Around the vertices of projective polytopes

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New trends from Classical Theorems in Geometry, Combinatorics, and Topology

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McMullen problem

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A projective transformation $P : \mathbb{R}^d \to \mathbb{R}^d$ is such that $P(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$ where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$ such that at least one of $c \neq 0$ or $\delta \neq 0$.

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Theorem (Las Vergnas,1985) $n(d) \leq d(d + 1)/2$ for any $d \geq 2$.

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An oriented matroid is uniform if $\chi(B) = +$ or $-$ for any base B.

Topological representation

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Theorem (Folkman and Lawrence, 1978) Any loop-free oriented matroid of rank $d+1$ (up to isomorphism) are in one-to-one correspondence with signed arrangements of pseudo-spheres in S^d (up to topological equivalence).

Notions and Facts

• Any configuration of points in $\mathbb{R}^d$ induce an oriented matroid in the affine space of rank $r = d + 1$ where the signed set of circuits are the coefficients of minimal affine dependencies.

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• The set of acyclic reorientations of M are in bijection with the set of cells of the corresponding arrangement of pseudospheres.

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Theorem (R.A. 2001) $n(d) \leq 2d + \lceil \frac{d}{2} \rceil$ for any $d \geq 2$.

Lawrence oriented matroid

A Lawrence oriented matroid M of rank r on the totally ordered set $E = \{1, \ldots, n\}$, $r \leq n$, is a uniform oriented matroid obtained as the union of r uniform oriented matroids $M_1, \ldots, M_r$ of rank 1 on $(E, <)$.

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The chirotope χ corresponds to some Lawrence oriented matroid M_A if and only if there exists a matrix $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ with entries from $\{+1, -1\}$ (where the i-th row corresponds to the chirotope of the oriented matroid M_i) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where B is an ordered r-tuple $j_1 \leq \ldots \leq j_r$ elements of E.

elements

rank	1	2	3	4	5	6	7	
1	+	−	−	+	+	+	+	← $\chi(M_1)$
2	+	−	+	+	+	+	+	← $\chi(M_2)$
3	+	+	+	+	+	+	+	
4	+	−	+	+	+	+	+	

Matrix A arising a Lawrence oriented matroid $M = \bigcup_{i=1}^{n} M_i$.

elements

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1	+	−	−	+	+	**−**	+	← $\chi(M_1)$
2	+	−	+	+	+	**−**	+	← $\chi(M_2)$
3	+	+	+	+	+	**−**	+	
4	+	−	+	+	+	**−**	+	

Reorientation of element 6 arising a Lawrence oriented matroid $-_6M$.

	1	2	3	4	5	6	7
1	+	−	−	+	+	+	+
2	+	−	+	+	+	+	+
3	+	+	+	+	+	+	+
4	+	−	+	+	+	+	+

We define Top Travel $[TT]$ and the Bottom Travel $[BT]$ on the entries of A, both formed by horizontal and vertical movements.

	1	2	3	4	5	6	7
1	+	−	−	+	+	+	+
2	+	−	+	+	+	+	+
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4	+	−	+	+	+	+	+

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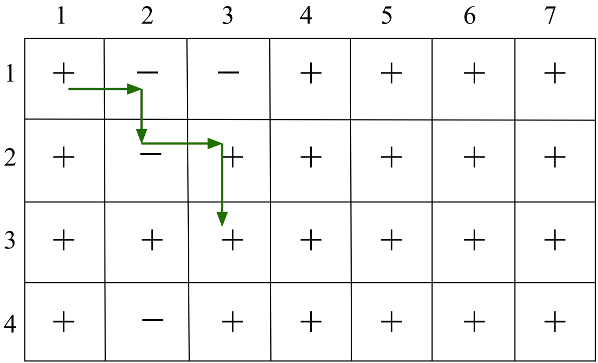

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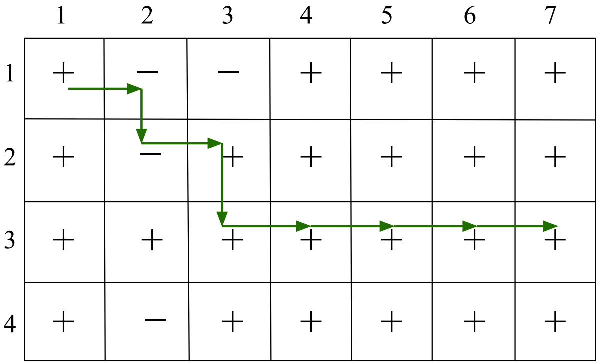

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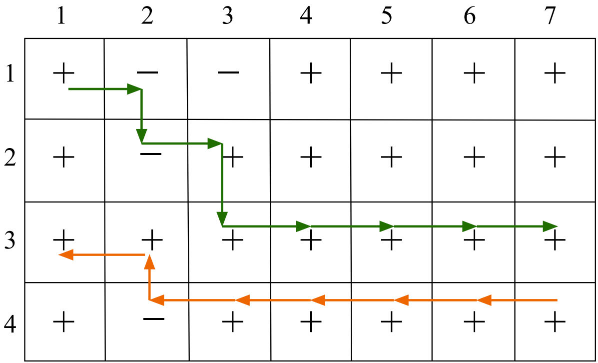

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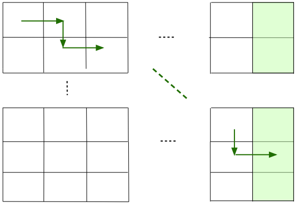
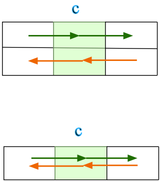
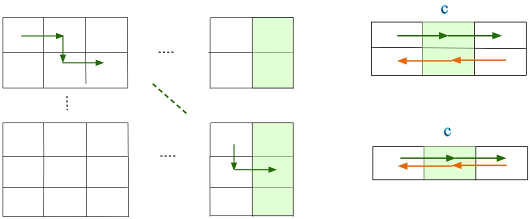


TT ends at last column

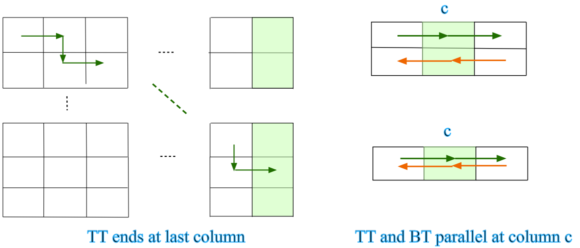


TT and BT parallel at column c



TT ends at last column   TT and BT parallel at column c

- M_A is acyclic iff TT arrives at the last column of A.



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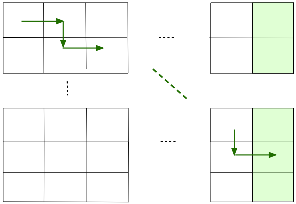
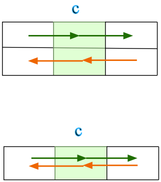
- c is interior in M_A iff TT and BT are parallel at column c.



TT ends at last column



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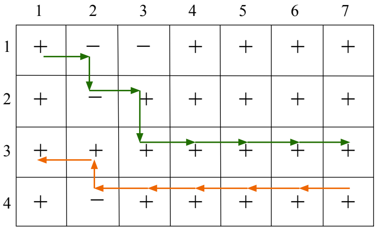
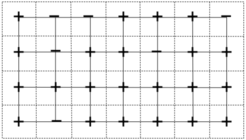
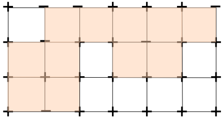


M_A is acyclic and 4, 5 and 6 are interior elements.

Chessboard

+	−	−	+	+	+	−
+	−	+	+	−	+	+
+	+	+	+	+	+	+
+	−	+	+	+	+	+

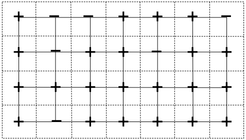


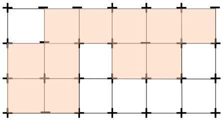


Chessboard of matrix A invariant under reorientations

Chessboard

+	−	−	+	+	+	−
+	−	+	+	−	+	+
+	+	+	+	+	+	+
+	−	+	+	+	+	+





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The upper bound $n(d) \leq 2d + \lceil \frac{d}{2} \rceil$ for any $d \geq 2$ comes from ...

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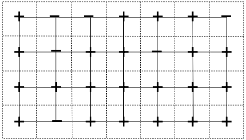
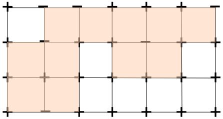
+	−	−	+	+	+	−
+	−	+	+	−	+	+
+	+	+	+	+	+	+
+	−	+	+	+	+	+





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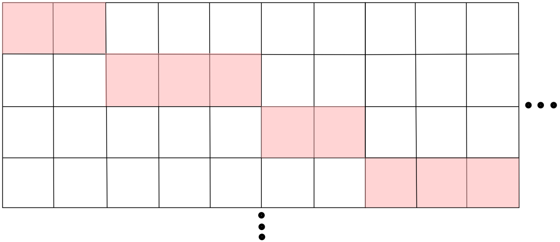

McMullen problem - Neighbourly version

A d-polytope is k-neighbourly if for $k \leq \lceil \frac{d}{2} \rceil$ fixed, every subset of at most k vertices of the vertex set of the polytope is a face of the polytope.

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Neighbourly version What is the larges integer $v(d,k)$ be the largest integer such that any $v(d,k)$ points in general position in $\mathbb{R}^d$ can be mapped by a permissible projective transformation onto points onto the vertices of a k-neighbourly convex polytope ?

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Theorem (García-Colin, 2014) Let $2 \leq k \leq \lceil \frac{d}{2} \rceil$. Then,

$$d + \left\lfloor \frac{d}{k} \right\rfloor + 1 \leq v(d,k) < 2d - k + 1.$$

Projective k-faces

Let $X \subset \mathbb{R}^d$ be a set of points in general position. Let

$$h_k(X, d) = \max_T \{f_k(\text{conv}(T(X)))\},$$

maximum taken over all possible permissible projective transformations T of X and $f_k(P)$ denotes the number of k-faces of a polytope P.

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We consider

$$H_k(n,d) = \min_{X \subset \mathbb{R}^d, |X|=n} \{h_k(X,d)\}.$$

Generalizing McMullen

Generalized version Let $t \geq 0$ be an integer. What is the largest integer $n(t, d)$ such that any set of n points in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation onto the vertices of a convex polytope with at most t points in its interior ?

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$$n(0,d) = n(d)$$

The function $n(t,d)$ will allow us to study $H_0(n,d)$ in a more general setting since

$$H_0(n(t,d),d) = n(t,d) - t$$

Upper bounds

Theorem (García-Colin, Montejano, R.A., 2023) Let $d, t \geq 1$ and $n \geq 2$ be integers. Then,

$$H_0(n,d) \begin{cases} = 2 & \text{if } d = 1,\ n \geq 2, \\ = 5 & \text{if } d = 2,\ n \geq 5, \\ \leq 7 & \text{if } d = 3,\ n \geq 7, \\ \leq n - 1 - t & \text{if } d \geq 4,\ n \geq 2d + t(d-2) + 2, t \geq 1. \end{cases}$$

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By the Upper Bound Theorem we have

$$H_k(n,d) \leq f_k(C_d(H_0(n,d))) \text{ for all } n \geq 1 \text{ and any } k \geq 1$$

where $C_d(n)$ is the d-dimensional cyclic polytope with n vertices.

Minimal Randon partition

Let $X = A \cup B$ be any partition of the set of points X in general position in $\mathbb{R}^d$.
$r_X(A, B) :=$ the number of $(d+2)$-element subsets $S \subset X$ such that $\text{conv}(A \cap S) \cap \text{conv}(B \cap S) \neq \emptyset$

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Consider the functions

$$r(X) := \max_{\{(A,B) | A \cup B = X\}} r_X(A, B) \quad \text{and} \quad r(n, d) := \min_{X \subset \mathbb{R}^d, |X| = n} r(X).$$

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Consider the functions

$$r(X) := \max_{\{(A,B) | A \cup B = X\}} r_X(A, B) \quad \text{and} \quad r(n, d) := \min_{X \subset \mathbb{R}^d, |X| = n} r(X).$$

Theorem (García-Colin, Montejano, R.A., 2023) Let $d, n \geq 1$ be integers. Then, $r(n, d) = H_{d'-1}(n, d')$ where $d' = n - d - 2$.

2-Randon partition

Theorem (García-Colin, Montejano, R.A., 2023) Let $n \geq 4$ be an integer. Then,

$$r(n,2) \begin{cases} = 2 & \text{if } n = 5, \\ = 5 & \text{if } n = 6, \\ = 10 & \text{if } n = 7, \\ \leq 2\binom{\frac{n-1}{2}+2}{\frac{n-1}{2}-2} & \text{if } n \geq 7, n\text{-odd}, \\ \leq \binom{\frac{n}{2}+2}{\frac{n}{2}-2} + \binom{\frac{n}{2}+1}{\frac{n}{2}-3} & \text{if } n \geq 8, n\text{-even}. \end{cases}$$

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Question : $r(9,3) =?$

Colored points in the plane

Problem (Pach and Szegedy, 2003) : Given n points in general position in the plane, coloured red and blue, maximize the number of multicoloured 4-tuples with the property that the convex hull of its red elements and the convex hull of its blue elements have at least one point in common. In particular, **show that when the maximum is attained, the number of red and blue elements are roughly the same**.

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Theorem (García-Colin, Montejano, R.A., 2023) Let $X \subset \mathbb{R}^2$ be a set of points in general position with $|X| = n \geq 8$. Then, for any partition A, B of X such that $r_X(A, B) = r(X)$, we have that $|A|, |B| \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Tolerence

$\lambda(t,d) :=$ the smallest number λ such that for any set X of λ points in $\mathbb{R}^d$ there exists a partition of $X = A \cup B$ and a subset $P \subseteq X$ of cardinality $\lambda - i$, for some $0 \leq i \leq t$, such that

$$\operatorname{conv}(A \setminus y) \cap \operatorname{conv}(B \setminus y) \begin{cases} \neq \emptyset & \text{if } y \in P, \\ = \emptyset & \text{if } y \in X \setminus P. \end{cases}$$

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Theorem (García-Colin, Montejano, R.A., 2023) Let $t \geq 0$ and $d \geq 1$ be integers. Then,

$$n(t,d) = \max_{m \in \mathbb{N}} \{m \mid \lambda(t, m-d-1) \leq m\}$$

and

$$\lambda(t,d) = \min_{m \in \mathbb{N}} \{m \mid m \leq n(t, m-d-1)\}.$$

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Question 3 : Is it true that any simple arrangement of $n \geq 2d+1$ (pseudo)hyperplanes in $\mathbb{P}^d$ contains a cell of size at least $2d+1$? Moreover, is it true that for any $n \geq 2d+1$, there exists a simple arrangement of n (pseudo)hyperplanes in $\mathbb{P}^d$ with every cell of size at most $2d+1$?

Thanks for your attention !