On the ball number of links

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(joint work with I. Rasskin)

Knots, Surfaces, and 3-manifolds

CMO-BIRS, Oaxaca, June 12th, 2023

Diagrams

Some diagrams

Trivial knot
0_1

Trefoil knot
3_1

Figure-eight knot
4_1

Pentafoil knot
5_1

Trivial link
0_1^2

Hopf link
2_1^2

Solomon link
4_1^2

Borromean link
6_2^3

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The crossing number of a link L, denoted by $cr(L)$, is the minimum number of crossings among all the diagrams of L.

Necklace representation

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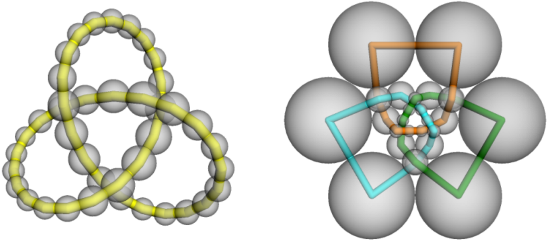
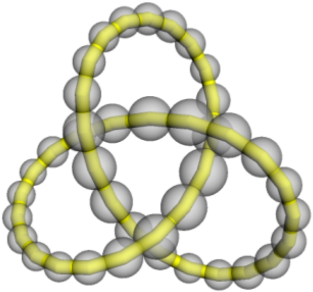


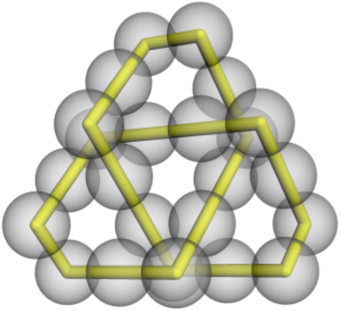
Necklace representations of the Trefoil and the Borromean link.

Necklace representation



40 spheres

Necklace representation



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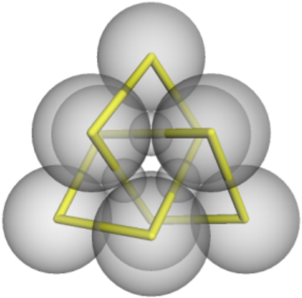
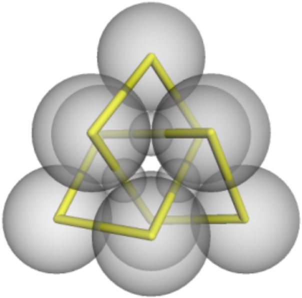


12 spheres

Necklace representation



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Question What is the minimum number of spheres among all necklace representations of a given link ?

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Conjecture (Maehara, 2007) $ball($$) = 12$

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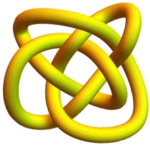

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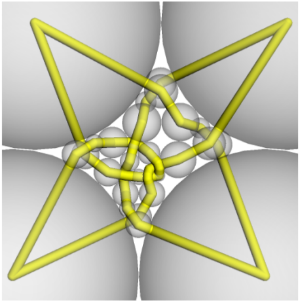

The *Lorentzian space* $\mathbb{L}^{d+1,1}$, of dimension $d+2$, is the vector space of dimension $d+2$ equipped with *Lorentzian product*

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_{d+1} y_{d+1} - x_{d+2} y_{d+2}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{L}^{d+1,1}$$

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Some Lorenzian theory

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$$\langle v_b, v_{b'} \rangle \begin{cases} > 1 & \text{if } b \text{ and } b' \text{ are nested} \\ = 1 & \text{if } b \text{ and } b' \text{ are internally tangent} \\ = 0 & \text{if } b \text{ and } b' \text{ are orthogonal} \\ = -1 & \text{if } b \text{ and } b' \text{ are externally tangent} \\ < -1 & \text{if } b \text{ and } b' \text{ are disjoint} \end{cases}$$

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G is disk packable if there is a disk packing in the plane whose contact graph is isomorphic to G

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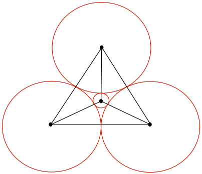

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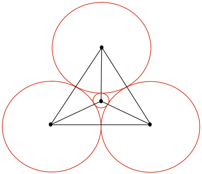

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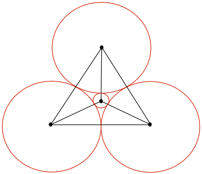

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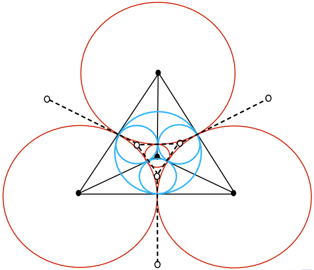

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Remark : Lorentz geometry (and other building blocks) is used to verify that the construction works well.

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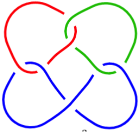

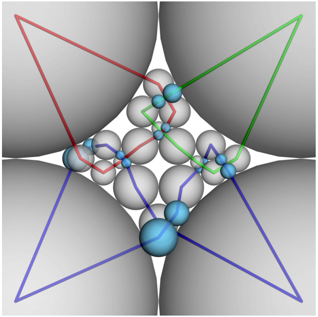

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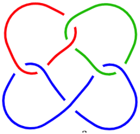

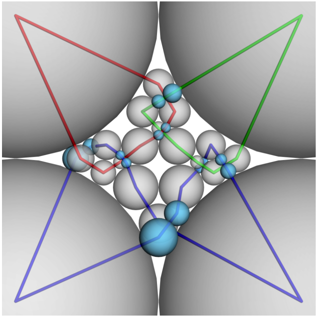

Conjecture (Rasskin + R.A., 2021) $ball(L) \leq 4cr(L)$ for any link L. Moreover, the equality holds if L is alternating.

Apollonius' theorem

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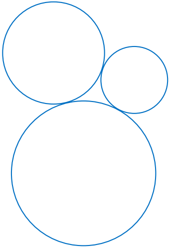

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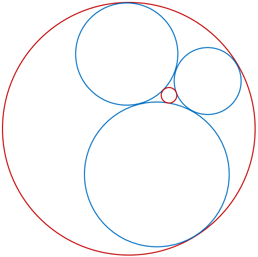

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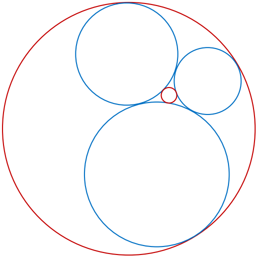
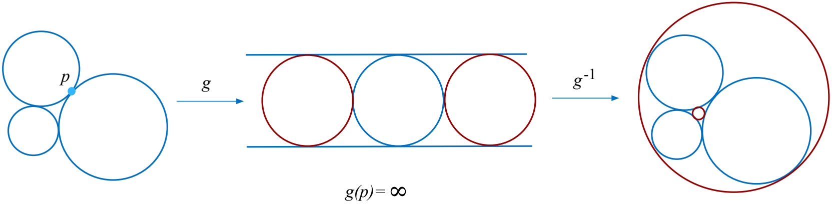


Proof (idea) :



Apollonian gasket

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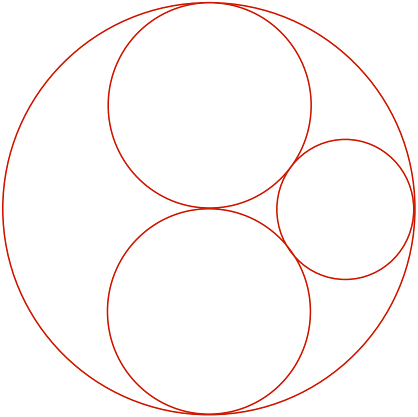

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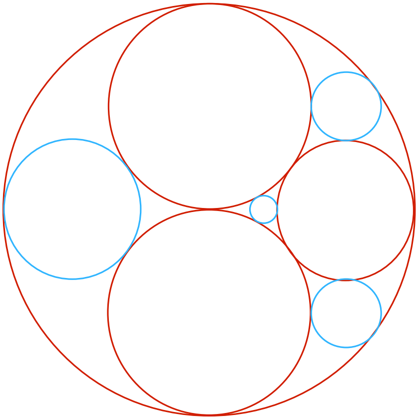

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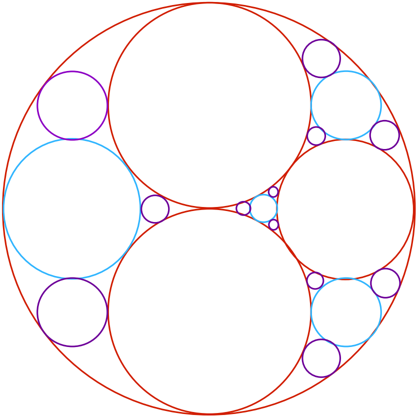
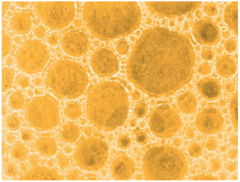
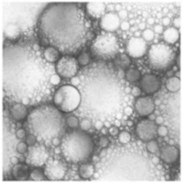


Motivation

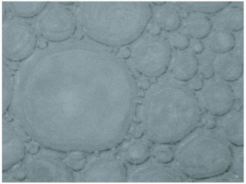
Apollonian packings are attractive to study :



Granular systems



Fluid emulsion



Foam bubbles

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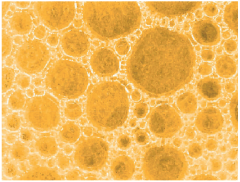	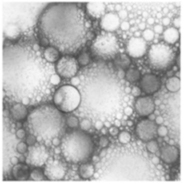	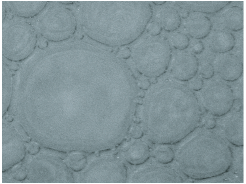
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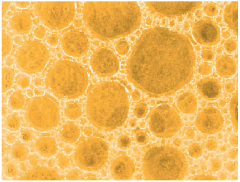

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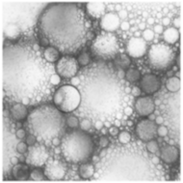

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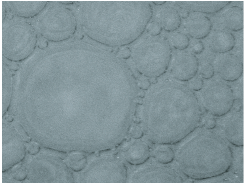

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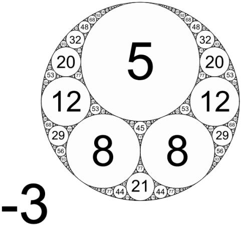
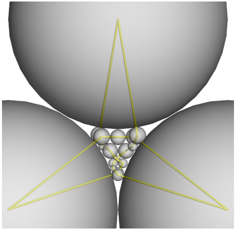


Number theory



Knot theory

Inversion with respect to a circle

The inverse of a point Q with respect to a circle with center O and radius r is the point Q' lying on the segment $[O, Q]$ such that $d(O, Q) \cdot d(O, Q') = r^2$.

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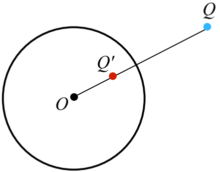
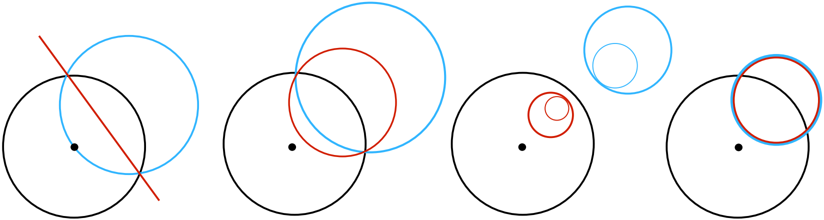
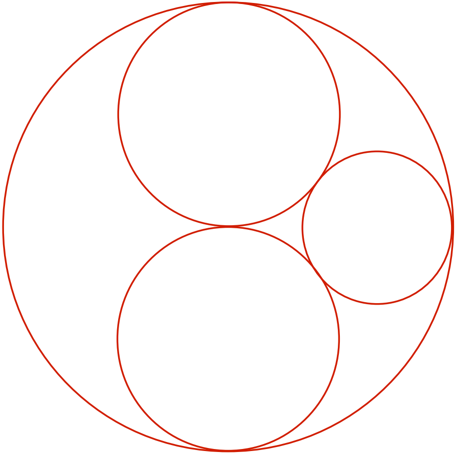




From the Tetrahedron



Packings by using inversions

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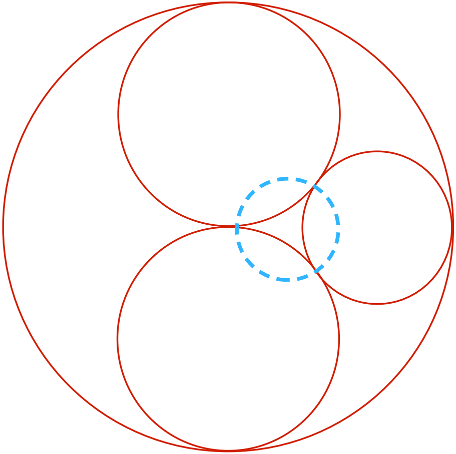

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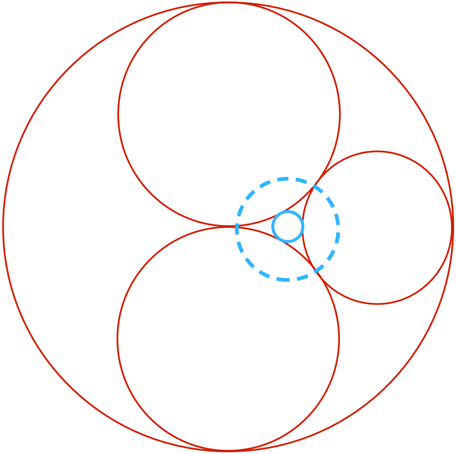

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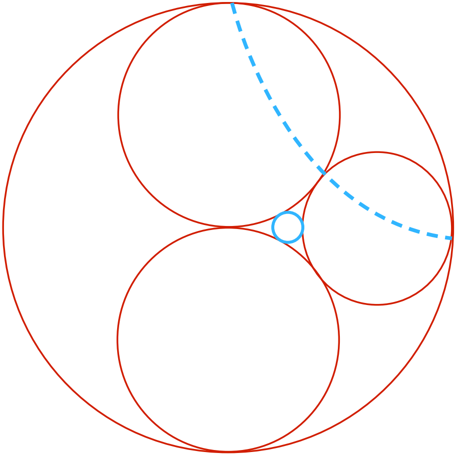

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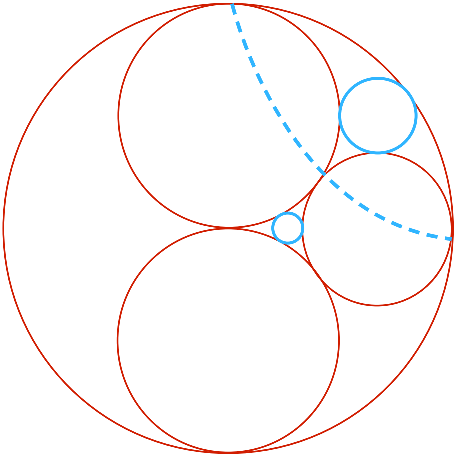

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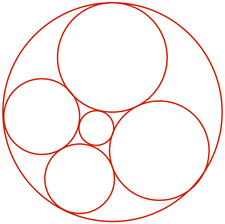

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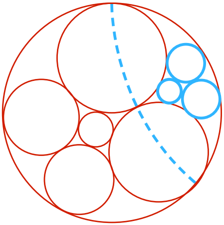

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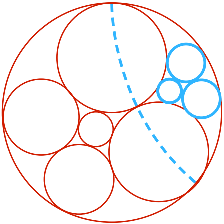
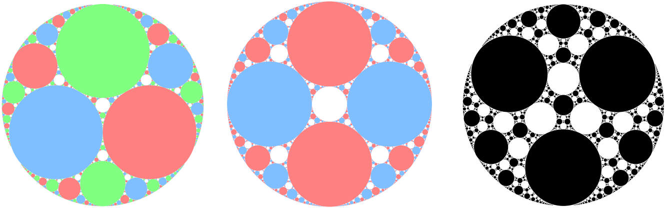


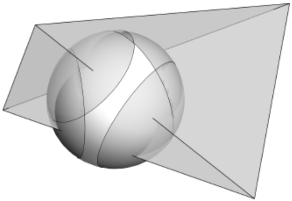
Gaskets from the Tetrahedron, the Octahedron and the Cube

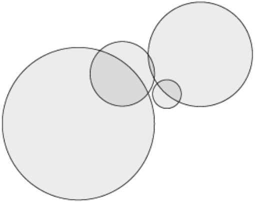


Let P be a $(d+1)$-polytope *sphere-exterior* (vertices outside the sphere)

Ball arrangements

Let P be a $(d+1)$-polytope *sphere-exterior* (vertices outside the sphere)
The *projected ball arrangement* $B(P)$ of P, is the collection of d-balls whose *light sources* are the vertices of P.





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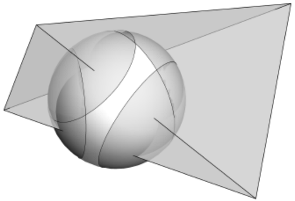

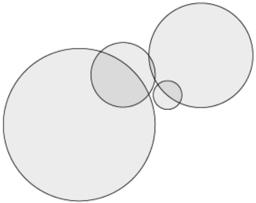

If P is a $(d+1)$-polytope is *edge-scribible* (i.e., all the edges of P are tangent to $\mathbb{S}^d$) then $B(P)$ is a *d-ball packing*, denoted by B_P and called polytopal packing.

Polytopal sphere packings

- Not all the sphere packings are polytopal packings

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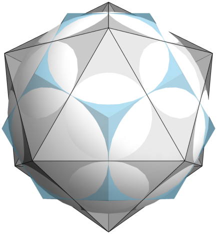
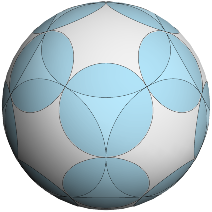
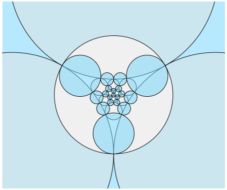
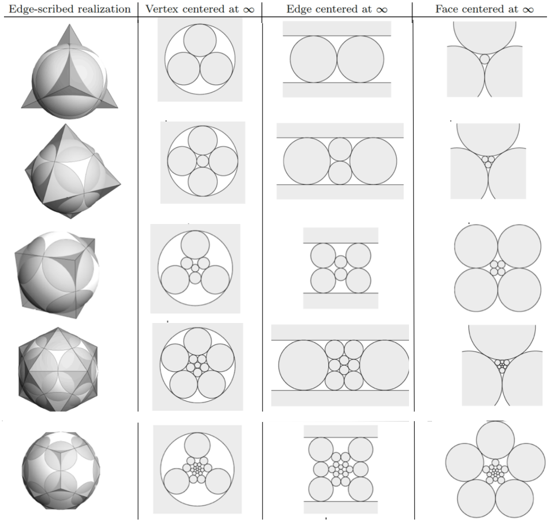
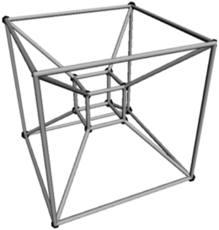
An edge-scribible icosahedron and its polar (dodecahedron).

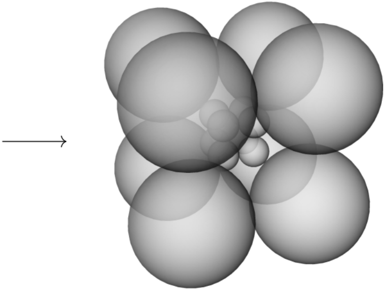




Stereographic projections



Hypercube (cube in dimension 4)





3-ball polytopal packings

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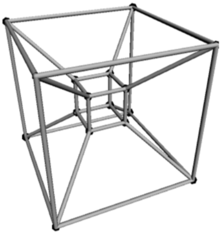

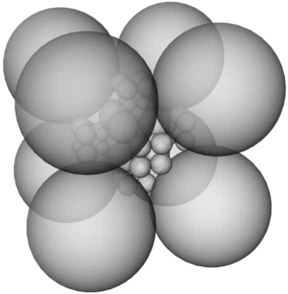

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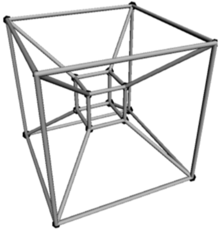
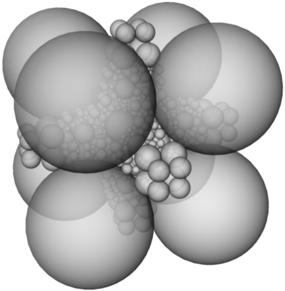


$\longrightarrow$



Orthoplicial representation

Theorem (Rasskin + R.A., 2023) Every link admits a necklace representation in $B_{\mathcal{O}^4}$.

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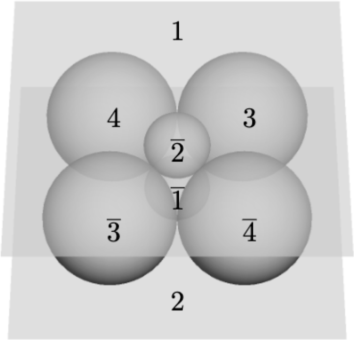

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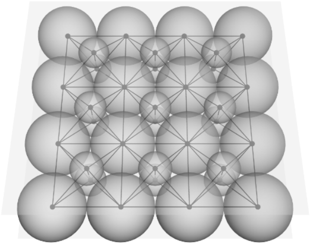

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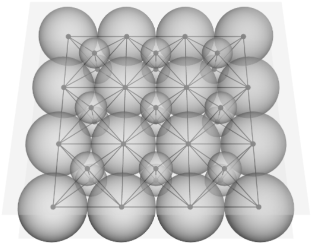

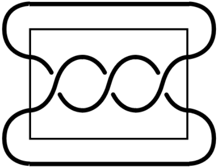

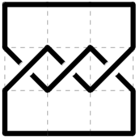

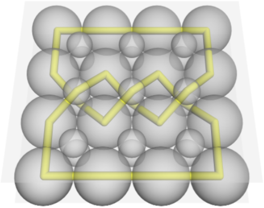

Tetrahedral and cubic representations

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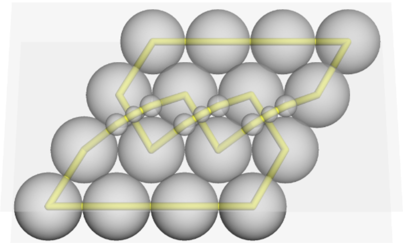
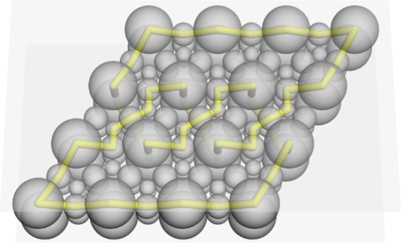
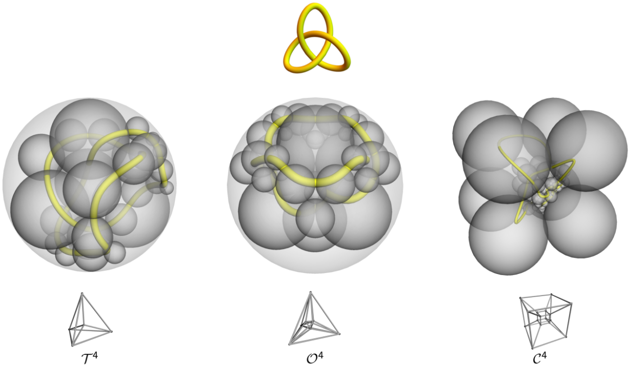




Apollonian representations

Theorem (Rasskin + R.A., 2023) Every link admits a necklace representation in $B_{\mathcal{O}^4}$, $B_{\mathcal{T}^4}$ and $B_{\mathcal{C}^4}$.



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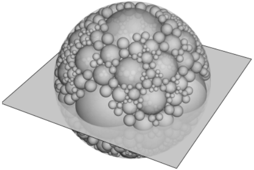

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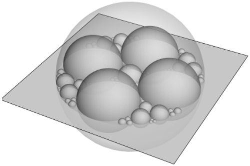

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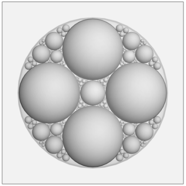

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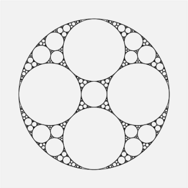

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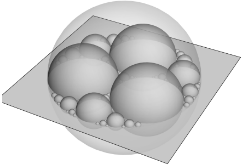

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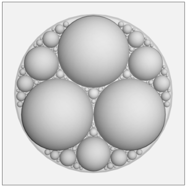

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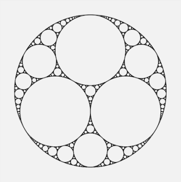

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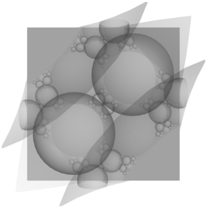

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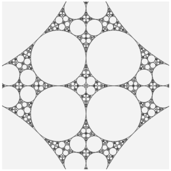
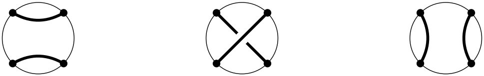


2-tangles

A 2-tangle



Elementary 2-tangle

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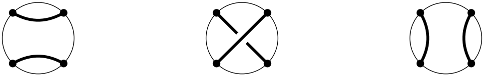

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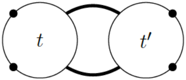

Sum of tangles t and t'

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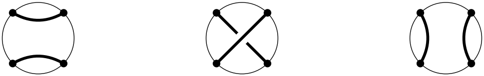

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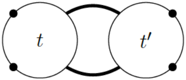

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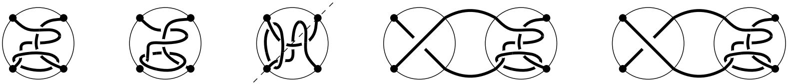

t $-t$ $F(t)$ $H^+(t)$ $H^-(t)$

Operations with tangles

Rational tangles

Let $a_1, \ldots, a_n$ be integers $a_i \neq 0$. Let $t(a_1, \ldots, a_n)$ the rational tangle given by Conway's algorithm :

$$t(a_1, \ldots, a_n) = H^{a_1} F \cdots H^{a_n} F(t_\infty)$$

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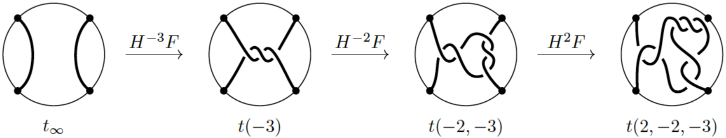
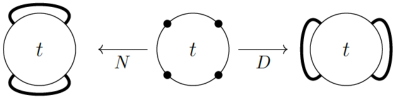
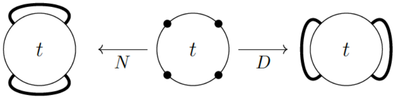


$t(2, -2, -3)$



Tangle closures : Denominator and Numerator

Rational links



Tangle closures : Denominator and Numerator

The slope of a rational tangle $t(a_1, \dots, a_n)$ is the rational number p/q obtained by the continued fraction expansion

$$[a_1, \dots, a_n] := a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}} = \frac{p}{q}.$$

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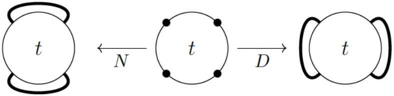

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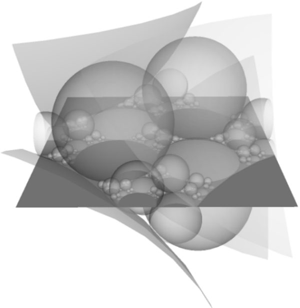
Theorem (Conway 1970) Two rational tangles are equivalent if and only if they have the same slope.



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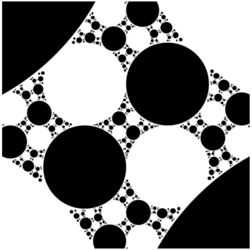
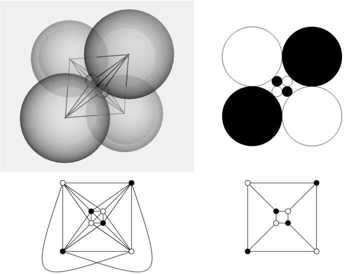
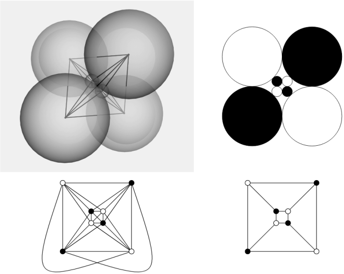


Orthoplicial packing $B_{\mathcal{O}^4}$ Cubic section $B_{\mathcal{C}^3}$



Associated graph to B_{C^3}



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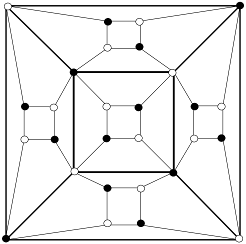
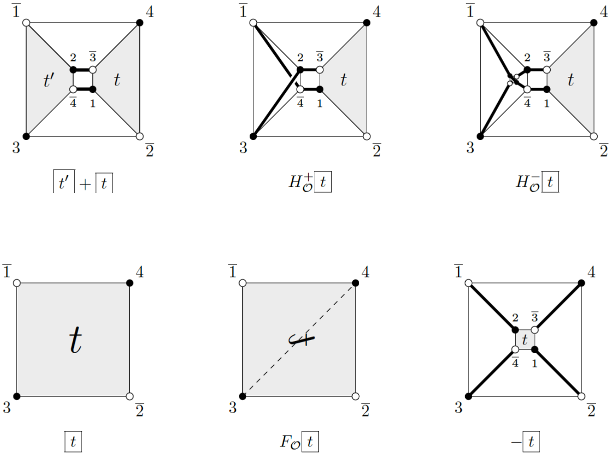


Tangles : cubic diagrams



Imitating tangle operations in the graph of B_{C^3}

Tangles : cubic diagram

Theorem (Rasskin + R.A., 2023) Any rational link admits an orthocubic representation (cubic diagram) and therefore there is a necklace representation contained in a section of B_{O^4}.

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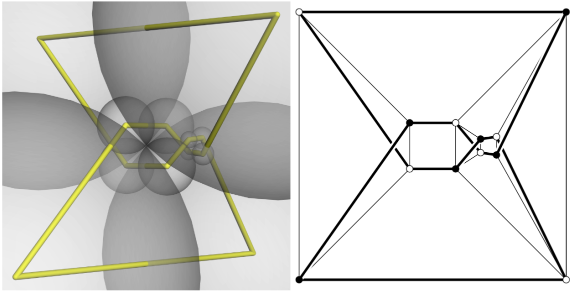

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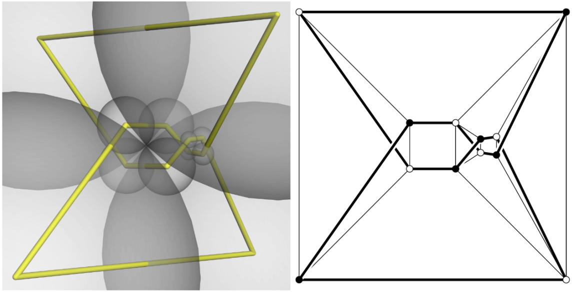

Theorem (Rasskin + R.A., 2023) Let L be an algebraic link obtained by the closure of the algebraic tangle $t_{p_1/q_1} + \cdots + t_{p_m/q_m}$ where all the p_i/q_i have same sign. Then, $ball(L) \leq 4cr(L)$.

No tightness for non-alternating links

Pretzel links $P(q_1, \ldots, q_n)$ are the tangles $t_{1/q_1} + \cdots + t_{1/q_m}$.

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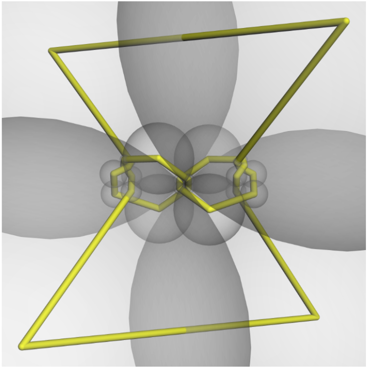

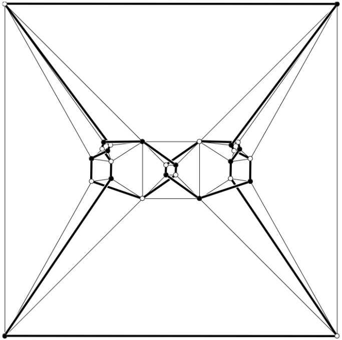

Orthocubic point

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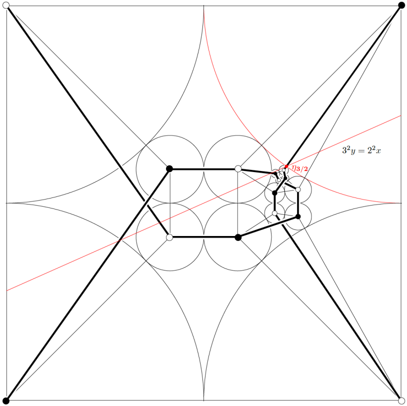

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Proof (idea) : Calculate the inversive coordinates of the orthocubic point of every rational tangle

$$\mathrm{i}(\eta_{p/q}) = \begin{pmatrix} p^2 \\ q^2 \\ (p-q)^2 \\ \sqrt{2}(p^2 - pq + q^2) \end{pmatrix}$$

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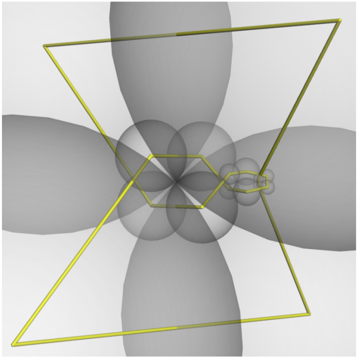
By computing

$$\langle i(\eta_{p/q}), i(\eta_{p/q}) \rangle = 0 \Leftrightarrow \underbrace{p}_{a}{}^4 + \underbrace{q}_{b}{}^4 + \underbrace{(p-q)}_{c}{}^4 = 2\underbrace{(p^2 - pq + q^2)}_{d}{}^2$$

we produce the solution $a^4 + b^4 + c^4 = 2d^2$

Figure eight knot

$\mathbf{5}^4 + \mathbf{2}^4 + \mathbf{3}^4 = 2 \times \mathbf{19}^2$

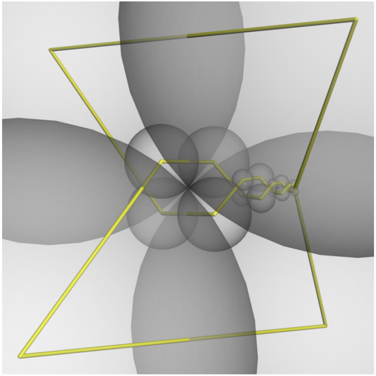


16 spheres$= 4cr(4_1)$

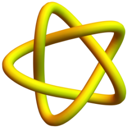
Knot $4_1 \leftrightarrow \frac{\mathbf{5}}{\mathbf{2}}$

Torus knot

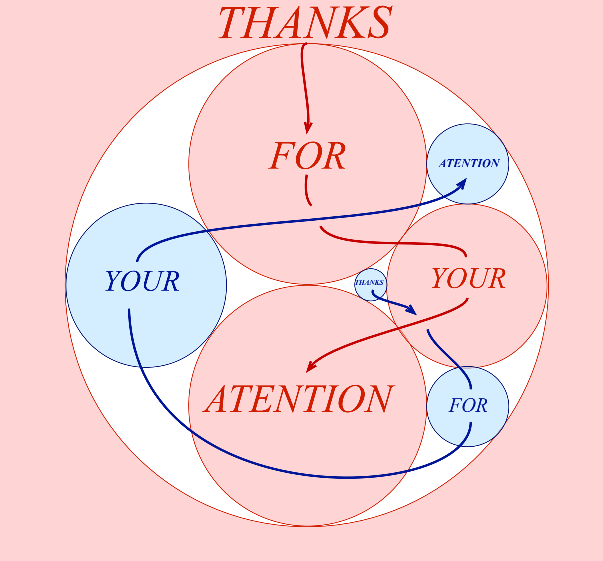
$\mathbf{5}^4 + \mathbf{1}^4 + \mathbf{4}^4 = 2 \times \mathbf{21}^2$



20 spheres$= 4cr(5_1)$



Knot $5_1 \leftrightarrow \frac{\mathbf{5}}{\mathbf{1}}$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