

# MATROID TORIC IDEALS: COMPLETE INTERSECTION, MINORS AND MINIMAL SYSTEMS OF GENERATORS

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ABSTRACT. In this paper, we investigate three problems concerning the toric ideal associated to a matroid. Firstly, we list all matroids  $\mathcal{M}$  such that its corresponding toric ideal  $I_{\mathcal{M}}$  is a complete intersection. Secondly, we handle the problem of detecting minors of a matroid  $\mathcal{M}$  from a minimal set of binomial generators of  $I_{\mathcal{M}}$ . In particular, given a minimal set of binomial generators of  $I_{\mathcal{M}}$  we provide a necessary condition for  $\mathcal{M}$  to have a minor isomorphic to  $\mathcal{U}_{d,2d}$  for  $d \geq 2$ . This condition is proved to be sufficient for  $d = 2$  (leading to a criterion for determining whether  $\mathcal{M}$  is binary) and for  $d = 3$ . Finally, we characterize all matroids  $\mathcal{M}$  such that  $I_{\mathcal{M}}$  has a unique minimal set of binomial generators.

## 1. INTRODUCTION

Let  $\mathcal{M}$  be a matroid on a finite ground set  $E = \{1, \dots, n\}$ , we denote by  $\mathcal{B}$  the set of bases of  $\mathcal{M}$ . Let  $k$  be an arbitrary field and consider  $k[x_1, \dots, x_n]$  a polynomial ring over  $k$ . For each base  $B \in \mathcal{B}$ , we introduce a variable  $y_B$  and we denote by  $R$  the polynomial ring in the variables  $y_B$ , i.e.,  $R := k[y_B \mid B \in \mathcal{B}]$ . A *binomial* in  $R$  is a difference of two monomials, an ideal generated by binomials is called a *binomial ideal*.

We consider the homomorphism of  $k$ -algebras  $\varphi : R \longrightarrow k[x_1, \dots, x_n]$  induced by  $y_B \mapsto \prod_{i \in B} x_i$ . The image of  $\varphi$  is a standard graded  $k$ -algebra, which is called the *bases monomial ring of the matroid  $\mathcal{M}$*  and it is denoted by  $S_{\mathcal{M}}$ . By [23, Theorem 5],  $S_{\mathcal{M}}$  has Krull dimension  $\dim(S_{\mathcal{M}}) = n - c + 1$ , where  $c$  is the number of connected components of  $\mathcal{M}$ . The number  $c$  of connected components is the largest integer  $k$  such that  $E$  is the disjoint union of the nonempty sets  $E_1, \dots, E_k$  and  $\mathcal{M}$  is the direct sum of some matroids  $\mathcal{M}_1, \dots, \mathcal{M}_k$ , where  $\mathcal{M}_i$  has ground set  $E_i$ . The kernel of  $\varphi$ , which is the presentation ideal of  $S_{\mathcal{M}}$ , is called the *toric ideal of  $\mathcal{M}$*  and is denoted by  $I_{\mathcal{M}}$ . It is well known that  $I_{\mathcal{M}}$  is a prime, binomial and homogeneous ideal, see, e.g., [20]. Since  $R/I_{\mathcal{M}} \simeq S_{\mathcal{M}}$ , it follows that the height of  $I_{\mathcal{M}}$  is  $\text{ht}(I_{\mathcal{M}}) = |\mathcal{B}| - \dim(S_{\mathcal{M}})$ .

In [24], White posed several conjectures concerning basis exchange properties on matroids. One of these combinatorial conjectures turned out to be equivalent to decide if  $I_{\mathcal{M}}$  is always generated by quadratics. This algebraic version of the conjecture motivated several authors to study  $I_{\mathcal{M}}$ . Despite this conjecture is still open, it has been proved to be true by means of this algebraic approach for several families of matroids (see [13] and the references there). Even more, it is not even known if for every matroid its corresponding toric ideal admits a quadratic Gröbner basis.

In this paper we study the algebraic structure of toric ideals of matroids. We study three different problems concerning  $I_{\mathcal{M}}$ .

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**1.1. Complete intersection.** The first problem is to characterize the matroids  $\mathcal{M}$  such that  $I_{\mathcal{M}}$  is a complete intersection. The toric ideal  $I_{\mathcal{M}}$  is a *complete intersection* if  $\mu(I_{\mathcal{M}}) = \text{ht}(I_{\mathcal{M}})$ , where  $\mu(I_{\mathcal{M}})$  denotes the minimal number of generators of  $I_{\mathcal{M}}$ . Equivalently,  $I_{\mathcal{M}}$  is a complete intersection if and only if there exists a set of homogeneous binomials  $g_1, \dots, g_s \in R$  such that  $s = \text{ht}(I_{\mathcal{M}})$  and  $I_{\mathcal{M}} = (g_1, \dots, g_s)$ .

Complete intersection toric ideals were first studied by Herzog in [11]. Since then, they have been extensively studied by several authors. In the context of toric ideals associated to combinatorial structures, the complete intersection property has been widely studied for graphs, see, e.g., [2, 22, 10]. In this work we address this problem in the context of toric ideals of matroids and prove that there are essentially three matroids whose corresponding toric ideal is a complete intersection; namely, the rank 2 matroids without loops or coloops on a ground set of 4 elements.

**1.2. Minors.** Many of the most celebrated results on matroids make reference to minors, for this reason it is convenient to have tools to detect whether a matroid has a certain minor or not. In this work we study the problem of detecting whether a matroid  $\mathcal{M}$  has a minor isomorphic to  $\mathcal{U}_{d,2d}$  with  $d \geq 2$ , where  $\mathcal{U}_{r,n}$  denotes the uniform matroid of rank  $r$  on  $E = \{1, \dots, n\}$ . More precisely, we prove that whenever a matroid contains a minor isomorphic to  $\mathcal{U}_{d,2d}$ , then there exist  $B_1, B_2 \in \mathcal{B}$  such that  $\Delta_{\{B_1, B_2\}} = \binom{2d-1}{d}$ ; where, for every  $B_1, B_2 \in \mathcal{B}$ ,  $\Delta_{\{B_1, B_2\}}$  denotes the number of pairs of bases  $\{D_1, D_2\}$  such that  $B_1 \cup B_2 = D_1 \cup D_2$  as multisets. This condition is also proved to be sufficient for  $d = 2$  and  $d = 3$ . Since  $\mathcal{U}_{2,4}$  is the only excluded minor for a matroid to be binary, the result for  $d = 2$  provides a new criterion for detecting whether a matroid is binary. Moreover, we provide an example to show that for  $d = 5$  this condition is no longer sufficient. These results are presented in purely combinatorial terms, nevertheless whenever one knows a minimal set of binomial generators of  $I_{\mathcal{M}}$ , one can easily compute  $\Delta_{\{B_1, B_2\}}$  for all  $B_1, B_2 \in \mathcal{B}$ . Thus, these results give a method to detect if a matroid has a minor isomorphic to  $\mathcal{U}_{2,4}$  or  $\mathcal{U}_{3,6}$  provided one knows a minimal set of binomial generators of  $I_{\mathcal{M}}$ .

**1.3. Minimal systems of generators.** Minimal systems of binomial generators of toric ideals have been studied in several papers; see, e.g., [4, 8]. In general, for a toric ideal it is possible to have more than one minimal system of generators formed by binomials. Given a toric ideal  $I$ , we denote by  $\nu(I)$  the number of minimal sets of binomial generators of  $I$ , where the sign of a binomial does not count. A recent problem arising from algebraic statistics (see [21]) is to characterize when a toric ideal  $I$  possesses a unique minimal system of binomial generators; i.e., when  $\nu(I) = 1$ . The problems of determining  $\nu(I)$  and characterizing when  $\nu(I) = 1$  for a toric ideal  $I$  were studied in [6, 15], also in [9, 12] in the context of toric ideals associated to affine monomial curves and in [16, 19] for toric ideals of graphs. In this paper we also handle these problems in the context of toric ideals of matroids. More precisely, we characterize all matroids  $\mathcal{M}$  such that  $\nu(I_{\mathcal{M}}) = 1$ . This result follows as a consequence of a lower bound we obtain for  $\nu(I_{\mathcal{M}})$ . This bound turns to be an equality whenever  $I_{\mathcal{M}}$  is generated by quadratics.

The paper is organized as follows. In the next section, we recall how the operations of deletion and contraction on a matroid  $\mathcal{M}$  reflect into  $I_{\mathcal{M}}$ . We prove that the complete intersection property is preserved under taking minors (Proposition 2.1). We then give a complete list of all matroids whose corresponding toric ideal is a complete intersection (Theorem 2.3). To this end, we first give such a list for matroids of rank 2 (Proposition 2.2), which is based on results given in [2]. In Section 3, we provide a necessary condition for a matroid to contain a minor isomorphic to  $\mathcal{U}_{d,2d}$  for  $d \geq 2$  in terms of the values  $\Delta_{\{B_1, B_2\}}$  for  $B_1, B_2 \in \mathcal{B}$  (Proposition 3.3). We also prove that this condition is also sufficient when  $d = 2$  or  $d = 3$  (Theorems 3.4 and 3.5). Moreover, we show that this condition is no longer sufficient for  $d = 5$ . In the last section we focus on giving formulas for the values  $\mu(I_{\mathcal{M}})$  and  $\nu(I_{\mathcal{M}})$ . In particular, we give a lower bound for these in terms of the values  $\Delta_{\{B_1, B_2\}}$  for  $B_1, B_2 \in \mathcal{B}$  (Theorem 4.1). Moreover, this lower bound turns to be exact provided

$I_{\mathcal{M}}$  is generated by quadratics. Finally, we characterize all those matroids whose toric ideal has a unique minimal binomial generating set (Theorem 4.2).

## 2. COMPLETE INTERSECTION TORIC IDEALS OF MATROIDS

We begin this section by setting up some notation and recalling some results about matroids which are useful in the sequel. For a general background on matroids we refer the reader to [18].

Let  $\mathcal{M}$  be a matroid on the ground set  $E = \{1, \dots, n\}$  and rank  $r$ . Let  $\mathcal{B}$  denote the set of bases of  $\mathcal{M}$ . By definition  $\mathcal{B}$  is not empty and satisfies the following *exchange axiom*:

For every  $B_1, B_2 \in \mathcal{B}$  and for every  $e \in B_1 \setminus B_2$ , there exists  $f \in B_2 \setminus B_1$  such that  $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ .

Brualdi proved in [5] that the exchange axiom is equivalent to the *symmetric exchange axiom*:

For every  $B_1, B_2$  in  $\mathcal{B}$  and for every  $e \in B_1 \setminus B_2$ , there exists  $f \in B_2 \setminus B_1$  such that both  $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$  and  $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$ .

Now we recall some basic facts and results over toric ideals of matroids needed later on. Firstly, we observe that for  $B_1, \dots, B_s, D_1, \dots, D_s \in \mathcal{B}$ , the homogeneous binomial  $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$  belongs to  $I_{\mathcal{M}}$  if and only if  $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$  as multisets. Since  $I_{\mathcal{M}}$  is a homogeneous binomial ideal, it follows that

$$I_{\mathcal{M}} = \left( \{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \text{ as multisets}\} \right).$$

From this expression one easily derives that whenever  $r \in \{0, 1, n-1, n\}$ , then  $I_{\mathcal{M}} = (0)$  and  $I_{\mathcal{M}}$  is a complete intersection. Thus, we only consider the case  $2 \leq r \leq n-2$ .

Now we prove that the operations of taking duals, deletion, contraction and taking minors of  $\mathcal{M}$  preserve the property of being a complete intersection on  $I_{\mathcal{M}}$ . For more details on how these operations affect  $I_{\mathcal{M}}$  we refer the reader to [3, Section 2].

We denote by  $\mathcal{M}^*$  the dual matroid of  $\mathcal{M}$ . It is straightforward to check that  $\sigma(I_{\mathcal{M}}) = I_{\mathcal{M}^*}$ , where  $\sigma$  is the isomorphism of  $k$ -algebras  $\sigma : R \longrightarrow k[y_{E \setminus B} \mid B \in \mathcal{B}]$  induced by  $y_B \mapsto y_{E \setminus B}$ . Thus,  $I_{\mathcal{M}}$  is a complete intersection if and only if  $I_{\mathcal{M}^*}$  also is.

For every  $A \subset E$ ,  $\mathcal{M} \setminus A$  denotes the *deletion of  $A$  from  $\mathcal{M}$*  and  $\mathcal{M}/A$  denotes the *contraction of  $A$  from  $\mathcal{M}$* . For  $E' \subset E$ , the restriction of  $\mathcal{M}$  to  $E'$  is denoted by  $\mathcal{M}|_{E'}$ .

**Proposition 2.1.** *Let  $\mathcal{M}'$  be a minor of  $\mathcal{M}$ . If  $I_{\mathcal{M}}$  is a complete intersection, then  $I_{\mathcal{M}'}$  also is.*

*Proof.* Take  $e \in E$  and let us prove that  $I_{\mathcal{M} \setminus \{e\}}$  is a complete intersection. If  $e$  is a loop, then  $\mathcal{B}$  is the set of bases of both  $\mathcal{M}$  and  $\mathcal{M} \setminus \{e\}$  and, hence,  $I_{\mathcal{M}} = I_{\mathcal{M} \setminus \{e\}}$ . Assume that  $e$  is not a loop and take  $\mathcal{G}$  a binomial generating set of  $I_{\mathcal{M}}$ . By [2, Lemma 2.2] or [17],  $I_{\mathcal{M} \setminus \{e\}}$  is generated by the set  $\mathcal{G}' := \mathcal{G} \cap k[y_B \mid e \notin B \in \mathcal{B}]$ . Hence,  $I_{\mathcal{M} \setminus \{e\}}$  is a complete intersection (see [2, Proposition 2.3]). An iterative application of this result proves that for all  $A \subset E$ ,  $I_{\mathcal{M} \setminus A}$  is a complete intersection.

For every  $A \subset E$ , it suffices to observe that  $\mathcal{M}/A = (\mathcal{M}^* \setminus A)^*$  to deduce that  $I_{\mathcal{M}/A}$  is also a complete intersection whenever  $I_{\mathcal{M}}$  is. Thus, the result follows.  $\square$

As we mentioned in the proof of Proposition 2.1, if  $e$  is a loop then  $I_{\mathcal{M}} = I_{\mathcal{M} \setminus \{e\}}$ . Moreover, if  $e$  is a coloop of  $\mathcal{M}$ , then  $I_{\mathcal{M}}$  is essentially equal to  $I_{\mathcal{M}/\{e\}}$ . Indeed, if one considers the isomorphism of  $k$ -algebras  $\tau : R \longrightarrow k[y_{B \setminus \{e\}} \mid B \in \mathcal{B}]$  induced by  $y_B \mapsto y_{B \setminus \{e\}}$ , then  $\tau(I_{\mathcal{M}}) = I_{\mathcal{M}/\{e\}}$ . For this reason we may assume without loss of generality that  $\mathcal{M}$  has no loops or coloops.

Now we study the complete intersection property for  $I_{\mathcal{M}}$  when  $\mathcal{M}$  has rank 2. In this case, we associate to  $\mathcal{M}$  the graph  $\mathcal{H}_{\mathcal{M}}$  with vertex set  $E$  and edge set  $\mathcal{B}$ . It turns out that  $I_{\mathcal{M}}$  coincides with the toric ideal of the graph  $\mathcal{H}_{\mathcal{M}}$  (see, e.g., [2]). In particular, from [2, Corollary 3.9], we have that whenever  $I_{\mathcal{M}}$  is a complete intersection, then  $\mathcal{H}_{\mathcal{M}}$  does not contain  $\mathcal{K}_{2,3}$  as subgraph, where

$\mathcal{K}_{2,3}$  denotes the complete bipartite graph with partitions of sizes 2 and 3. The following result characterizes the complete intersection property for toric ideals of rank 2 matroids.

**Proposition 2.2.** *Let  $\mathcal{M}$  be a rank 2 matroid on a ground set of  $n \geq 4$  elements without loops or coloops. Then,  $I_{\mathcal{M}}$  is a complete intersection if and only if  $n = 4$ .*

*Proof.* ( $\Rightarrow$ ) Assume that  $n \geq 5$  and let us prove that  $I_{\mathcal{M}}$  is not a complete intersection. Since  $\mathcal{M}$  has rank 2 and has no loops or coloops, we may assume that it has two disjoint basis, namely  $B_1 = \{1, 2\}, B_2 = \{3, 4\} \in \mathcal{B}$ . Moreover, 5 is not a coloop, so we may also assume that  $B_3 = \{1, 5\} \in \mathcal{B}$ . Since  $B_1, B_2 \in \mathcal{B}$ , by the symmetric exchange axiom, we can also assume that  $B_4 = \{1, 3\}, B_5 = \{2, 4\} \in \mathcal{B}$ . If  $\{4, 5\} \in \mathcal{B}$ , then  $\mathcal{H}_{\mathcal{M}}$  has a subgraph  $\mathcal{K}_{2,3}$  and  $I_{\mathcal{M}}$  is not a complete intersection. Let us suppose that  $\{4, 5\} \notin \mathcal{B}$ . By the exchange axiom for  $B_2$  and  $B_3$  we have  $B_6 := \{3, 5\} \in \mathcal{B}$ . Again by the exchange axiom for  $B_5$  and  $B_6$  we get that  $B_7 := \{2, 5\} \in \mathcal{B}$ . Thus,  $\mathcal{H}_{\mathcal{M}}$  has  $\mathcal{K}_{2,3}$  as a subgraph and  $I_{\mathcal{M}}$  is not a complete intersection.

( $\Leftarrow$ ) There are three non isomorphic rank 2 matroids without loops or coloops and  $n = 4$ . Namely,  $\mathcal{M}_1$  with set of bases  $\mathcal{B}_1 = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$ ,  $\mathcal{M}_2$  with set of bases  $\mathcal{B}_2 = \mathcal{B}_1 \cup \{\{1, 4\}\}$  and  $\mathcal{M}_3 = \mathcal{U}_{2,4}$ . For  $i = 1, 2$  one can easily check that  $\text{ht}(I_{\mathcal{M}_i}) = 1$  and that  $I_{\mathcal{M}_i} = (y_{\{1,2\}}y_{\{3,4\}} - y_{\{1,3\}}y_{\{2,4\}})$ ; thus both  $I_{\mathcal{M}_1}$  and  $I_{\mathcal{M}_2}$  are complete intersections. Moreover,  $\text{ht}(I_{\mathcal{M}_3}) = 2$  and a direct computation with SINGULAR [7] or COCOA [1] yields that  $I_{\mathcal{M}_3} = (y_{\{1,2\}}y_{\{3,4\}} - y_{\{1,3\}}y_{\{2,4\}}, y_{\{1,4\}}y_{\{2,3\}} - y_{\{1,3\}}y_{\{2,4\}})$ ; thus  $I_{\mathcal{M}_3}$  is also a complete intersection.  $\square$

Now, we apply Proposition 2.2 to give the list of all matroids  $\mathcal{M}$  such that  $I_{\mathcal{M}}$  is a complete intersection.

**Theorem 2.3.** *Let  $\mathcal{M}$  be a matroid without loops or coloops and with  $2 \leq r \leq n - 1$ . Then,  $I_{\mathcal{M}}$  is a complete intersection if and only if  $n = 4$  and  $\mathcal{M}$  is the matroid whose set of bases is:*

- (1)  $\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$ ,
- (2)  $\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$ , or
- (3)  $\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}$ , i.e.,  $\mathcal{M} = \mathcal{U}_{2,4}$ .

*Proof.* By Proposition 2.2 it only remains to prove that  $I_{\mathcal{M}}$  is not a complete intersection provided  $r \geq 3$ . Since  $n > r + 1$  and  $\mathcal{M}$  has no loops or coloops, we can take  $B_1, B_2 \in \mathcal{B}$  such that  $|B_1 \setminus B_2| = 2$  and consider  $f \in B_1 \cap B_2$ . Since  $f$  is not a coloop, there exists  $B' \in \mathcal{B}$  such that  $f \notin B'$ . Moreover, since  $B_1, B' \in \mathcal{B}$ , by the exchange axiom there exists  $e \in B'$  such that  $B_3 := (B_1 \setminus \{f\}) \cup \{e\} \in \mathcal{B}$ . We observe that  $|B_2 \setminus B_3| \in \{2, 3\}$ . Setting  $A := B_1 \cap B_2 \cap B_3$ , we can assume without loss of generality that  $f = 1$  and that  $B_1 = A \cup \{1, 2, 3\}$ ,  $B_2 = A \cup \{1, 4, 5\}$  and  $B_3 = A \cup \{2, 3, e\}$ , where  $e \in \{5, 6\}$ . We have two cases.

*Case 1:*  $e = 5$ . We consider the matroid  $(\mathcal{M}')^*$ , the dual matroid of  $\mathcal{M}' := (\mathcal{M}/A)|_{E'}$ , with  $E' = \{1, 2, 3, 4, 5\}$ . We observe that  $\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 5\}$  are bases of  $\mathcal{M}'$  and hence  $\{4, 5\}, \{2, 3\}, \{1, 4\}$  are bases of  $(\mathcal{M}')^*$ . Thus  $(\mathcal{M}')^*$  is a rank 2 matroid without loops or coloops and, by Proposition 2.2,  $I_{(\mathcal{M}')^*}$  is not a complete intersection. Hence, by Proposition 2.1, we conclude that  $I_{\mathcal{M}}$  is not a complete intersection.

*Case 2:*  $e = 6$ . We consider the minor  $\mathcal{M}' := (\mathcal{M}/A)|_{E'}$ , where  $E' = \{1, 2, 3, 4, 5, 6\}$  and observe that  $\{1, 4, 5\}, \{1, 2, 3\}, \{2, 3, 6\}$  are bases of  $\mathcal{M}'$ . By the symmetric exchange axiom, we may also assume that  $\{1, 2, 4\}, \{1, 3, 5\}$  are also bases of  $\mathcal{M}'$ . We claim that for every base  $B$  of  $\mathcal{M}$ , either  $1 \in B$  or  $6 \in B$ , but not both. Indeed, if there exists a base  $B$  of  $\mathcal{M}'$  such that  $\{1, 6\} \subset B$  then the rank 2 matroid  $\mathcal{M}_1 := \mathcal{M}'/\{1\}$  on the set  $E' \setminus \{1\}$  has no loops or coloops. Thus, by Proposition 2.2,  $I_{\mathcal{M}_1}$  is not a complete intersection and, by Proposition 2.1, neither is  $I_{\mathcal{M}}$ . If there exists a base of  $\mathcal{M}'$  such that  $1 \notin B$  and  $6 \notin B$ , the rank 2 matroid  $\mathcal{M}_2 := (\mathcal{M}' \setminus \{6\})^*$  on the set  $E' \setminus \{6\}$  has no loops or coloops. Thus again by Proposition 2.2, we get that  $I_{\mathcal{M}_2}$  is not a complete intersection and, by Proposition 2.1, neither is  $I_{\mathcal{M}}$ . Analogously, one can prove that for every base  $B$  of  $\mathcal{M}'$  either  $2 \in B$  or  $5 \in B$  but not both, and that either  $3 \in B$  or  $4 \in B$  but not

both. Hence,  $\mathcal{M}'$  is the transversal matroid with presentation  $(\{1, 6\}, \{2, 5\}, \{3, 4\})$ . Since  $\mathcal{M}'$  has 8 bases and 3 connected components, then  $I_{\mathcal{M}'}$  has height 4. Moreover, a direct computation yields that  $I_{\mathcal{M}'}$  is minimally generated by 9 binomials; thus,  $I_{\mathcal{M}'}$  is not a complete intersection and the proof is finished.  $\square$

### 3. FINDING MINORS IN A MATROID

In this section we investigate a characterization for a matroid to contain certain minors in terms of a set of binomial generators of its corresponding toric ideal. In particular, we focus our attention to detect if a matroid  $\mathcal{M}$  contains a minor  $\mathcal{U}_{d,2d}$  for  $d \geq 2$ . We consider the following binary equivalence relation  $\sim$  on the set of pairs of bases:

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ as multisets,}$$

and we denote by  $\Delta_{\{B_1, B_2\}}$  the cardinality of the equivalence class of  $\{B_1, B_2\}$ .

For two sets  $A, B$  we denote by  $A \triangle B$  the *symmetric difference* of  $A$  and  $B$ , i.e.,  $A \triangle B := (A \setminus B) \cup (B \setminus A)$ .

We now introduce two lemmas concerning the values  $\Delta_{\{B_1, B_2\}}$ . The first one provides some bounds on the values of  $\Delta_{\{B_1, B_2\}}$ . In the proof of this lemma we use the so called *multiple symmetric exchange property* (see [25]):

For every  $B_1, B_2$  in  $\mathcal{B}$  and for every  $A_1 \subset B_1$ , there exists  $A_2 \subset B_2$  such that  $(B_1 \cup A_2) \setminus A_1 \in \mathcal{B}$  and  $(B_2 \cup A_1) \setminus A_2$  are in  $\mathcal{B}$ .

**Lemma 3.1.** *For every  $B_1, B_2 \in \mathcal{B}$ , then  $2^{d-1} \leq \Delta_{\{B_1, B_2\}} \leq \binom{2d-1}{d}$ , where  $d := |B_1 \setminus B_2|$ .*

*Proof.* Take  $e \in B_1 \setminus B_2$ . By the multiple symmetric exchange property, for every  $A_1$  such that  $e \in A_1 \subset (B_1 \setminus B_2)$ , there exists  $A_2 \subset B_2$  such that both  $B'_1 := (B_1 \cup A_2) \setminus A_1$  and  $B'_2 := (B_2 \cup A_1) \setminus A_2$  are bases. Since  $B_1 \cup B_2 = B'_1 \cup B'_2$  as multisets, we derive that  $\Delta_{\{B_1, B_2\}}$  is greater or equal to the number of sets  $A_1$  such that  $e \in A_1 \subset (B_1 \setminus B_2)$ , which is exactly  $2^{d-1}$ .

We set  $A := B_1 \cap B_2$ ,  $C := B_1 \triangle B_2$  and take  $e \in B_1 \setminus B_2$ . Take  $B_3, B_4 \in \mathcal{B}$  such that  $B_1 \cup B_2 = B_3 \cup B_4$  as multisets and assume that  $e \in B_4$ . Then,  $B_3 \setminus A \subset C \setminus \{e\}$  with  $|B_3 \setminus A| = |B_1 \setminus B_2| = d$  elements; thus,  $\Delta_{\{B_1, B_2\}} \leq \binom{2d-1}{d}$ .  $\square$

Moreover, the bounds of Lemma 3.1 are sharp for every  $d \geq 2$ . Indeed, if one considers the transversal matroid on the set  $\{1, \dots, 2d\}$  with presentation  $(\{1, d+1\}, \dots, \{d, 2d\})$ , and takes the bases  $B_1 = \{1, \dots, d\}$ ,  $B_2 = \{d+1, \dots, 2d\}$ , then  $|B_1 \setminus B_2| = d$  and  $\Delta_{\{B_1, B_2\}} = 2^{d-1}$ . Also, if we consider the uniform matroid  $\mathcal{U}_{d,2d}$  then for any base  $B$  we have that  $\Delta_{\{B, E \setminus B\}} = \binom{2d-1}{d}$ .

The second lemma interprets the values of  $\Delta_{\{B_1, B_2\}}$  in terms of the number of bases-cobases of a certain minor of  $\mathcal{M}$ . Recall that a base  $B \in \mathcal{B}$  is a *base-cobase* if  $E \setminus B$  is also a base of  $\mathcal{M}$ .

**Lemma 3.2.** *Let  $B_1, B_2 \in \mathcal{B}$  of a matroid  $\mathcal{M}$  and consider the matroid  $\mathcal{M}' := (\mathcal{M}/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$  on the ground set  $B_1 \triangle B_2$ . Then, the number of bases-cobases of  $\mathcal{M}'$  is equal to  $2\Delta_{\{B_1, B_2\}}$ .*

*Proof.* Set  $t := \Delta_{\{B_1, B_2\}}$  and consider  $B_3, B_4, \dots, B_{2t} \in \mathcal{B}$  such that  $B_1 \cup B_2 = B_{2i-1} \cup B_{2i}$  as multisets for all  $i \in \{1, \dots, t\}$ . Take  $i \in \{1, \dots, t\}$ , then  $B_1 \cap B_2 \subset B_{2i-1}, B_{2i} \subset B_1 \cup B_2$  and, thus,  $B_{2i-1} \setminus (B_1 \cap B_2)$  and  $B_{2i} \setminus (B_1 \cap B_2)$  are complementary bases-cobases of  $\mathcal{M}'$ . This proves that  $2t$  is less or equal to the number of bases-cobases of  $\mathcal{M}'$ .

Conversely, take  $D'_1$  a base-cobase of  $\mathcal{M}'$  and denote by  $D'_2$  its complementary base-cobase of  $\mathcal{M}'$ , i.e.,  $D'_1 \cup D'_2 = B_1 \triangle B_2$ . Moreover, if we set  $D_i := D'_i \cup (B_1 \cap B_2) \in \mathcal{B}$  for  $i = 1, 2$ , then  $D_1 \cup D_2 = B_1 \cup B_2$  as multisets. This proves that  $2t$  is greater or equal to the number of bases-cobases of  $\mathcal{M}'$ .  $\square$

The following result provides a necessary condition for a matroid to have a minor isomorphic to  $\mathcal{U}_{d,2d}$ .

**Proposition 3.3.** *If  $\mathcal{M}$  has a minor  $\mathcal{M}' \simeq \mathcal{U}_{d,2d}$  for some  $d \geq 2$ , then there exist  $B_1, B_2 \in \mathcal{B}$  such that  $\Delta_{\{B_1, B_2\}} = \binom{2d-1}{d}$ .*

*Proof.* Let  $A, C \subset E$  be disjoint sets such that  $\mathcal{M}' := (\mathcal{M} \setminus A)/C \simeq \mathcal{U}_{d,2d}$  and denote  $E' := E \setminus (A \cup C)$ . Since  $\mathcal{M}' = (\mathcal{M} \setminus A)/C$ , then there exist  $e_1, \dots, e_{r-d} \in A \cup C$  such that  $B' \cup \{e_1, \dots, e_{r-d}\} \in \mathcal{B}$  for every  $B'$  base of  $\mathcal{M}'$  (notice that the set  $\{e_1, \dots, e_{r-d}\}$  might not only have elements of  $C$ ). We take any  $D \subset E'$  with  $d$  elements, we have that  $B_1 = D \cup \{e_1, \dots, e_{r-d}\} \in \mathcal{B}$ ,  $B_2 = (E' \setminus D) \cup \{e_1, \dots, e_{r-d}\} \in \mathcal{B}$  and  $B_1 \cup B_2 = E' \cup \{e_1, \dots, e_{r-d}\}$ . Thus,  $\Delta_{\{B_1, B_2\}} \geq \binom{2d}{d}/2 = \binom{2d-1}{d}$ . Since  $|B_1 \setminus B_2| = d$ , by Lemma 3.1 we are done.  $\square$

Since  $\mathcal{U}_{2,4}$  is the only forbidden minor for a matroid to be binary, (see, e.g., [18, Theorem 6.5.4]) the following result gives a criterion for  $\mathcal{M}$  to be binary by proving the converse of Proposition 3.3 for  $d = 2$ .

**Theorem 3.4.**  *$\mathcal{M}$  is binary if and only if  $\Delta_{\{B_1, B_2\}} \neq 3$  for every  $B_1, B_2 \in \mathcal{B}$ .*

*Proof.* ( $\Rightarrow$ ) Assume that there exist  $B_1, B_2 \in \mathcal{B}$  such that  $\Delta_{\{B_1, B_2\}} = 3$ . Let us denote  $d := |B_1 \setminus B_2|$ . By Lemma 3.1 we observe that  $d = 2$ . If we set  $C := B_1 \cap B_2$  and  $A = E \setminus (B_1 \cup B_2)$ , then  $\mathcal{M}' := (\mathcal{M} \setminus A)/C$  is a rank 2 matroid on a ground set of 4 elements and, by Lemma 3.2, it has 6 bases-cobases, thus  $\mathcal{M}' \simeq \mathcal{U}_{2,4}$  and  $\mathcal{M}$  is not binary.

( $\Leftarrow$ ) Assume that  $\mathcal{M}$  is not binary, then  $\mathcal{M}$  has a minor  $\mathcal{M}' \simeq \mathcal{U}_{2,4}$  and the result follows from Proposition 3.3.  $\square$

We also prove that the converse of Proposition 3.3 also holds for  $d = 3$ . In order to prove this we make use of the database of matroids available at

[www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/index.html](http://www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/index.html)

which is based on [14]. This database includes all matroids with  $n \leq 9$  and all matroids with  $n = 10$  and  $r \neq 5$ .

**Theorem 3.5.**  *$\mathcal{M}$  has a minor  $\mathcal{M}' \simeq \mathcal{U}_{3,6}$  if and only if  $\Delta_{\{B_1, B_2\}} = 10$  for some  $B_1, B_2 \in \mathcal{B}$ .*

*Proof.* ( $\Rightarrow$ ) It follows from Proposition 3.3.

( $\Leftarrow$ ) Assume that there exist  $B_1, B_2 \in \mathcal{B}$  such that  $\Delta_{\{B_1, B_2\}} = 10$ . We denote  $d := |B_1 \setminus B_2|$  and, by Lemma 3.1, we observe that  $d \in \{3, 4\}$ . We set  $C := B_1 \cap B_2$ ,  $A = E \setminus (B_1 \cup B_2)$  and  $\mathcal{M}' := (\mathcal{M} \setminus A)/C$ , the rank  $d$  matroid on the ground set  $E' = (B_1 \cup B_2) \setminus C$  with  $2d$  elements. Moreover, by Lemma 3.2,  $\mathcal{M}'$  has exactly 20 bases-cobases. An exhaustive computer aided search among the 940 non-isomorphic rank 4 matroids on a set of 8 elements proves that there does not exist such a matroid. Therefore  $d = 3$ , and  $\mathcal{M}'$  is a rank 3 matroid on a ground set of 6 elements with 20 bases-cobases, thus  $\mathcal{M}' \simeq \mathcal{U}_{3,6}$ .  $\square$

In view of Theorems 3.4 and 3.5, one might wonder if the condition  $\Delta_{\{B_1, B_2\}} = \binom{2d-1}{d}$  for some  $B_1, B_2 \in \mathcal{B}$  is also sufficient to have  $\mathcal{U}_{d,2d}$  as a minor. For  $d = 4$ , we do not know it is true or not. Nevertheless, Example 3.6 shows that for  $d = 5$  this is no longer true. That is to say, there exists a matroid  $\mathcal{M}$  with two bases  $B_1, B_2$  such that  $\Delta_{\{B_1, B_2\}} = \binom{9}{5} = 126$  and  $\mathcal{M}$  has not a minor isomorphic to  $\mathcal{U}_{5,10}$ . To prove this result we use the fact that there exist rank 3 matroids with exactly  $k$  bases-cobases for  $k = 14$  and for  $k = 18$ . We have found these matroids by an exhaustive search among the 36 non-isomorphic matroids of rank 3 on a set of 6 elements.

**Example 3.6.** Let  $\mathcal{M}_1, \mathcal{M}_2$  be rank 3 matroids on the sets  $E_1$  and  $E_2$  with exactly 14 and 18 bases-cobases respectively. Consider the matroid  $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2$ , i.e., the direct sum of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

It is easy to check that  $\mathcal{M}$  has exactly  $14 \cdot 18 = 252$  bases-cobases. Therefore, if we take  $B$  a base-cobase of  $\mathcal{M}$  and denote by  $B'$  its complementary base-cobase, then  $\Delta_{\{B, B'\}} = 252/2 = 126$ . Let us see now that  $\mathcal{M}$  has not a minor isomorphic to  $\mathcal{U}_{5,10}$ . Suppose that there exist  $A, B \subset E_1 \cup E_2$  such that  $\mathcal{U}_{5,10} \simeq (\mathcal{M} \setminus A)/B$ . We observe that  $A \cup B$  has two elements and if we denote  $A_i := A \cap E_i$  and  $B_i := B \cap E_i$  for  $i = 1, 2$ , then  $\mathcal{U}_{5,10} \simeq (\mathcal{M} \setminus A)/B = ((\mathcal{M}_1 \setminus A_1)/B_1) \oplus ((\mathcal{M}_2 \setminus A_2)/B_2)$ , but this is not possible since  $\mathcal{U}_{5,10}$  has only one connected component.

One of the interests in Proposition 3.3 and Theorems 3.4 and 3.5 comes from the fact that for every  $B_1, B_2 \in \mathcal{B}$ , the values of  $\Delta_{\{B_1, B_2\}}$  can be directly computed from a minimal set of generators of  $I_{\mathcal{M}}$  formed by binomials. The following proposition can be obtained as a consequence of [6, Theorems 2.5 and 2.6]. However, we find it convenient to include a direct proof of this fact.

**Proposition 3.7.** *Let  $\{g_1, \dots, g_s\}$  be a minimal set of binomial generators of  $I_{\mathcal{M}}$ . Then,*

$$\Delta_{\{B_1, B_2\}} = 1 + |\{g_i = y_{B_{i_1}} y_{B_{i_2}} - y_{B_{i_3}} y_{B_{i_4}} \mid B_{i_1} \cup B_{i_2} = B_1 \cup B_2 \text{ as a multiset}\}|$$

for every  $B_1, B_2 \in \mathcal{B}$ .

*Proof.* Set  $\mathcal{H} := \{g_1, \dots, g_s\}$  and take  $B_1, B_2 \in \mathcal{B}$ . Assume that  $g_1, \dots, g_t \in \mathcal{H}$  are of the form  $g_i = y_{B_{i_1}} y_{B_{i_2}} - y_{B_{i_3}} y_{B_{i_4}}$  with  $B_{i_1} \cup B_{i_2} = B_1 \cup B_2$  as a multiset. We consider the graph  $\mathcal{G}$  with vertices  $\{B_j, B_k\} \subset \mathcal{B}$  such that  $B_j \cup B_k = B_1 \cup B_2$  as multisets and, for every  $i \in \{1, \dots, t\}$ , if  $g_i = y_{B_{i_1}} y_{B_{i_2}} - y_{B_{i_3}} y_{B_{i_4}}$  then  $f_i$  is the edge connecting  $\{B_{i_1}, B_{i_2}\}$  and  $\{B_{i_3}, B_{i_4}\}$ . We observe that  $\mathcal{G}$  has  $\Delta_{\{B_1, B_2\}}$  vertices and  $t$  edges; to conclude that  $\Delta_{\{B_1, B_2\}} = t + 1$  we prove that  $\mathcal{G}$  is a tree. Assume that  $\mathcal{G}$  has a cycle and suppose that the sequence of edges  $(f_1, \dots, f_k)$  forms a cycle. After replacing  $g_i$  by  $-g_i$  if necessary, we get that  $g_1 + \dots + g_k = 0$ , which contradicts the minimality of  $\mathcal{H}$ . Assume now that  $\mathcal{G}$  is not connected and denote by  $\mathcal{G}_1$  one of its connected components. We take  $\{B_{j_1}, B_{j_2}\}$  a vertex of  $\mathcal{G}_1$ ,  $\{B_{k_1}, B_{k_2}\}$  a vertex which is not in  $\mathcal{G}_1$  and consider  $q := y_{B_{j_1}} y_{B_{j_2}} - y_{B_{k_1}} y_{B_{k_2}} \in I_{\mathcal{M}}$ . We claim that  $q$  can be written as a combination of  $g_1, \dots, g_t$ , i.e.,  $q = \sum_{i=1}^t q_i g_i$  for some  $q_1, \dots, q_t \in R$ . Indeed, the matroid  $\mathcal{M}$  induces a grading on  $R$  by assigning to  $y_B$  the degree  $\deg_{\mathcal{M}}(y_B) := \sum_{i \in \mathcal{B}} e_i \in \mathbb{N}^n$ , where  $\{e_1, \dots, e_n\}$  is the canonical basis of  $\mathbb{Z}^n$ . Since  $I_{\mathcal{M}}$  is a graded ideal with respect to this grading, whenever  $q \in I_{\mathcal{M}}$  one may assume that  $q$  can be written as a combination of the  $g_i$  such that  $\deg_{\mathcal{M}}(g_i)$  is componentwise less or equal to  $\deg_{\mathcal{M}}(q)$ . By construction of  $q$ , we have that  $\deg_{\mathcal{M}}(g_i)$  is componentwise less or equal to  $\deg_{\mathcal{M}}(q)$  if and only if  $i \in \{1, \dots, t\}$  and the claim is proved. Moreover, if we consider  $\mathcal{B}_1 := \cup_{\{B, B'\} \in V(\mathcal{G}_1)} \{B, B'\}$  and the homomorphism of  $k$ -algebras  $\rho : R \rightarrow k[y_B \mid B \in \mathcal{B}_1]$  induced by  $y_B \mapsto y_B$  if  $B \in \mathcal{B}_1$ , or  $y_B \mapsto 0$  otherwise, then  $y_{B_{j_1}} y_{B_{j_2}} = \rho(q) = \sum_{f_i \in E(\mathcal{G}_1)} \rho(q_i) g_i$ , which is not possible. Thus, we conclude that  $\mathcal{G}$  is connected and that  $\Delta_{\{B_1, B_2\}} = t + 1$ .  $\square$

#### 4. MATROIDS WITH A UNIQUE SET OF BINOMIAL GENERATORS

In general, for a toric ideal it is possible to have more than one minimal system of generators formed by binomials. For example, as we saw in the proof of Proposition 2.2, the matroid  $\mathcal{U}_{2,4}$  is minimally generated by  $\{f_1, f_2\}$ , where  $f_1 := y_{\{1,2\}} y_{\{3,4\}} - y_{\{1,3\}} y_{\{2,4\}}$  and  $f_2 := y_{\{1,4\}} y_{\{2,3\}} - y_{\{1,3\}} y_{\{2,4\}}$ ; nevertheless, if we consider  $f_3 := y_{\{1,2\}} y_{\{3,4\}} - y_{\{1,4\}} y_{\{2,3\}}$  one can easily check that  $I_{\mathcal{M}}$  is also minimally generated by  $\{f_1, f_3\}$  and by  $\{f_2, f_3\}$ . Thus,  $\mu(I_{\mathcal{U}_{2,4}}) = 2$  and  $\nu(I_{\mathcal{U}_{2,4}}) \geq 3$ .

In this section we begin by giving some bounds for the values of  $\mu(I_{\mathcal{M}})$  and  $\nu(I_{\mathcal{M}})$  in terms of the values  $\Delta_{\{B_1, B_2\}}$  for  $B_1, B_2 \in \mathcal{B}$ . Moreover, this lower bounds turn out to be the exact values if  $I_{\mathcal{M}}$  is generated by quadratics.

**Theorem 4.1.** *Let  $R = \{\{B_1, B_2\}, \dots, \{B_{2s-1}, B_{2s}\}\}$  be a set of representatives of  $\sim$  and set  $r_i := \Delta_{\{B_{2i-1}, B_{2i}\}}$  for all  $i \in \{1, \dots, s\}$ . Then,*

- (1)  $\mu(I_{\mathcal{M}}) \geq (b^2 - b - 2s)/2$ , where  $b := |\mathcal{B}|$ , and
- (2)  $\nu(I_{\mathcal{M}}) \geq \prod_{i=1}^s r_i^{r_i-2}$ .

Moreover, in both cases equality holds whenever  $I_{\mathcal{M}}$  is generated by quadratics.

*Proof.* From Proposition 3.7, we deduce that  $\mu(I_{\mathcal{M}}) \geq \sum_{i=1}^s (\Delta_{\{B_{2i-1}, B_{2i}\}} - 1)$  with equality if and only if  $I_{\mathcal{M}}$  is generated by quadratics. It suffices to observe that  $\sum_{i=1}^s \Delta_{\{B_{2i-1}, B_{2i}\}} = b(b-1)/2$  to prove (I).

For each  $i \in \{1, \dots, s\}$  we consider the complete graph  $\mathcal{G}_i$  with vertices  $\{B_{j_1}, B_{j_2}\}$  such that  $B_{2i-1} \cup B_{2i} = B_{j_1} \cup B_{j_2}$  as multiset. We consider  $\mathcal{T}_i$  a spanning tree of  $\mathcal{G}_i$  and define  $\mathcal{H}_i := \{y_{B_{j_1}}y_{B_{j_2}} - y_{B_{j_3}}y_{B_{j_4}} \mid \text{the vertices } \{B_{j_1}, B_{j_2}\} \text{ and } \{B_{j_3}, B_{j_4}\} \text{ are connected by an edge in } \mathcal{T}_i\}$  and  $\mathcal{H} := \cup_{i=1}^s \mathcal{H}_i$ . Since  $\mathcal{H}$  is formed by degree 2 polynomials which are  $k$ -linearly independent, then  $\mathcal{H}$  can be extended to a minimal set of generators of  $I_{\mathcal{M}}$ . Since  $\mathcal{G}_i$  has exactly  $r_i$  vertices, then there are exactly  $r_i^{r_i-2}$  different spanning trees of  $\mathcal{G}_i$  that lead to different minimal systems of generators and, thus,  $\nu(I_{\mathcal{M}}) \geq \prod_{i=1}^s r_i^{r_i-2}$ . Moreover, if  $I_{\mathcal{M}}$  is generated by quadratics, let us see that the set  $\mathcal{H}$  is a set of generators itself. Indeed, let  $f \in I_{\mathcal{M}}$  be a binomial of degree two, then  $f = y_{B_{k_1}}y_{B_{k_2}} - y_{B_{k_3}}y_{B_{k_4}}$ . We take  $i \in \{1, \dots, s\}$  such that  $\{B_{k_1}, B_{k_2}\} \simeq \{B_{k_3}, B_{k_4}\} \simeq \{B_{2i-1}, B_{2i}\}$  and there exists a path in  $\mathcal{T}_i$  connecting the vertices  $\{B_{k_1}, B_{k_2}\}$  and  $\{B_{k_3}, B_{k_4}\}$ , the edges in this path correspond to binomials in  $\mathcal{H}$  and  $f$  is a combination of these binomials.  $\square$

We end by characterizing all matroids whose toric ideal has a unique minimal binomial generating set. We recall that the *basis graph of a matroid*  $\mathcal{M}$  is the undirected graph  $\mathcal{G}_{\mathcal{M}}$  with vertex set  $\mathcal{B}$  and edges  $\{B, B'\}$  such that  $|B \setminus B'| = 1$ . We also recall that the *diameter of a graph* is the maximum distance between two vertices of the graph.

**Theorem 4.2.** *Let  $\mathcal{M}$  be a rank  $r \geq 2$  matroid. Then,  $\nu(I_{\mathcal{M}}) = 1$  if and only if  $\mathcal{M}$  is binary and the diameter of  $\mathcal{G}_{\mathcal{M}}$  is at most 2.*

*Proof.* ( $\Rightarrow$ ) By Theorem 4.1, we have that  $\Delta_{\{B_1, B_2\}} \in \{1, 2\}$  for all  $B_1, B_2 \in \mathcal{B}$ . By Lemma 3.1 and Theorem 3.4, this is equivalent to  $\mathcal{M}$  is binary and  $|B_1 \setminus B_2| \in \{1, 2\}$  for all  $B_1, B_2 \in \mathcal{B}$ . Clearly this implies that the diameter of  $\mathcal{G}_{\mathcal{M}}$  is less or equal to 2.

( $\Leftarrow$ ) Assume that the diameter of  $\mathcal{G}_{\mathcal{M}}$  is  $\leq 2$ , we claim that  $\mathcal{M}$  is strongly base orderable. Recall that a matroid is strongly base orderable if for any two bases  $B_1$  and  $B_2$  there is a bijection  $\pi : B_1 \rightarrow B_2$  such that  $(B_1 \setminus C) \cup \pi(C)$  is a basis for all  $C \subset B_1$ . We take  $B_1, B_2 \in \mathcal{B}$  and observe that  $|B_1 \setminus B_2| \in \{1, 2\}$ . If  $B_1 \setminus B_2 = \{e\}$  and  $B_2 \setminus B_1 = \{f\}$  it suffices to consider the bijection  $\pi : B_1 \rightarrow B_2$  which is the identity on  $B_1 \cap B_2$  and  $\pi(e) = f$ . Moreover, if  $B_1 \setminus B_2 = \{e_1, e_2\}$  and  $B_2 \setminus B_1 = \{f_1, f_2\}$ , we denote  $A := B_1 \cap B_2$  and, by the symmetric exchange axiom, we can assume that both  $A \cup \{e_1, f_1\}$  and  $A \cup \{e_2, f_2\}$  are basis of  $\mathcal{M}$ ; then it suffices to consider  $\pi : B_1 \rightarrow B_2$  the identity on  $A$ ,  $\pi(e_1) = f_2$  and  $\pi(e_2) = f_1$  to conclude that  $\mathcal{M}$  is strongly base orderable. So, by [13, Theorem 2],  $I_{\mathcal{M}}$  is generated by quadratics. Moreover, from Lemma 3.1 and Theorem 3.4 we deduce that  $\Delta_{\{B_1, B_2\}} \in \{1, 2\}$  for all  $B_1, B_2 \in \mathcal{B}$ . Hence, the result follows by Theorem 4.1.  $\square$

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