THE SQUARE FROBENIUS NUMBER

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Abstract. Let $S = \langle s_1, \ldots, s_n \rangle$ be a numerical semigroup generated by the relatively prime positive integers $s_1, \ldots, s_n$. Let $k \geqslant 2$ be an integer. In this paper, we consider the following k-power variant of the Frobenius number of S defined as

$$^{k}r(S) := \text{ the largest } k\text{-power integer not belonging to } S.$$

In this paper, we investigate the case $k = 2$. We give an upper bound for $^{2}r(S_A)$ for an infinite family of semigroups S_A generated by *arithmetic progressions*. The latter turns out to be the exact value of $^{2}r(\langle s_1, s_2 \rangle)$ under certain conditions. We present an exact formula for $^{2}r(\langle s_1, s_1 + d \rangle)$ when $d = 3, 4$ and 5, study $^{2}r(\langle s_1, s_1 + 1 \rangle)$ and $^{2}r(\langle s_1, s_1 + 2 \rangle)$ and put forward two relevant conjectures. We finally discuss some related questions.

1. Introduction

Let $s_1, \ldots, s_n$ be relatively prime positive integers. Let

$$S = \langle s_1, \ldots, s_n \rangle = \left\{ \sum_{i=1}^{n} x_i s_i \;\middle|\; x_i \text{ integer, } x_i \geqslant 0 \right\}$$

be the numerical semigroup generated by $s_1, \ldots, s_n$. The largest integer which is not an element of S, denoted by $g(S)$ or $g(\langle s_1, \ldots, s_n \rangle)$, is called the *Frobenius number* of S. It is well known that $g(\langle s_1, s_2 \rangle) = s_1 s_2 - s_1 - s_2$. However, calculating $g(S)$ is a difficult problem in general. In [1] was shown that computing $g(S)$ is *NP-hard*. We refer the reader to [2] for an extensive literature on the Frobenius number.

Throughout this paper, the set of non-negative integers is denoted by $\mathbb{N}$. The non-negative integers not in S are called the *gaps* of S. The number of gaps of S, denoted by $N(S)$ (that is, $N(S) = \#(\mathbb{N} \setminus S)$) is called the *genus* of S. We recall that the *multiplicity* of S is the smallest positive element belonging to S.

Given a particular (arithmetical, number theoretical, etc.) *Property* P, one might consider the following two *P-type functions* of a semigroup S:

$$^{P}r(S) := \text{ the largest integer having property } P \text{ not belonging to } S$$

and

$$_{P}r(S) := \text{ the smallest integer having property } P \text{ belonging to } S.$$

Notice that the multiplicity and the Frobenius number are P-type functions where P is the property of being a positive integer[1].

In this spirit, we consider the property of being *perfect k-power* integer (that is, integers of the form m^k for some integers $m, k > 1$). Let $k \geqslant 2$ be an integer, we define

$$^{k\text{-power}}r(S) := \text{ the largest perfect } k\text{-power integer not belonging to } S.$$

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[1] *P-type functions* were introduced by the second author (often mentioned during his lectures) with the hope to better understand certain properties P in terms of linear forms.

This k-power variant of $g(S)$ is called the *k-power Frobenius number* of S, we may write ${}^{k}r(S)$ for short.

In this paper we investigate the 2-power Frobenius number, we call it the *square Frobenius number.*

In Section 2, we study the square Frobenius number of semigroups S_A generated by *arithmetic progressions*. We give an upper bound for ${}^{2}r(S_A)$ for an infinite family (Theorem 2.7) which turns out to be the exact value when the arithmetic progression consists of two generators (Corollary 2.9).

In Section 3, we present exact formulas for ${}^{2}r(\langle a, a+3\rangle)$ where $a \geqslant 3$ is an integer not divisible by 3 (Theorem 3.1), for ${}^{2}r(\langle a, a+4\rangle)$ where $a \geqslant 3$ is an odd integer (Theorem 3.2) and for ${}^{2}r(\langle a, a+5\rangle)$ where $a \geqslant 2$ is an integer not divisible by 5 (Theorem 3.3).

In Sections 4 and 5, we turn our attention to the cases $\langle a, a+1\rangle$ where $a \geqslant 2$ and $\langle a, a+2\rangle$ where $a \geqslant 3$ is an odd integer. We present formulas for the corresponding square Frobenius number in the case when neither of the generators are square integers (Propositions 4.1 and 5.1). We also put forward two conjectures on the values of ${}^{2}r(\langle a, a+1\rangle)$ and ${}^{2}r(\langle a, a+2\rangle)$ in the case when one of the generators is a square integer (Conjectures 4.2 and 5.2). The conjectured values have an unexpected close connection with a known recursive sequence (Equation (14)) and in which $\sqrt{2}$ and $\sqrt{3}$ (strangely) appear. A number of computer experiments support our conjectures.

Finally, Section 6 contains some concluding remarks.

2. Arithmetic progression

Let a, d and k be positive integers such that a and d are relatively prime. Throughout this section, we denote by S_A the semigroup generated by the *arithmetic progression* whose first element is a, with common difference d and of length $k+1$, that is,

$$S_A = \langle a, a+d, a+2d, \ldots, a+kd\rangle \, .$$

Note that the integers $a, a+d, \ldots, a+kd$ are relatively prime if and only if $\gcd(a, d) = 1$.

We shall start by giving a necessary and sufficient condition for a square to belong to S_A.

For any integer x coprime to d, a *multiplicative inverse modulo* d of x is an integer y such that $xy \equiv 1 \bmod d$.

Proposition 2.1. *Let i be an integer and let λ_i be the unique integer in $\{0, 1, \ldots, d-1\}$ such that $\lambda_i a + i^2 \equiv 0 \bmod d$. In other words, the integer λ_i is the remainder in the Euclidean division of $-a^{-1}i^2$ by d, where a^{-1} is a multiplicative inverse of a modulo d. Then,*

$$(a-i)^2 \in S_A \text{ if and only if } (i+kd)^2 \leqslant \left(\left(\left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor + k\right) d - \lambda_i\right)(a+kd).$$

A key step for the proof of this result is the following lemma, which can be thought as a variant of a result given in [3], see [4, Lemma 1] for a short proof. The arguments for the proof of this variant are similar to those used in the latter.

Lemma 2.2. *Let M be a non-negative integer and let x and y be the unique integers such that $M = ax + dy$, with $0 \leqslant y \leqslant a-1$. Then,*

$$M \in S_A \text{ if and only if } y \leqslant kx \text{ (with } x \geqslant 0).$$

Proof. First, suppose that $M \in S_A$ and let $x_0, x_1, \ldots, x_k$ be non-negative integers such that $M = \sum_{i=0}^{k} x_i(a+id)$. Then, we have that

$$M = \sum_{i=0}^{k} x_i a + \sum_{i=0}^{k} i x_i d = x'a + y'd,$$

with $x' = \sum_{i=0}^{k} x_i \in \mathbb{N}$ and $y' = \sum_{i=0}^{k} i x_i \in \mathbb{N}$. It follows that

$$y' = \sum_{i=0}^{k} i x_i \leqslant k \sum_{i=0}^{k} x_i = kx'.$$

Moreover, since $M = xa + yd$ with $y \in \{0, 1, \ldots, a-1\}$, we obtain that there exists a non-negative integer λ such that

$$y' = y + \lambda a \quad \text{and} \quad x' = x - \lambda a.$$

This leads to the inequality

$$y = y' - \lambda a \leqslant kx' - \lambda a = kx - \lambda(k+1)a \leqslant kx.$$

Conversely, suppose now that $y \leqslant kx$. Obviously, since $y \geqslant 0$, we know that $x \geqslant 0$. Let

$$y = qk + r$$

be the Euclidean division of y by k, with $q \in \mathbb{N}$ and $r \in \{0, 1, \ldots, k-1\}$. If $r = 0$, then we have that $0 \leqslant q \leqslant x$ since $y = qk \leqslant kx$. It follows that

$$M = xa + qkd = (x-q)a + q(a+kd) \in S_A.$$

Finally, if $r > 0$, then we have that $0 \leqslant q \leqslant x - 1$ since $y = qk + r \leqslant kx$. It follows that

$$M = xa + (qk+r)d = (x-q-1)a + q(a+kd) + (a+rd) \in S_A.$$

This completes the proof. □

We may now prove Proposition 2.1.

Proof of Proposition 2.1. Let i be an integer and let $\lambda_i \in \{0, 1, \ldots, d-1\}$ such that $\lambda_i a + i^2 \equiv 0 \bmod d$. We have that

$$\begin{aligned}
(a-i)^2 &= (a-2i)a + i^2 \\
&= (a - 2i - \lambda_i)a + \frac{i^2 + \lambda_i a}{d} d \\
&= \left(a - 2i - \lambda_i + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d\right) a + \left(\frac{i^2 + \lambda_i a}{d} - \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor a\right) d.
\end{aligned}$$

We thus have, by Lemma 2.2, that the square $(a-i)^2$ is in S_A if and only if

$$\frac{i^2+\lambda_i a}{d} - \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor a \leqslant k\left(a - 2i - \lambda_i + \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor d\right)$$

$$\iff \frac{i^2+\lambda_i a}{d} \leqslant k\left(a - 2i - \lambda_i\right) + \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor (a+kd)$$

$$\iff i^2+\lambda_i a \leqslant kd\left(a - 2i - \lambda_i\right) + \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor d(a+kd)$$

$$\iff i^2+2ikd \leqslant kda - \lambda_i(a+kd) + \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor d(a+kd)$$

$$\iff i^2+2ikd+k^2d^2 \leqslant kd(a+kd) - \lambda_i(a+kd) + \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor d(a+kd)$$

$$\iff (i+kd)^2 \leqslant \left(\left(\left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor + k\right) d - \lambda_i\right)(a+kd).$$

This completes the proof. □

Remark 1. We have that $\lambda_0 = 0$ and $\lambda_i > 0$ for all integers i such that $\gcd(i,d) = 1$ with $d \geqslant 2$. Moreover, $\lambda_i = \lambda_{d-i}$ for all $i \in \{1, 2, \ldots, d-1\}$.

The above characterization permits us to obtain an upper-bound of ${}^2r(S_A)$ when a is larger enough compared to $d \geqslant 3$.

Definition 2.3. Let λ^* be the integer defined by

$$\lambda^* = \max_{0\leqslant i\leqslant d-1}\left\{\lambda_i \in \{0,1,\ldots,d-1\} \;\mid \lambda_i a + i^2 \equiv 0 \bmod d\right\}.$$

Let $\{\alpha_1 < \ldots < \alpha_n\} \subseteq \{0,1,\ldots,d-1\}$ such that $\lambda_{\alpha_j} = \lambda^*$ and take $\alpha_{n+1} = d + \alpha_1$. Let $j \in \{1,\ldots,n\}$ be the index such that

$$(\mu d + \alpha_j)^2 \leqslant (kd - \lambda^*)(a+kd) < (\mu d + \alpha_{j+1})^2, \tag{1}$$

for some integer $\mu \geqslant 0$.

Remark 2.
(a) The above index j exists and it is unique. Indeed, we clearly have that there is an integer μ such that

$$\mu d \leqslant \sqrt{(kd-\lambda^*)(a+kd)} < (\mu+1)d.$$

Since $0 \leqslant \alpha_1 < \cdots < \alpha_n \leqslant d-1$, then the interval $[\mu d, (\mu+1)d[$ can be refined into intervals of the form $[\mu d + \alpha_i, \mu d + \alpha_{i+1}[$ for each $i = 1,\ldots,n-1$. Therefore, there is a unique index j verifying equation (1).
(b) We have that $\mu d + \alpha_{n+1} = (\mu+1)d + \alpha_1$.

The following two propositions give us useful information on the sequence of indices $\alpha_1,\ldots,\alpha_n$.

Proposition 2.4. *We have that $\alpha_i + \alpha_{n+1-i} = d$, for all $i \in \{1,\ldots,n\}$.*

Proof. Since $\{i \in \{1,\ldots,n\} \mid \lambda_i = \lambda^*\} = \{\alpha_1,\ldots,\alpha_n\}$, with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, and since $\lambda_{d-i} = \lambda_i$, for all $i \in \{1,\ldots,d-1\}$, by Remark 1. □

Proposition 2.5. *If $d \geqslant 3$ then $n \geqslant 2$ and $1 \leqslant \alpha_1 < \frac{d}{2} < \alpha_n \leqslant d-1$.*

Proof. Suppose that $n = 1$ and hence $\alpha_n = \alpha_1$. Since $d = \alpha_1 + \alpha_n = 2\alpha_1$ by Proposition 2.4, it follows that d is even and $\alpha_1 = \frac{d}{2}$.

If d is divisible by 4 then

$$\left(\frac{d}{2}\right)^2 = \frac{d}{4} \cdot d \equiv 0 \pmod{d}.$$

Therefore, $\lambda^* = \lambda_{\alpha_1} = \lambda_{\frac{d}{2}} = 0$. Moreover, since $\gcd(1, d) = 1$, we know that $\lambda_1 > 0$. It follows that $\lambda_1 > \lambda^*$, in contradiction with the maximality of λ^*.

If d is even, not divisible by 4, then $\frac{d}{2}$ is odd and

$$\left(\frac{d}{2}\right)^2 = \frac{d}{2} \cdot \frac{d}{2} = \frac{\frac{d}{2} - 1}{2} d + \frac{d}{2} \equiv \frac{d}{2} \pmod{d}.$$

Since a is coprime to d, we know that there exists a multiplicative inverse a^{-1} modulo d such that $aa^{-1} \equiv 1 \bmod d$. Since d is even, it follows that a^{-1} is odd and we obtain that

$$\lambda_{\frac{d}{2}} \equiv -a^{-1} \left(\frac{d}{2}\right)^2 \equiv -a^{-1} \frac{d}{2} \equiv \frac{d}{2} \pmod{d}.$$

Therefore, $\lambda_{\frac{d}{2}} = \frac{d}{2}$. Moreover, for any $i \in \left\{0, \ldots, \frac{d}{2} - 1\right\}$, since

$$\left(i + \frac{d}{2}\right)^2 = i^2 + id + \left(\frac{d}{2}\right)^2 \equiv i^2 + \frac{d}{2} \pmod{d}$$

and since a^{-1} is odd, it follows that

$$\lambda_{i+\frac{d}{2}} \equiv -a^{-1} \left(i + \frac{d}{2}\right)^2 \equiv -a^{-1} i^2 - a^{-1} \frac{d}{2} \equiv \lambda_i + \frac{d}{2} \pmod{d},$$

for all $i \in \left\{0, \ldots, \frac{d}{2} - 1\right\}$. Since $d \geqslant 3$, we have that $1 < \frac{d}{2} < 1 + \frac{d}{2} < d$. Finally, since $\lambda_1 > 0$, we deduce that

$$\max\left\{\lambda_1, \lambda_{1+\frac{d}{2}}\right\} > \frac{d}{2} = \lambda_{\frac{d}{2}},$$

in contradiction with the maximality of $\lambda_{\frac{d}{2}}$.

We thus have that if $d \geqslant 3$ then $n \geqslant 2$ and $\alpha_1 < \alpha_n$. Since $\alpha_1 + \alpha_n = d$, by Proposition 2.4, we deduced that $\alpha_1 < \frac{d}{2} < \alpha_n$. This completes the proof. ☐

Definition 2.6. Let us now consider the integer function $h(a, d, k)$ defined as

$$h(a, d, k) := (a - ((\mu - k)d + \alpha_{j+1}))^2.$$

Remark 3. We notice that the function $h(a, d, k)$ can always be computed for any relatively prime integers a and d and any positive integer k. It is enough to calculate λ_i for each $i = 0, \ldots, d - 1$, from which λ^* and the set of α_i's can be obtained and thus the desired μ and α_{j+1} can be computed.

Theorem 2.7. *Let $d \geqslant 3$ and $a + kd \geqslant 4kd^3$. Then,*

$${}^2r(S_A) \leqslant h(a, d, k)\,.$$

We need the following lemma before proving Theorem 2.7.

Lemma 2.8. *If $d \geqslant 3$ then*

$$\alpha_{i+1} - \alpha_i \leqslant d - 1 \quad \textit{and} \quad \alpha_i + \alpha_{i+1} \leqslant 2d$$

for all $i \in \{1, \ldots, n\}$.

Proof. First, let $i \in \{1,\ldots,n-1\}$. Since $1 \leqslant \alpha_j \leqslant d-1$, for all $j \in \{1,\ldots,n\}$, from Remark 1, it follows that

$$\alpha_{i+1} - \alpha_i < \alpha_{i+1} \leqslant d-1 \quad \text{and} \quad \alpha_i + \alpha_{i+1} < 2d.$$

Finally, for $i = n$, since $n \geqslant 2$ and $\alpha_n > \alpha_1$ by Proposition 2.5, it follows that

$$\alpha_{n+1} - \alpha_n = d + \alpha_1 - \alpha_n < d.$$

Moreover, since $\alpha_n = d - \alpha_1$ by Proposition 2.4, we obtain that

$$\alpha_n + \alpha_{n+1} = (d-\alpha_1) + (d+\alpha_1) = 2d.$$

This completes the proof. □

We now have all the ingredients to prove Theorem 2.7.

Proof of Theorem 2.7. It is known [3] that

$$g(S_A) = \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + (d-1)(a-1) - 1.$$

Since $a^2 > \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a,\ 2akd > (d-1)(a-1)$ and $(kd)^2 > 0$ then

$$g(S_A) < a^2 + 2kda + (kd)^2 = (a-(-kd))^2.$$

Therefore, it is enough to show that $(a-i)^2 \in S$ for all $-kd \leqslant i < (\mu - k)d + \alpha_{j+1}$.

We have two cases.

Case 1. $-kd \leqslant i \leqslant (\mu-k)d + \alpha_j$.

We have that

$$\begin{array}{lll} (i+kd)^2 & \leqslant (\mu d + \alpha_j)^2 & (\text{since } i \leqslant (\mu-k)d+\alpha_j) \\ & \leqslant (kd-\lambda^*)(a+kd) & (\text{by definition}) \\ & \leqslant \left(\left(\left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor + k\right) d - \lambda_i\right)(a+kd) & \left(\text{since } \lambda^* \geqslant \lambda_i \text{ and } \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor \geqslant 0\right) \end{array}$$

Therefore, by Proposition 2.1, we obtain that $(a-i)^2 \in S_A$.

Case 2. $(\mu-k)d+\alpha_j < i < (\mu-k)d + \alpha_{j+1}$.

In this case we have that $\alpha_j < i \bmod d < \alpha_{j+1}$ implying that $\lambda_i \leqslant \lambda^* - 1$ and thus

$$(kd-\lambda_i)(a+kd) \geqslant (kd-\lambda^*)(a+kd) + (a+kd). \tag{2}$$

Moreover,

$$\begin{aligned} (i+kd)^2 &< (\mu d + \alpha_{j+1})^2 \\ &= ((\mu d+\alpha_j) + (\alpha_{j+1}-\alpha_j))^2 \\ &= (\mu d + \alpha_j)^2 + (\alpha_{j+1}-\alpha_j)\left(2\left(\mu d + \alpha_j\right) + (\alpha_{j+1}-\alpha_j)\right) \\ &= (\mu d+\alpha_j)^2 + (\alpha_{j+1}-\alpha_j)\left(2\mu d + \alpha_j + \alpha_{j+1}\right). \end{aligned} \tag{3}$$

Now, from Lemma 2.8, we have that

$$\alpha_{\ell+1} - \alpha_\ell < d \quad \text{and} \quad \alpha_\ell + \alpha_{\ell+1} \leqslant 2d \tag{4}$$

for all $\ell \in \{1,\ldots,n\}$. Therefore, combining (3) and (4), we obtain

$$(i+kd)^2 < (\mu d+\alpha_j)^2 + d\left(2\mu d + 2d\right) = (\mu d + \alpha_j)^2 + 2d^2(\mu+1) \tag{5}$$

for a $j \in \{1,\ldots,n\}$.

Since

$$(\mu d + \alpha_j)^2 \overset{(by\ definition)}{\leqslant} (kd-\lambda^*)(a+kd) \overset{(2)}{\leqslant} (kd-\lambda_i)(a+kd) - (a+kd)$$

then

$$(i+kd)^2 < (kd-\lambda_i)\,(a+kd) + 2d^2\,(\mu+1) - (a+kd). \tag{6}$$

We claim that

$$2d^2(\mu+1) \leqslant a+kd. \tag{7}$$

We have two subcases

Subcase i) For $j \in \{1,\dots,n-1\}$. Since $(\mu d+\alpha_j)^2 \leqslant (kd-\lambda^*)(a+kd) < (\mu d+\alpha_{j+1})^2$, $\alpha_j \geqslant 1$ and $\alpha_{j+1} \leqslant \alpha_n < d$ then

$$\mu = \left\lfloor \frac{\sqrt{(kd-\lambda^*)(a+kd)}}{d} \right\rfloor.$$

Moreover, since $a+kd \geqslant 4kd^3 > 4(kd-\lambda^*)d^2$, it follows that

$$\mu \geqslant 2(kd-\lambda^*) \text{ with } \lambda^* > 0. \tag{8}$$

If $\mu = 2(kd-\lambda^*)$, then we have

$$2d^2(\mu+1) = 4kd^3 + 2(1-2\lambda^*)d^2 \leqslant 4kd^3 \leqslant a+kd,$$

as announced. Otherwise, if $\mu > 2(kd-\lambda^*)$, it follows that

$$(kd-\lambda^*)(a+kd) \geqslant (\mu d+\alpha_j)^2 \overset{(\alpha_j\geqslant 1)}{>} \mu^2 d^2 > \left(\mu^2-1\right)d^2 = (\mu-1)\,(\mu+1)\,d^2 \geqslant 2(kd-\lambda^*)(\mu+1)d^2,$$

obtaining the claimed inequality (7) for $j \in \{1,2,\dots,n-1\}$.

Subcase ii) For $j = n$. Since $(\mu d+\alpha_n)^2 \leqslant (kd-\lambda^*)(a+kd) < (\mu d+\alpha_{n+1})^2$, where $\alpha_n = d-\alpha_1$ and $\alpha_{n+1} = d+\alpha_1$, we obtain

$$((\mu+1)d-\alpha_1)^2 \leqslant (kd-\lambda^*)(a+kd) < ((\mu+1)d+\alpha_1)^2.$$

Since $\alpha_1 < d$, we have

$$\left\lfloor \frac{\sqrt{(kd-\lambda^*)(a+kd)}}{d} \right\rfloor \in \{\mu,\mu+1\}.$$

Moreover, since $a+kd \geqslant 4kd^3 > 4(kd-\lambda^*)d^2$, it follows that

$$\mu+1 \geqslant 2(kd-\lambda^*). \tag{9}$$

If $\mu+1 = 2(kd-\lambda^*)$, then we have

$$2d^2(\mu+1) = 4(kd-\lambda^*)d^2 < 4kd^3 \leqslant a+kd,$$

obtaining the claimed inequality (7). Otherwise, if $\mu+1 > 2(kd-\lambda^*)$, since $\alpha_1 < \frac{d}{2}$ from Proposition 2.5, we obtain

$$(kd-\lambda^*)(a+kd) \geqslant ((\mu+1)d-\alpha_1)^2 > \left((\mu+1)d-\frac{d}{2}\right)^2 = \left(\mu^2+\mu+\frac{1}{4}\right)d^2$$

$$> \mu\,(\mu+1)\,d^2 \overset{\mu\geqslant 2(kd-\lambda^*)}{\geqslant} 2(kd-\lambda^*)(\mu+1)d^2,$$

obtaining the claimed inequality (7) when $j = n$.

Finally, since inequality (7) is true for any $j \in \{1,\dots,n\}$ then, from equation (6) we have

$$(i+kd)^2 < (kd-\lambda_i)\,(a+kd) + 2d^2\,(\mu+1) - (a+kd) \leqslant (kd-\lambda_i)\,(a+kd).$$

We deduce, by Proposition 2.1, that $(a-i)^2 \in S_A$.

This completes the proof. □

Remark 4. The above proof can be adapted if we consider the weaker condition $a+kd > 4(kd-\lambda^*)d^2+d^2$ instead of $a+kd \geqslant 4kd^3$.

We believe that the upper bound $h(a,d,k)$ of ${}^2r(S_A)$ given in Theorem 2.7 is actually an equality. We are able to establish the latter in the case when $k=1$ for any $d \geqslant 3$.

Corollary 2.9. *Let $d \geqslant 3$ and $a+d \geqslant 4d^3$. Then,*

$$ {}^2r(\langle a, a+d\rangle) = h(a,d,1)\,. $$

Proof. By Theorem 2.7, we have ${}^2r(\langle a,a+d\rangle) \leqslant (a-((\mu-1)d+\alpha_{j+1}))^2$. It is thus enough to show that $(a-((\mu-1)d+\alpha_{j+1}))^2 \notin \langle a, a+d\rangle$.

Let $i=(\mu-1)d+\alpha_{j+1}$. We have

$$ \begin{aligned} i^2 &= ((\mu-1)d+\alpha_{j+1})^2 \\ &= ((\mu d+\alpha_j)-(d+\alpha_j-\alpha_{j+1}))^2 \\ &= (\mu d+\alpha_j)^2-(d+\alpha_j-\alpha_{j+1})\left(2\left(\mu d+\alpha_j\right)-(d+\alpha_j-\alpha_{j+1})\right) \\ &= (\mu d+\alpha_j)^2-(d+\alpha_j-\alpha_{j+1})\left((2\mu-1)d+\alpha_j+\alpha_{j+1}\right). \end{aligned} $$

Since $d+\alpha_j-\alpha_{j+1} \geqslant 1$, by Lemma 2.8, and $(2\mu-1)d+\alpha_j+\alpha_{j+1} > (2\mu-1)d$, it follows that

$$ i^2 < (\mu d+\alpha_j)^2-(2\mu-1)d \leqslant (d-\lambda^*)(a+d)-(2\mu-1)d. \tag{10} $$

Since $a+d \geqslant 4d^3$, we already know that $\mu+1 \geqslant 2(d-\lambda^*)$ (see equations (8) and (9) with $k=1$). It follows that $\mu \geqslant 2(d-\lambda^*)-1 \geqslant 1$ and then

$$ d-\lambda^* \leqslant \frac{\mu+1}{2} \leqslant 2\mu-1. \tag{11} $$

By combining equations (10) and (11) we obtain

$$ i^2 < (d-\lambda^*)(a+d)-(d-\lambda^*)d = (d-\lambda^*)a $$

and

$$ \frac{i^2+\lambda_i a}{ad} = \frac{i^2+\lambda^* a}{ad} < \frac{(d-\lambda^*)a+\lambda^* a}{ad} = 1. $$

We may thus deduce that

$$ \left\lfloor \frac{i^2+\lambda_i a}{ad} \right\rfloor = 0. $$

Finally, since

$$ (i+d)^2 = (\mu d+\alpha_{j+1})^2 > (d-\lambda^*)(a+d) = \left(\left(\left\lfloor \frac{i^2+\lambda_i a}{ad}\right\rfloor+1\right)d-\lambda_i\right)(a+d), $$

we deduce, from Proposition 2.1, that $(a-i)^2 \notin \langle a, a+d\rangle$, as desired. □

Unfortunately, the value of ${}^2r(\langle a,a+d\rangle)$ given in the above corollary does not hold in general (if the condition $a+d \geqslant 4d^3$ is not satisfied). However, as we will see below, the number of values of a not holding the equality ${}^2r(\langle a,a+d\rangle) = h(a,d,1)$ is finite for each fixed d.

3. Formulas for $\langle a, a+d\rangle$ with small $d \geqslant 3$

In this section, we investigate the value of ${}^2r(\langle a, a+d\rangle)$ when d is small.

For any positive integer $d \geqslant 3$, we may define the set $E(d)$ to be the set of integers a coprime to d not holding the equality of Corollary 2.9, that is,

$$E(d) := \left\{a \in \mathbb{N} \setminus \{0,1\} \;\middle|\; \gcd(a,d) = 1 \text{ and } {}^2r(\langle a, a+d\rangle) \neq h(a,d,1)\right\}.$$

Since $\lambda^* \leqslant d-1$ then, from Corollary 2.9, we obtain that $E(d) \subset [2, 4d^3-1] \cap \mathbb{N}$. We completely determine the set $E(d)$ for a few values of $d \geqslant 3$ by computer calculations, see Table 1.

d	$\lvert E(d)\rvert$	$E(d)$
3	0	$\emptyset$
4	0	$\emptyset$
5	5	$\{2,4,13,27,32\}$
6	0	$\emptyset$
7	10	$\{2,3,4,9,16,18,19,23,30,114\}$
8	5	$\{5,9,21,45,77\}$
9	5	$\{2,4,7,8,16\}$
10	14	$\{3,9,13,23,27,33,43,123,133,143,153,163,333,343\}$
11	14	$\{2,3,4,5,7,8,9,14,16,18,25,36,38,47\}$
12	9	$\{13,19,25,31,67,79,139,151,235\}$

Table 1. $E(d)$ for the first values of $d \geqslant 3$.

The exact values of ${}^2r(\langle a, a+d\rangle)$ when $a \in E(d)$, for $d \in \{3,\dots,12\}$, are given in Appendix A.

For each value $d \in \{3,\dots,12\}$, an explicit formula for ${}^2r(\langle a, a+d\rangle)$ can be presented excluding the values given in Table 1. The latter can be done by using (essentially) the same arguments as those applied in the proofs of Theorem 2.7 and Corollary 2.9. We present the proof for the case $d = 3$.

Theorem 3.1. *Let $a \geqslant 2$ be an integer not divisible by 3 and let $S = \langle a, a+3\rangle$. Then,*

$${}^2r(S) = \begin{cases} (a-(3b-1))^2 & \text{if either } (3b+1)^2 \leqslant a+3 < (3b+2)^2 \text{ and } a \equiv 1 \bmod 3 \\ & \quad\text{or } (3b+1)^2 \leqslant 2(a+3) < (3b+2)^2 \text{ and } a \equiv 2 \bmod 3, \\ (a-(3b+1))^2 & \text{if either } (3b+2)^2 \leqslant a+3 < (3b+4)^2 \text{ and } a \equiv 1 \bmod 3 \\ & \quad\text{or } (3b+2)^2 \leqslant 2(a+3) < (3b+4)^2 \text{ and } a \equiv 2 \bmod 3. \end{cases}$$

Proof. Since $g(S) = (a-1)(a+2) - 1 = a^2 + a - 3 < (a+1)^2$ then

$${}^2r(S) \leqslant (a-1)^2.$$

By Proposition 2.1, we know that

$$(a-i)^2 \in S \iff (i+3)^2 \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a}\right\rfloor + 3 - \lambda_i\right)(a+3), \tag{12}$$

where $\lambda_i \in \{0,1,2\}$ such that $\lambda_i a + i^2 \equiv 0 \bmod 3$, that is,

$$\lambda_i = \begin{cases} 0 & \text{if } i \equiv 0 \bmod 3 \text{ and } a \equiv 1,2 \bmod 3, \\ 1 & \text{if } i \equiv 1,2 \bmod 3 \text{ and } a \equiv 2 \bmod 3, \\ 2 & \text{if } i \equiv 1,2 \bmod 3 \text{ and } a \equiv 1 \bmod 3. \end{cases}$$

We have four cases.

Case 1. Suppose that $a \equiv 1 \bmod 3$ with $(3b+1)^2 \leqslant a+3 < (3b+2)^2$. Note that $b \geqslant 1$ since $a+3 \geqslant 19$.
If $i \leqslant 3b-2$ then

$$(i+3)^2 \leqslant (3b+1)^2 \leqslant a+3 \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3),$$

obtaining, by equation (12), that $(a-i)^2 \in S$.

If $i = 3b-1$ then

$$i^2 = (3b-1)^2 = 9b^2 - 6b + 1 \overset{b\geqslant 1}{<} 9b^2 + 6b - 2 = (3b+1)^2 - 3 \leqslant a,$$

obtaining that

$$0 \leqslant \frac{i^2+\lambda_i a}{3a} = \frac{i^2+2a}{3a} < 1 \quad (\text{since } 3b-1 \equiv 2 \bmod 3)$$

and thus

$$\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3) = a+3 < (3b+2)^2 = (i+3)^2,$$

by equation (12), we have that $(a-i)^2 \notin S$.

Case 2. Suppose that $a \equiv 1 \bmod 3$ with $(3b+2)^2 \leqslant a+3 < (3b+4)^2$. If $b=0$, we have $(a-1) \notin S$ since

$$\left(3\left\lfloor \frac{1+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3) = a+3 < 4^2.$$

Therefore ${}^2r(S) = (a-1)^2$ in this case. Suppose now that $b \geqslant 1$.
If $i \leqslant 3b-1$, then

$$(i+3)^2 \leqslant (3b+2)^2 \leqslant a+3 \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3),$$

obtaining, by equation (12), that $(a-i)^2 \in S$.

If $i = 3b$ then, using that $\lambda_i = 0$,

$$(i+3)^2 = (3b+3)^2 \leqslant 3(3b+2)^2 \leqslant 3(a+3) \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3),$$

obtaining, by equation (12), that $(a-i)^2 \in S$.

If $i = 3b+1$ then

$$i^2 = (3b+1)^2 = 9b^2 + 6b + 1 \overset{b\geqslant 1}{<} 9b^2 + 12b + 1 = (3b+2)^2 - 3 \leqslant a,$$

obtaining that

$$0 \leqslant \frac{i^2+\lambda_i a}{3a} = \frac{i^2+2a}{3a} < 1 \quad (\text{since } 3b+1 \equiv 1 \bmod 3)$$

and thus

$$\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3) = a+3 < (3b+4)^2 = (i+3)^2,$$

therefore, by equation (12), we have that $(a-i)^2 \notin S$.

Case 3. Suppose that $a \equiv 2 \bmod 3$ with $(3b+1)^2 \leqslant 2(a+3) < (3b+2)^2$. Note that $b \geqslant 1$ since $2(a+3) \geqslant 16$.
If $i \leqslant 3b-2$ then

$$(i+3)^2 \leqslant (3b+1)^2 \leqslant 2(a+3) \overset{\lambda_i \leqslant 1}{\leqslant} \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3),$$

obtaining, by equation (12), that $(a-i)^2 \in S$.

If $i = 3b-1$ then

$$i^2 = (3b-1)^2 = 9b^2 - 6b + 1 \overset{b \geqslant 1}{<} 9b^2 + 6b - 5 = (3b+1)^2 - 6 \leqslant 2a,$$

obtaining that

$$0 \leqslant \frac{i^2+\lambda_i a}{3a} = \frac{i^2+a}{3a} < 1 \quad (\text{since } 3b-1 \equiv 2 \bmod 3)$$

and thus

$$\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3) = 2(a+3) < (3b+2)^2 = (i+3)^2,$$

therefore, by equation (12), we have that $(a-i)^2 \notin S$.

Case 4. Suppose that $a \equiv 2 \bmod 3$ with $(3b+2)^2 \leqslant 2(a+3) < (3b+4)^2$. If $b=0$, we have $(a-1) \notin S$ since

$$\left(3\left\lfloor \frac{1+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3) = 2(a+3) < 4^2.$$

Therefore ${}^2r(S) = (a-1)^2$ in this case. Suppose now that $b \geqslant 1$.
If $i \leqslant 3b-1$ then

$$(i+3)^2 \leqslant (3b+2)^2 \leqslant 2(a+3) \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3),$$

obtaining, by equation (12), that $(a-i)^2 \in S$.

If $i = 3b$ then, using that $\lambda_i = 0$,

$$(i+3)^2 = (3b+3)^2 \overset{b \geqslant 1}{<} \frac{3}{2}(3b+2)^2 \leqslant 3(a+3) \leqslant \left(3\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right)(a+3).$$

Therefore, by equation (12), we have $(a-i)^2 \in S$.

If $i = 3b+1$ then

$$i^2 = (3b+1)^2 = 9b^2 + 6b + 1 \overset{b \geqslant 1}{<} 9b^2 + 12b - 2 = (3b+2)^2 - 6 \leqslant 2a,$$

when $b \geqslant 1$ and clearly $i^2 = 1 < 2a$ when $b = 0$, obtaining that

$$0 \leqslant \frac{i^2+\lambda_i a}{3a} = \frac{i^2+a}{3a} < 1$$

and

$$\left\lfloor \frac{i^2+\lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3\left\lfloor\frac{i^2+\lambda_i a}{3a}\right\rfloor+3-\lambda_i\right)(a+3)=2(a+3)<(3b+4)^2=(i+3)^2,$$

therefore, by equation(12), we have that $(a-i)^2\notin S$.

□

The proofs of the following two theorems are completely analogous to that of Theorem 3.1 with a larger number of cases to be analyzed (in each case, the appropriate inequality is obtained in order to apply Proposition 2.1).

Theorem 3.2. *Let $a\geqslant 3$ be an odd integer and let $S=\langle a,a+4\rangle$. Then,*

$$ {}^2r(S)=\begin{cases}(a-(4b-1))^2 & \text{if either } (4b+1)^2\leqslant a+4<(4b+3)^2 \quad \text{and } a\equiv 1 \bmod 4\\ & \qquad \text{or } \ (4b+1)^2\leqslant 3(a+4)<(4b+3)^2 \text{and } a\equiv 3 \bmod 4,\\ & \\ (a-(4b+1))^2 & \text{if either } (4b+3)^2\leqslant a+4<(4b+5)^2 \quad \text{and } a\equiv 1 \bmod 4\\ & \qquad \text{or } \ (4b+3)^2\leqslant 3(a+4)<(4b+5)^2 \text{and } a\equiv 3 \bmod 4.\end{cases}$$

Theorem 3.3. *Let $a\geqslant 2$ be an integer not divisible by 5 and let $S=\langle a,a+5\rangle$. Then,*

$$ {}^2r(S)=\begin{cases}1 & \text{if } a=2 \text{ or } 4,\\ 10^2 & \text{if } a=13,\\ (a-6)^2 & \text{if } a=27 \text{ or } 32,\\ (a-(5b-2))^2 & \text{if either } (5b+2)^2\leqslant a+5<(5b+3)^2 \quad \text{and } a\equiv 4 \bmod 5\\ & \qquad \text{or } \ (5b+2)^2\leqslant 2(a+5)<(5b+3)^2 \text{and } a\equiv 2 \bmod 5,\\ & \\ (a-(5b-1))^2 & \text{if either } (5b+1)^2\leqslant a+5<(5b+4)^2 \quad \text{and } a\equiv 1 \bmod 5\\ & \qquad \text{or } \ (5b+1)^2\leqslant 2(a+5)<(5b+4)^2 \text{and } a\equiv 3 \bmod 5, a\neq 13,\\ & \\ (a-(5b+1))^2 & \text{if either } (5b+4)^2\leqslant a+5<(5b+6)^2 \quad \text{and } a\equiv 1 \bmod 5\\ & \qquad \text{or } \ (5b+4)^2\leqslant 2(a+5)<(5b+6)^2 \text{and } a\equiv 3 \bmod 5,\\ & \\ (a-(5b+2))^2 & \text{if either } (5b+3)^2\leqslant a+5<(5b+7)^2 \quad \text{and } a\equiv 4 \bmod 5, a\neq 4\\ & \qquad \text{or } \ (5b+3)^2\leqslant 2(a+5)<(5b+7)^2 \text{and } a\equiv 2 \bmod 5, a\neq 2,27,32.\end{cases}$$

4. Study of $\langle a,a+1\rangle$

We investigate the square Frobenius number of $\langle a,a+1\rangle$ with $a\geqslant 2$. We first study the case when neither a nor $a+1$ is a square integer.

Proposition 4.1. *Let a be a positive integer such that $b^2<a<a+1<(b+1)^2$ for some integer $b\geqslant 1$. Then,*

$$ {}^2r(\langle a,a+1\rangle)=(a-b)^2.$$

Proof. Since $g(\langle a,a+1\rangle)=a^2-a-1$ then

$$(a-1)^2\leqslant g(\langle a,a+1\rangle)<a^2.$$

We thus have that ${}^2r(\langle a,a+1\rangle)<a^2$. We shall show that $(a-i)^2\in\langle a,a+1\rangle$ for $i\in\{1,2,\ldots,b-1\}$.

We first observe that

$$(a-i)^2=a^2-2ai+i^2=(a-2i)a+i^2=(a-2i-i^2)a+i^2(a+1), \tag{13}$$

for any integer i.

Since for any $i\in\{1,2,\ldots,b-1\}$ we have

$$a-2i-i^2=a-i(i+2)\geqslant a-(b-1)(b+1)=a-b^2+1>0$$

and

$$i^2 > 0$$

then, by (13), we deduce that $(a-i)^2 \in \langle a, a+1\rangle$ for any $i \in \{1, 2, \ldots, b-1\}$.

Finally, since $a+1 < (b+1)^2$ (implying that $a - 2b - b^2 < 0$) and $0 < b^2 < a$ then we may deduce, from (13), that $(a-b)^2 \notin \langle a, a+1\rangle$. □

Let $(u_n)_{n\geqslant 1}$ be the recursive sequence defined by

(14) $u_1 = 1, u_2 = 2, u_3 = 3, u_{2n} = u_{2n-1} + u_{2n-2}$ and $u_{2n+1} = u_{2n} + u_{2n-2}$ for all $n \geqslant 2$.

The first few values of $(u_n)_{n\geqslant 1}$ are

$$1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239, 408, 577, 985, \ldots\ldots$$

This sequence appears in a number of other contexts. For instance, it corresponds to the *denominators of Farey fraction approximations to* $\sqrt{2}$, where the fractions are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{7}{5}$, $\frac{10}{7}$, $\frac{17}{12}$, $\frac{24}{17}\ldots$, see [5].

We pose the following conjecture in the case when either a or $a+1$ is a square integer.

Conjecture 4.2. *Let* $(u_n)_{n\geqslant 1}$ *be the recursive sequence given in* (14).
If $a = b^2$ *for some integer* $b \geqslant 1$ *then*

$$^2r(\langle a, a+1\rangle) = \begin{cases} \left(a - \lfloor b\sqrt{2}\rfloor\right)^2 & \text{if } b \notin \bigcup_{n\geqslant 0} \{u_{4n+1}, u_{4n+2}\}, \\ \left(a - \lfloor b\sqrt{3}\rfloor\right)^2 & \text{if } b \in \bigcup_{n\geqslant 0} \{u_{4n+1}, u_{4n+2}\}. \end{cases}$$

If $a+1 = b^2$ *for some integer* $b \geqslant 1$ *then*

$$^2r(\langle a, a+1\rangle) = \begin{cases} \left(a - \lfloor b\sqrt{2}\rfloor\right)^2 & \text{if } b \notin \bigcup_{n\geqslant 1} \{u_{4n-1}, u_{4n}\}, \\ \left(a - \lfloor b\sqrt{3}\rfloor\right)^2 & \text{if } b \in \bigcup_{n\geqslant 1} \{u_{4n}, u_{4n+3}\}, \\ 2^2 & \text{if } b = u_3 = 3. \end{cases}$$

The formulas of Conjecture 4.2 have been verified by computer for all integers $a \geqslant 2$ up to 10^6.

5. Study of $\langle a, a+2\rangle$

We investigate the square Frobenius number of $\langle a, a+2\rangle$ with $a \geqslant 3$ odd. We first study the case when neither a nor $a+2$ is a square integer.

Proposition 5.1. *Let* $a \geqslant 3$ *be an odd integer such that* $(2b+1)^2 < a < a+2 < (2b+3)^2$ *for some integer* $b \geqslant 1$. *Then,*

$$^2r(\langle a, a+2\rangle) = (a - (2b+1))^2.$$

Proof. Since $g(\langle a, a+2\rangle) = (a-1)(a+1) - 1 = a^2 - 2$ then

$$(a-1)^2 < g(\langle a, a+2\rangle) < a^2.$$

We thus have that $^2r(\langle a, a+2\rangle) < a^2$. We shall show that $(a-i)^2 \in \langle a, a+2\rangle$ for $i \in \{1, 2, \ldots, 2b\}$.

We first observe that for any integer i, we have

$$(a-2i)^2 = a^2 - 4ai + 4i^2 = (a-4i)a + 4i^2 = (a-4i-2i^2)a + 2i^2(a+2). \tag{15}$$

Since for any $i \in \{1, 2, \ldots, b\}$ we have

$$a - 4i - 2i^2 = a - 2i(i+2) \geqslant a - 2i(2i+1) > a - (2i+1)^2 \geqslant a - (2b+1)^2 > 0$$

and

$$2i^2 > 0$$

then, by (15), it follows that $(a-2i)^2 \in \langle a, a+2\rangle$ for any $i \in \{1, 2, \ldots, b\}$.

Moreover, for any integer i, we have

$$\begin{aligned}(a-(2i+1))^2 &= a^2 - 2a(2i+1) + (2i+1)^2 = (a-2(2i+1))a + (2i+1)^2\\ &= (a-4i-3)a + (2i+1)^2 + a\\ &= \left(a - 4i - 3 - \tfrac{(2i+1)^2+a}{2}\right) a + \tfrac{(2i+1)^2+a}{2}(a+2)\\ &= \tfrac{a-4i^2-12i-7}{2}a + \tfrac{(2i+1)^2+a}{2}(a+2)\\ &= \tfrac{a+2-(2i+3)^2}{2}a + \tfrac{(2i+1)^2+a}{2}(a+2).\end{aligned} \tag{16}$$

Note that $a+2-(2i+3)^2$ and $(2i+1)^2+a$ are even because a is odd.

Since, for any $i \in \{0, 1, \ldots, b-1\}$ we have

$$\frac{a+2-(2i+3)^2}{2} \geqslant \frac{a+2-(2b+1)^2}{2} > 0$$

and

$$\frac{(2i+1)^2+a}{2} > 0$$

then it follows, from (16) ,that $(a-(2i+1))^2 \in \langle a, a+2\rangle$, for any $i \in \{0, 1, \ldots, b-1\}$.

Finally, since

$$0 < \frac{(2b+1)^2+a}{2} < a$$

and

$$\frac{a+2-(2b+3)^2}{2} < 0,$$

then we have, from (16), that $(a-(2b+1))^2 \notin \langle a, a+2\rangle$. □

We pose the following conjecture in the case when either a or $a+2$ is a square integer.

Conjecture 5.2. *Let $(u_n)_{n\geqslant 1}$ be the recursive sequence given in* (14).
If $a = (2b+1)^2$ for some integer $b \geqslant 1$ then

$$^2r(\langle a, a+2\rangle) = \begin{cases} \left(a - 2\left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor\right)^2 & \text{if } (2b+1) \notin \bigcup_{n\geqslant 1} \{u_{4n+1}\},\\ \left(a - \lfloor (2b+1)\sqrt{3} \rfloor\right)^2 & \text{if } (2b+1) \in \bigcup_{n\geqslant 2} \{u_{4n+1}\},\\ 38^2 & \text{if } 2b+1 = u_5 = 7.\end{cases}$$

If $a+2 = (2b+1)^2$ for some integer $b \geqslant 1$ then

$$^2r(\langle a, a+2\rangle) = \begin{cases} \left(a - 2\left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor\right)^2 & \text{if } (2b+1) \notin \bigcup_{n\geqslant 0} \{u_{4n+3}\},\\ \left(a - \lfloor (2b+1)\sqrt{3} \rfloor\right)^2 & \text{if } (2b+1) \in \bigcup_{n\geqslant 0} \{u_{4n+3}\}.\end{cases}$$

The formulas of Conjecture 5.2 have been verified by computer for all odd integers $a \geqslant 3$ up to 10^6.

6. Concluding remarks

In the process of investigating square Frobenius numbers different problems arose. We naturally consider the P-type function ${}_{k\text{-power}}r(S) = {}_k r(S)$ defined as,

$${}_k r(S) := \text{ the smallest perfect } k\text{-power integer belonging to } S.$$

It is clear that

$$s \leqslant {}_k r(S) \leqslant s^k \tag{17}$$

where s is the multiplicity of S.

Theorem 6.1. *Let $S_A = \langle a, a+d, \ldots, a+kd\rangle$ where a, d, k are positive integers with $\gcd(a,d) = 1$. If $d \leqslant \frac{ak}{1+2k}$ then*

$${}_2 r(S_A) \leqslant (a-d)^2.$$

Proof. We shall use the characterization given in Proposition 2.1 with $i = d$. In this case $\lambda_d = 0$ and $d \leqslant \frac{ak}{1+2k} < a$ thus

$$\left(\left(\left\lfloor \frac{d^2+0a}{ad} \right\rfloor + k\right) d - 0\right)(a+kd) = \left(\left(\left\lfloor \frac{d}{a} \right\rfloor + k\right) d - 0\right)(a+kd) = akd + (kd)^2.$$

Thus,

$$(d+kd)^2 \leqslant akd + (kd)^2 \iff d^2 + 2kd^2 \leqslant akd \iff d \leqslant \frac{ak}{1+2k}.$$

Therefore, by Proposition 2.1, $(a-d)^2 \in S_A$. □

Problem 1. *Let $k \geqslant 2$ be an integer and let S be a numerical semigroup. Investigate the computational complexity to determine ${}^k r(S)$ and/or ${}_k r(S)$.*

Or more ambitious,

Question 1. *Let $k \geqslant 2$ be an integer. Is there a closed formula for ${}^k r(S)$ and/or ${}_k r(S)$ for any semigroup S?*

Perhaps a first step on this direction might be the following.

Problem 2. *Give a formula for ${}^2 r(\langle F_i, F_j\rangle)$ and/or ${}_2 r(\langle F_i, F_j\rangle)$ with $\gcd(F_i, F_j) = 1$ where F_k denotes the k^{th} Fibonacci number. What about ${}^2 r(\langle a^2, b^2\rangle)$ where a and b are relatively prime integers ? We clearly have that ${}_2 r(\langle a^2, b^2\rangle) = a^2$ for $1 \leqslant a < b$.*

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Appendix A. Complement to formulas for $\langle a, a+d\rangle$ with small $d \geqslant 3$

In Tabular 2, we compare the exact values of ${}^2r(\langle a, a+d\rangle)$ and the formula $h(a, d, 1)$, when $a \in E(d)$ for $d \in \{3, \ldots, 12\}$.

d	a	${}^2r(\langle a, a+d\rangle)$	$h(a, d, 1)$	d	a	${}^2r(\langle a, a+d\rangle)$	$h(a, d, 1)$
5	2	1	0	10	43	40^2	39^2
5	4	1	2^2	10	123	110^2	109^2
5	13	10^2	9^2	10	133	120^2	119^2
5	27	21^2	20^2	10	143	130^2	129^2
5	32	26^2	25^2	10	153	140^2	139^2
7	2	1	5^2	10	163	150^2	149^2
7	3	2^2	0	10	333	310^2	309^2
7	4	5^2	6^2	10	343	320^2	319^2
7	9	7^2	6^2	11	2	3^2	4^2
7	16	14^2	13^2	11	3	5^2	9^2
7	18	17^2	16^2	11	4	5^2	6^2
7	19	14^2	13^2	11	5	7^2	9^2
7	23	21^2	20^2	11	7	4^2	2^2
7	30	28^2	27^2	11	8	7^2	12^2
7	114	105^2	104^2	11	9	8^2	12^2
8	5	4^2	3^2	11	14	13^2	19^2
8	9	10^2	12^2	11	16	14^2	20^2
8	21	16^2	15^2	11	18	15^2	13^2
8	45	36^2	35^2	11	25	22^2	20^2
8	77	64^2	63^2	11	36	33^2	31^2
9	2	3^2	4^2	11	38	36^2	34^2
9	4	3^2	6^2	11	47	44^2	42^2
9	7	6^2	11^2	12	13	14^2	18^2
9	8	6^2	4^2	12	19	17^2	16^2
9	16	9^2	12^2	12	25	26^2	30^2
10	3	2^2	7^2	12	31	29^2	28^2
10	9	7^2	12^2	12	67	59^2	58^2
10	13	10^2	9^2	12	79	71^2	70^2
10	23	20^2	19^2	12	139	125^2	124^2
10	27	26^2	25^2	12	151	137^2	136^2
10	33	30^2	29^2	12	235	215^2	214^2

Table 2. ${}^2r(\langle a, a+d\rangle)$ and $h(a, d, 1)$ when $a \in E(d)$ for $d \in \{3, \ldots, 12\}$

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