Frobenius problem: Algorithms and Complexity

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Diophantine Frobenius Problem

Let $a_1, \ldots, a_n$ be positive integers with $\gcd(a_1, \ldots, a_n) = 1$, find the largest integer (called the Frobenius number and denoted by $g(a_1, \ldots, a_n)$) that is not representable as a nonnegative integer combination of $a_1, \ldots, a_n$.

Example : If $a_1 = 3$ and $a_2 = 8$ then

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So, $g(3, 8) = 13$.

We denote by $\langle a_1, \ldots, a_n \rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$.

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We denote by $\langle a_1, \ldots, a_n \rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$.

Theorem $g(a_1,\ldots,a_n)$ exists and is finite.

Proof (sketch). Since $\gcd(a_1,\ldots,a_n)=1$ then

$$m_1a_1+\cdots+m_na_n=1 \text{ for some } m_i\in\mathbb{Z}$$

Let P and $-Q$ be the sum of positive and negative terms respectively (and so $P-Q=1$).

Let $k\geq 0$ then $(a_1-1)Q+k=(a_1-1)Q+ha_1+k'$ with $h\geq 0$ and $0<k'<a_1$

So, $(a_1-1)Q+k=ha_1+(a_1-1-k')Q+k'P$.

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Ideas

- Graph theory
- Discrete Optimisation problems (Knapsack problem)
- Additive number theory
- Index of primitivity of matrix
- Geometry of numbers (covering radius)
- Quantifier elimination
- Ehrhar polynomial
- Hilbert series
- Möbius function

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Theorem (Sylvester, 1882) $g(a, b) = ab - a - b$.

Theorem (R.A., 1996) Computing $g(a_1, \ldots, a_n)$ is $\mathcal{NP}$-hard (under Turing reductions).

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[IKP] Input : positive integers $a_1, \ldots, a_n$ and t,

Question : do there exist integers $x_i \geq 0$,

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Procedure

Find $g(a_1,\ldots,a_n)$

IF $t > g(a_1,\ldots,a_n)$ THEN **IKP** is answered affirmatively

ELSE

IF $t = g(a_1,\ldots a_n)$ THEN

IKP is answered negatively

ELSE

Find $g(\bar{a}_1,\ldots,\bar{a}_n,\bar{a}_{n+1})$, $\bar{a}_i = 2a_i$, $i = 1,\ldots,n$ and
$\bar{a}_{n+1} = 2g(a_1,\ldots,a_n)+1$ (note that $gcd(\bar{a}_1,\ldots,\bar{a}_n,\bar{a}_{n+1}) = 1$)
Find $g(\bar{a}_1,\ldots,\bar{a}_n,\bar{a}_{n+1},\bar{a}_{n+2})$, $\bar{a}_{n+2} = g(\bar{a}_1,\ldots,\bar{a}_n,\bar{a}_{n+1}) - 2t$
IKP is answered affirmatively if and only if
$g(\bar{a}_1,\ldots,\bar{a}_{n+2}) < g(\bar{a}_1,\ldots,\bar{a}_{n+1})$.

Methods

When $n = 3$

- Selmer and Bayer, 1978
- Rödseth, 1978
- Davison, 1994
- Scarf and Shallcross, 1993

When $n \geq 4$

- Heap and Lynn, 1964
- Wilf, 1978
- Nijenhuis, 1979
- Greenberg, 1980
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Kannan's result

Theorem (Kannan, 1992) There is a polynomial time algorithm to compute $g(a_1, \ldots, a_n)$ when $n \geq 2$ is fixed.

Let P be a closed bounded convex set in $\mathbb{R}^n$ and let L be a lattice of dimension n also in $\mathbb{R}^n$.

The least positive real t so that $tP + L$ equals $\mathbb{R}^n$ is called the covering radius of P with respect to L (denoted by $\mu(P, L)$).

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Theorem (Kannan, 1992) Let

$$L = \{(x_1,\ldots,x_{n-1}) | x_i \text{ integers and } \sum_{i=1}^{n-1} a_i x_i \equiv 0 \bmod a_n\}$$

and

$$S = \{(x_1,\ldots,x_{n-1}) | x_i \geq 0 \text{ reals and } \sum_{i=1}^{n-1} a_i x_i \leq 1\}.$$

Then, $\mu(S, L) = g(a_1,\ldots,a_n) + a_1 + \cdots + a_n$

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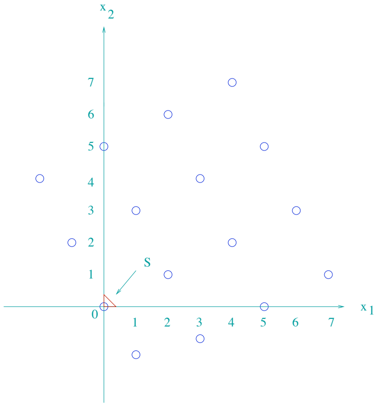

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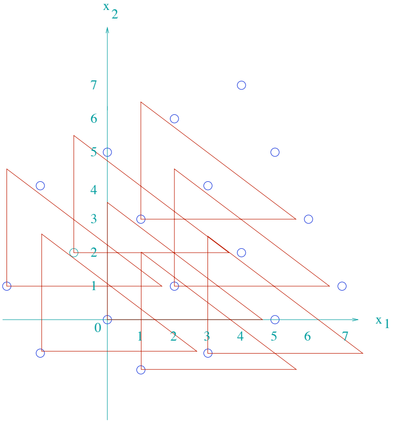

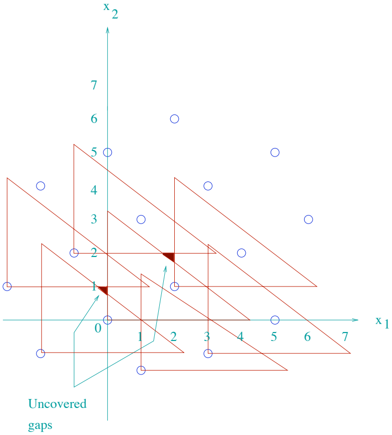

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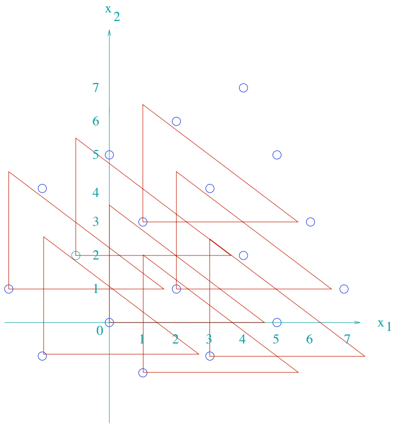

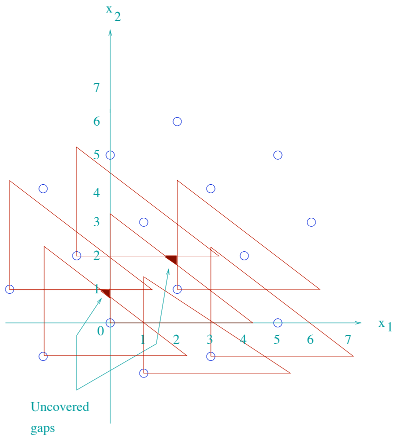

Notice that $(14)S$ covers the plane while $(13)S$ does not.

Example 2 : Let a, b be positive integers with $\gcd(a, b) = 1$.
Minimum integer t such that tS covers the interval $[0, b]$ is ab.
Thus, $g(a, b) = \mu(S, L) - a - b = ab - a - b$.

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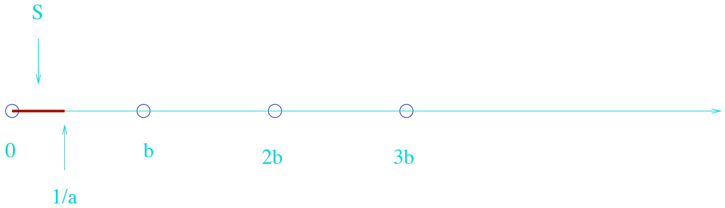

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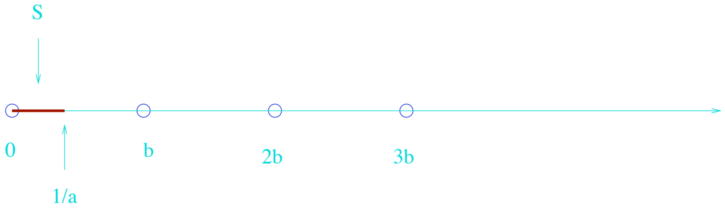

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Heap and Lynn's method

Let $B = (b_{i,j})$, $1 \leq, i, j \leq n$ be a positive matrix, that is, $b_{i,j} \geq 0$.
B reducible if there exists an $(n \times n)$ permutation matrix P such that

$$PBP^T = \begin{bmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{bmatrix}$$

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Let B be a real $(m \times m)$-matrix. Let $G(B)$ be the directed graph with $V(G) = \{1, \ldots, m\}$ and directed edge form i to j if and only if $b_{i,j} \neq 0$.

Theorem B irreducible if and only if $G(B)$ is strongly connected.

An irreducible nonnegative matrix B is primitive if $B^t > 0$ for some integer $t \geq 1$. The least integer $\gamma(B)$ such that $B^{\gamma(B)} > 0$ is called the index of primitivity of B.

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Lemma 1 (Heap and Lynn, 1964) Let B be a primitive matrix and let $0 < a_1 < \cdots < a_k$ be the distinct lengths of all *elementary* circuits of $G(B)$. Then, $\gcd(a_1, \ldots, a_n) = 1$ and the length L, of any circuit of $G(B)$ can be expressed in the form $L = \sum_{i=1}^{n} x_i a_i$ with $x_i \geq 0$ for all i.

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Lemma 3 (Heap and Lynn, 1964) Let B be a primitive matrix and let $0 < a_1 < \cdots < a_k$ be the distinct lengths of all elementary circuits of $G(B)$. Then, $g(a_1, \ldots, a_n) \leq \gamma(B) - 1$.

Let $B' = (b'_{i,j})$, $1 \leq, i, j \leq s = a_n + a_{n-1} - 1$ be the matrix defined by

$$b'_{i,j} = \begin{cases} 1 & \text{if } j = i+1 \text{ with } i = 1, \ldots, s-1 \text{ and } i \neq a_{n-1}, \\ 1 & \text{if } j = 1 \text{ with } i = s \text{ or } a_t,\ t = 1, \ldots, n-1, \\ 1 & \text{if } i = 1 \text{ and } j = a_{n-1}+1, \\ 0 & \text{otherwise.} \end{cases}$$

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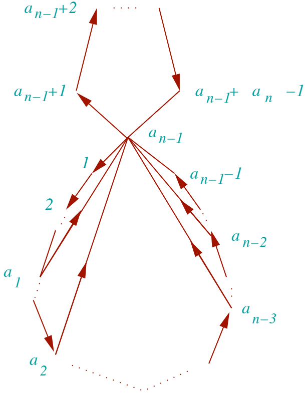

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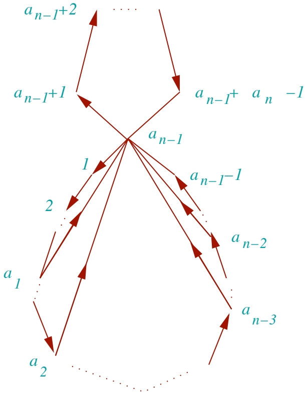

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Hilbert series and Apéry set

Let $A[S] = K[z^{a_1}, \ldots, z^{a_n}]$ be the semigroup ring over K (of characteristic 0) associated to the semigroup $S = \langle a_1, \ldots, a_n \rangle$. Then, the Hilbert series of $A[S]$ is

$$H(A[S], z) = \sum_{i \in S} z^s = \frac{Q(z)}{(1 - z^{a_1}) \cdots (1 - z^{a_n})}$$

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The Apéry set of $S = \langle a_1, \ldots, a_n \rangle$ for $m \in S$ is

$$Ap(S; m) = \{s \in S \mid s - m \notin S\}$$

$Ap(S; m)$ consititutes a complete set of residues $\pmod{m}$.

(there is a unique $w \in Ap(S; m)$ satisfying $w \equiv i \pmod{m}$)

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$Ap(S;m)$ consititutes a complete set of residues $\pmod{m}$.

(there is a unique $w \in Ap(S;m)$ satisfying $w \equiv i \pmod{m}$)

$$g(a_1, \ldots, a_n) = \max Ap(S;m) - m.$$

$$S = Ap(S;m) + m\mathbb{Z}_{\geq 0}$$

$$H(A[S],z) = \sum_{i \in S} z^s = \sum_{w \in Ap(S;m)} \sum_{i=0}^{\infty} z^{w+mi}$$
$$= \sum_{w \in Ap(S;m)} z^w \sum_{i=0}^{\infty} z^{mi}$$

So, $H(A[S],z) = \frac{1}{1-x^m} \sum_{w \in Ap(S;m)} z^w$.

Hence, $deg(H(A[S],z)) = \max Ap(S;m) - m = g(a_1,\ldots,a_n)$.

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Theorem (Herzog 1970, Morales 1987) Formula for $H(A[S], z)$ when $S = \langle a, b, c \rangle$.

Theorem (R.A. and Rødseth, 2009) $S = \langle a, a+d, \ldots, a+kd, c \rangle$

$$H(S; x) = \frac{F_{s_v}(a;x)(1-x^{c(P_{v+1}-P_v)}) + F_{s_v - s_{v+1}}(a;x)(x^{c(P_{v+1}-P_v)} - x^{cP_{v+1}})}{(1-x^a)(1-x^d)(1-x^{a+kd})(1-x^c)}$$

where $s_v, s_{v+1}, P_v, P_{v+1}$ are some *particular* integers.

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Algorithm Apéry

Input : a, d, c, k, s_0 Output : $s_v, s_{v+1}, P_v, P_{v+1}$

$r_{-1} = a, r_0 = s_0$

$r_{i-1} = \kappa_{i+1} r_i + r_{i+1}, \kappa_{i+1} = \lfloor r_{i-1}/r_i \rfloor, 0 = r_{\mu+1} < r_\mu < \cdots < r_{-1}$

$p_{i+1} = \kappa_{i+1} p_i + p_{i-1}, \quad p_{-1} = 0, \quad p_0 = 1$

$T_{i+1} = -\kappa_{i+1} T_i + T_{i-1}, \quad T_{-1} = a + kd, T_0 = \frac{1}{a}((a+kd)r_0 - kc)$

IF there is a minimal u such that $T_{2u+2} \leq 0$, THEN

$$\begin{pmatrix} s_v & P_v \\ s_{v+1} & P_{v+1} \end{pmatrix} = \begin{pmatrix} \gamma & 1 \\ \gamma - 1 & 1 \end{pmatrix} \begin{pmatrix} r_{2u+1} & -p_{2u+1} \\ r_{2u+2} & p_{2u+2} \end{pmatrix}, \gamma = \left\lfloor \frac{-T_{2u+2}}{T_{2u+1}} \right\rfloor + 1$$

ELSE $s_v = r_\mu, s_{v+1} = 0, P_v = p_\mu, P_{v+1} = p_{\mu+1}$.

Scarf and Shallcross' method

A body represented by $\{x \in \mathbb{R}^{n-1} : Ax \leq b\}$ with $b \in \mathbb{Z}^n$ and A a real $(n \times n-1)$-matrix is a maximal lattice free body if it contains no lattice points in its interior and if any strictly larger body by relaxing some of the inequalities does contain an interior point.

Let A be a $(n \times n-1)$-matrix whose columns generate the $(n-1)$-dimensional lattice h satisfying $a \cdot h = 0$ (that is, the columns forming a basis of $\{v \in \mathbb{Z}^n : a \cdot v = 0\}$).

$$g(a_1, \ldots, a_n) = \max\{a \cdot b | b \text{ is integral and } \{Ax \leq b\} \text{ maximal lattice free body}\}.$$

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Example 3 Let $a_1 = 3,\ a_2 = 5$ (and thus $n = 2$).

So, the set of vectors $h = (h_1, h_2)$ such that $(3, 5) \cdot (h_1, h_2) = 0$ is given by $(\pm 5r, \mp 3r)$, $r \geq 0$.

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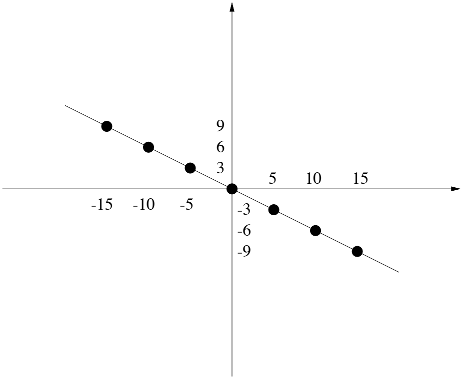
The 1-dimensional lattice generated by h



The one column matrix $A = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ generates the integer lattice of h.

So, we want $b = (b_1, b_2)$ such that

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Wilf's method

- Form a circle of a_n lights, labeled by $l_0, l_1, \ldots, l_{a_n-1}$ (initially light l_0 is on and the others off).
- Sweep around the circle starting from l_0 (clockwise) and as we encounter each light we will turn it on if any of the n lights which are situated at distance $a_1, \ldots, a_n$ back (*i.e.*, in counterclockwise sense) from the present one is on, we leave it on if it was already on, otherwise we leave it off.
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• Let $s(l_{a_i})$ be the number of times light l_{a_i} is visited during the procedure and let l_r be the last visited off light just before ending the process. Then, $g(a_1, \ldots, a_n) = r + (s(l_r) - 1)a_n$.

Exemple 4 : Let $a_1 = 5, a_2 = 6$ and $a_3 = 7$.

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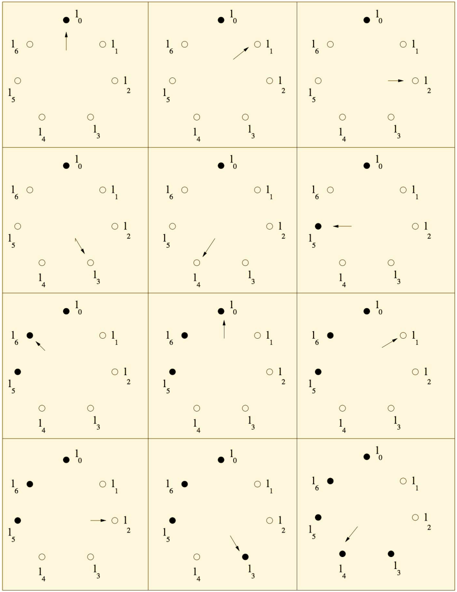

So, l_2 is the last visited off light and thus

$g(5,6,7) = 2 + (s(l_2) - 1)7 = 2 + (2-1)7 = 9.$

Fast Algorithms (computational algebraic methods)

Einstein, Lichtblau, Strzebonski and Wagon, 2007

Find $g(a_1, \ldots, a_4)$ involving 100-digit numbers in about one second

Find $g(a_1, \ldots, a_{10})$ involving 10-digit numbers in two days

Roune, 2008

Find $g(a_1, \ldots, a_4)$ involving 10, 000-digit numbers in few second

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Package

http ://www.broune.com/frobby/

http ://www.math.ruu.nl/people/beukers/frobenius/

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