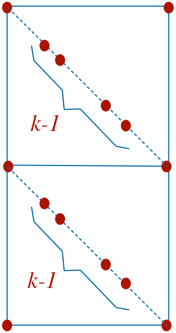
Integer points in LPM polytopes

Let us consider $kP_{U_{2,3}}$



Integer points in LPM polytopes

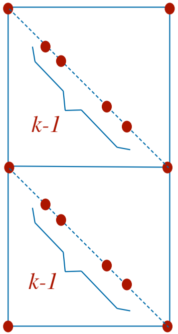
Let us consider $kP_{U_{2,3}}$



$\mathcal{C}^k_{U_{2,3}}$

Integer points in LPM polytopes

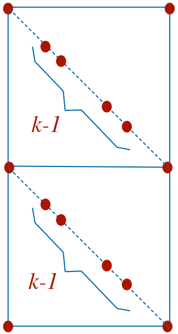
Let us consider $kP_{U_{2,3}}$



$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2)$

Integer points in LPM polytopes

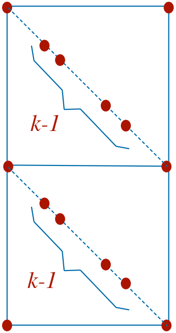
Let us consider $kP_{U_{2,3}}$



$$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1$$

Integer points in LPM polytopes

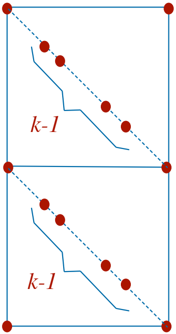
Let us consider $kP_{U_{2,3}}$



$$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1 = kP_{U_{2,3}} \cap \mathbb{Z}^3$$

Integer points in LPM polytopes

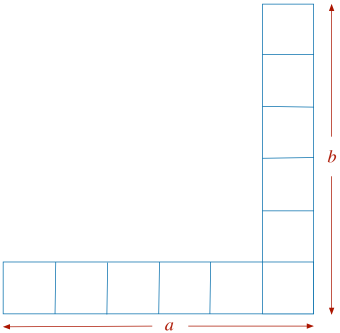
Let us consider $kP_{U_{2,3}}$



$$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1 = kP_{U_{2,3}} \cap \mathbb{Z}^3 = L_{P_{U_{2,3}}}(k)$$

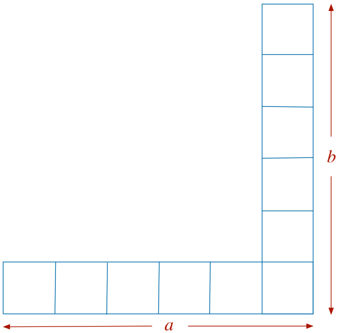
Integer points in LPM polytopes

Let $S(a,b)$ be the matroid associated to



Integer points in LPM polytopes

Let $S(a, b)$ be the matroid associated to



$$L_{S(a,b)}(k) = \sum_{i=0}^{k} \binom{a+k-1-i}{a-1} \binom{b+k-1-i}{b-1}$$

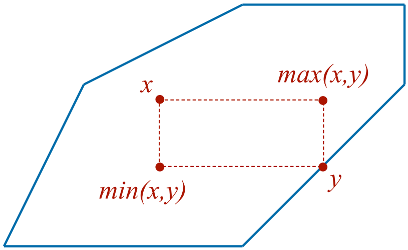
Distributive polytopes

A polytope $P \subseteq \mathbb{R}^n$ is called distributive if for all $x, y \in P$ also their componentwise maximum and minimum $\max(x, y)$ and $\min(x, y)$ are in P.

Distributive polytopes

A polytope $P \subseteq \mathbb{R}^n$ is called distributive if for all $x, y \in P$ also their componentwise maximum and minimum $\max(x, y)$ and $\min(x, y)$ are in P.

Example : A distributive polytope in $\mathbb{R}^2$.



Distributive polytopes

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a connected rank r LPM on $r + m$ elements.

Distributive polytopes

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a connected rank r LPM on $r + m$ elements. Then, there exists a bijective affine transformation taking $P_M \subset \mathbb{R}^{r+m}$ into a full-dimensional distributive integer polytope $Q_M \subset \mathbb{R}^{r+m-1}$ such that $L_{P_M}(t) = L_{Q_M}(t)$.

Distributive polytopes

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a connected rank r LPM on $r + m$ elements. Then, there exists a bijective affine transformation taking $P_M \subset \mathbb{R}^{r+m}$ into a full-dimensional distributive integer polytope $Q_M \subset \mathbb{R}^{r+m-1}$ such that $L_{P_M}(t) = L_{Q_M}(t)$.

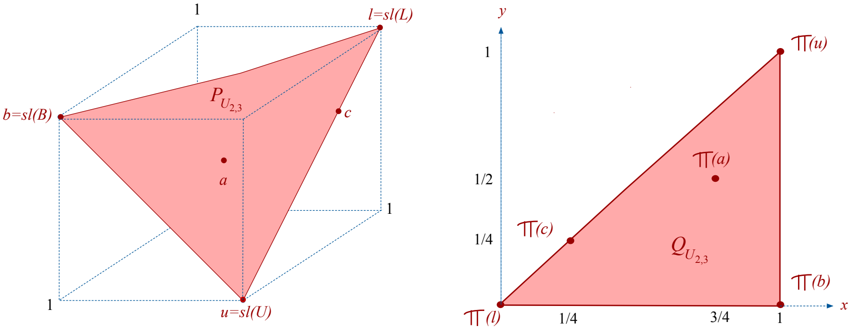
Proof (idea). Recall that $st(L) = (l_1, \ldots, l_{r+m})$. Check that

$$\begin{array}{llll} \pi : & P_M \subset \mathbb{R}^{r+m} & \longrightarrow & \mathbb{R}^{r+m-1} \\ & p = (p_1, \ldots, p_{r+m}) & \mapsto & (p_1 - l_1, \ldots, \sum_{j=1}^{r+m-1}(p_j - l_j)) \end{array}$$

is a suitable transformation.

Example

$P_{U_{2,3}}$



We have $\pi(a) = (\frac{3}{4}, \frac{1}{2}), \pi(b) = (1, 0)$ and $\pi(c) = (\frac{1}{4}, \frac{1}{4})$.

Order polytopes

Let X be a poset on $\{1, \ldots, n\}$ such that this labeling is natural, i.e., if $i <_X j$ then $i < j$.

Order polytopes

Let X be a poset on $\{1, \ldots, n\}$ such that this labeling is natural, i.e., if $i <_X j$ then $i < j$.

The order polytope $\mathcal{O}(X)$ of X is defined as the set of those $x \in \mathbb{R}^n$ such that

$$0 \leq x_i \leq 1, \text{ for all } i \in X \text{ and}$$
$$x_i \geq x_j, \text{ if } i \leq_X j$$

Order polytopes

Let X be a poset on $\{1, \ldots, n\}$ such that this labeling is natural, i.e., if $i <_X j$ then $i < j$.

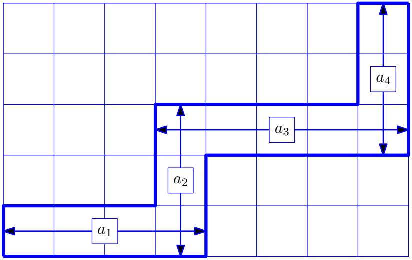
The order polytope $\mathcal{O}(X)$ of X is defined as the set of those $x \in \mathbb{R}^n$ such that

$$0 \leq x_i \leq 1, \text{ for all } i \in X \text{ and}$$
$$x_i \geq x_j, \text{ if } i \leq_X j$$

Remark. $\mathcal{O}(X)$ is a bounded convex polytope

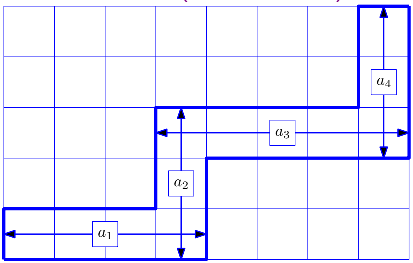
Snake polytopes

Snake $S(a_1, a_2, a_3, a_4)$



Snake polytopes

Snake $S(a_1, a_2, a_3, a_4)$



Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a_1, \ldots, a_k \geq 2$ be integers. Then, a connected LPM M is the snake $S(a_1, \ldots a_k)$ if and only if Q_M is the order polytope of the zig-zag chain poset on $a_1, \ldots, a_k$.

Snake polytopes

Recall that

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

Snake polytopes

Recall that

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

Conjecture (De Loera, Haws, Köppe, 2009) The h^*-vector of base matroid polytopes are unimodal, i.e.,

$$h_d^* \leq h_{d_1}^* \leq \cdots \leq h_j^* \geq h_{j+1}^* \geq \cdots \geq h_0^* \text{ for some } j$$

Snake polytopes

Recall that

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}$$

Conjecture (De Loera, Haws, Köppe, 2009) The h^*-vector of base matroid polytopes are unimodal, i.e.,

$$h_d^* \leq h_{d_1}^* \leq \cdots \leq h_j^* \geq h_{j+1}^* \geq \cdots \geq h_0^* \text{ for some } j$$

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a, b \geq 2$ be integers. The h^*-vectors of the snake polytopes $P_{S(a,\ldots,a)}$ and $P_{S(a,b)}$ are unimodal.