On Ramsey numbers of complete graphs with dropped stars

Jonathan Chappelon[*†1], Luis Pedro Montejano[‡1] and Jorge Luis Ramírez Alfonsín[§1]

[1]Université Montpellier 2, Institut de Mathématiques et de Modélisation de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France

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Abstract. Let $r(G, H)$ be the smallest integer N such that for any 2-coloring (say, red and blue) of the edges of K_n, $n \geqslant N$, there is either a red copy of G or a blue copy of H. Let $K_n - K_{1,s}$ be the complete graph on n vertices from which the edges of $K_{1,s}$ are dropped. In this note we present exact values for $r(K_m - K_{1,1}, K_n - K_{1,s})$ and new upper bounds for $r(K_m, K_n - K_{1,s})$ in numerous cases. We also present some results for the Ramsey number of Wheels versus $K_n - K_{1,s}$.

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1. Introduction

Let G and H be two graphs. Let $r(G, H)$ be the smallest integer N such that for any 2-coloring (say, red and blue) of the edges of K_n, $n \geqslant N$ there is either a red copy of G or a blue copy of H. Let $K_n - K_{1,s}$ be the complete graph on n vertices from which the edges of $K_{1,s}$ are dropped. We notice that $K_n - K_{1,1} = K_n - e$ (the complete graph on n vertices from which an edge is dropped) and $K_n - K_{1,2} = K_n - P_3$ (the complete graph on n vertices from which a path on three vertices is dropped).

In this note we investigate $r(K_m - e, K_n - K_{1,s})$ and $r(K_m, K_n - K_{1,s})$ for a variety of integers m, n and s. In the next section, we prove our main result (Theorem 1). In Section 3, we will present exact values for $r(K_m - e, K_n - K_{1,s})$ when $n = 3$ or 4 and some values of m and s. In Section 4, new upper bounds for $r(K_m, K_n - P_3)$ for several integers m and n are given. In Section 5, we give new upper bounds for $r(K_m, K_n - K_{1,s})$ when $m, s \geqslant 3$ and several values of n. In Section 6, we present some equalities for $r(K_4, K_n - K_{1,s})$ extending the validity of some results given in [3]. Finally, in Section 7, we will present results concerning the Ramsey number of the Wheel W_5 versus $K_n - K_{1,s}$. We present exact values for $r(W_5, K_6 - K_{1,s})$ when $s = 3$ and 4 and the equalities $r(W_5, K_n - K_{1,s}) = r(W_5, K_{n-1})$ when $n = 7$ and 8 for some values of s.
Some known values/bounds for specific $r(K_m, K_n)$ needed for this paper are given in the Appendix.

2. Main result

Let G be a graph and denote by G^v the graph obtained from G to which a new vertex v, incident to all the vertices of G, is added. Our main result is the following

Theorem 1. *Let n and s be positive integers. Let G_1 be any graph and let N be an integer such that $N \geqslant r(G_1^v, K_n)$. If $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant r(G_1, K_{n+1} - K_{1,s})$ then $r(G_1^v, K_{n+1} - K_{1,s}) \leqslant N$.*

*Corresponding author
†E-mail address: jonathan.chappelon@um2.fr
‡E-mail address: lpmontejano@gmail.com
§E-mail address: jramirez@um2.fr

Proof. Let K_N be a complete graph on N vertices and consider any 2-coloring of the edges of K_N (say, red and blue). We shall show that there is either a G_1^v red or a $K_{n+1} - K_{1,s}$ blue. Since $N \geqslant r(G_1^v, K_n)$ then K_N has a red G_1^v or a blue K_n. In the former case we are done, so let us suppose that K_N admit a blue K_n, that we will denote by H. We have two cases.

Case 1) There exists a vertex $u \in V(K_N \setminus H)$ such that $|N_H^r(u)| \leqslant s$ where $N_H^r(u)$ is the set of vertices in H that are joined to u by a red edge. In this case, we may construct the blue graph $G' = K_{n+1} - K_{1,|N_H^r(u)|}$, this is done by taking H (containing n vertices) and vertex u together with the blue edges between u and the vertices of H. Now, since $|N_H^r(u)| \leqslant s$ then the graph $K_{n+1} - K_{1,s}$ is contained in G' (and thus we found a blue $K_{n+1} - K_{1,s}$).

Case 2) $|N_H^r(u)| > s$ for every vertex $u \in V(K_N \setminus H)$. Then we have that the number of red edges $\{x, y\}$ with $x \in V(H)$ and $y \in V(K_N \setminus H)$ is at least $(N - n)(s + 1)$. So, by the pigeon hole principle, we have that there exists at least one vertex $v \in V(H)$ such that $d^r_{K_N \setminus H}(v) \geqslant \left\lceil \frac{(s+1)(N-n)}{n} \right\rceil$, where $d^r_{K_N \setminus H}(v) = \left| N^r_{K_N \setminus H}(v) \right|$ and $N^r_{K_N \setminus H}(v)$ denotes the set of vertices in $K_N \setminus H$ incident to v with a red edge. But since $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant r(G_1, K_{n+1} - K_{1,s})$ then the graph induced by $N^r_{K_N \setminus H}(v)$ has either a blue $K_{n+1} - K_{1,s}$ (and we are done) or a red G_1 to which we add vertex v to find a red G^v as desired. □

3. Some exact values for $r(K_m - e, K_n - K_{1,s})$

Let $s \geqslant 1$ be an integer. We clearly have that

$$r(K_3 - e, K_m) \leqslant r(K_3 - e, K_{m+1} - K_{1,s}).$$

Since

$$r(K_3 - e, K_{m+1} - K_{1,s}) \leqslant r(K_3 - e, K_{m+1} - e)$$

and (see [10])

$$r(K_3 - e, K_m) = r(K_3 - e, K_{m+1} - e) = 2m - 1$$

then

$$r(K_3 - e, K_{m+1} - K_{1,s}) = 2m - 1 \text{ for each } s = 1, \ldots, m - 1.$$

3.1. Case $m = 4$.

Corollary 1.
(a) $r(K_4 - e, K_5 - K_{1,3}) = 11$.
(b) $r(K_4 - e, K_6 - K_{1,s}) = 16$ *for any* $3 \leqslant s \leqslant 4$.
(c) $r(K_4 - e, K_7 - K_{1,s}) = 21$ *for any* $4 \leqslant s \leqslant 5$.

Proof. (a) It is clear that $r(K_4 - e, K_4) \leqslant r(K_4 - e, K_5 - K_{1,3})$. Since $r(K_4 - e, K_4) = 11$ (see [10]) then $11 \leqslant r(K_4 - e, K_5 - K_{1,3})$. We will now show that $r(K_4 - e, K_5 - K_{1,3}) \leqslant 11$. By taking $N = 11$, $s = 3$ and $n = 4$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4 \times 7}{4} \right\rceil = 7 = r(K_3 - e, K_5 - K_{1,3})$ and so, by Theorem 1, we have $r(K_4 - e, K_5 - K_{1,3}) \leqslant 11$, and the result follows.

The proofs for (b) and (c) are analogues. We just need to check that conditions of Theorem 1 are satisfied by taking : $N = r(K_4 - e, K_5) = 16$ for (b) and $N = r(K_4 - e, K_6) = 21$ for (c). □

We notice that Corollary 1(a) is claimed in [8] without a proof. Corollary 1(b) can also be obtained by using that $r(K_4-e, K_6-P_3)=16$ [9] since $16=r(K_4-e, K_6-P_3) \geqslant r(K_4-e, K_6-K_{1,s}) \geqslant r(K_4-e, K_5)=16$ for $s \in \{3,4\}$. Corollary 1(c) was first posed by Hoeth and Mengersen [9]. The best known upper bounds for $r(K_4-e, K_7-K_{1,3})$ and $r(K_4-e, K_7-P_3)$ are obtained by applying the following classical recursive formula :

$$(1) \qquad r(K_m-e, K_n-K_{1,s}) \leqslant r(K_{m-1}-e, K_n-K_{1,s})+r(K_m-e, K_{n-1}-K_{1,s}).$$

Hence

$$r(K_4-e, K_7-K_{1,3}) \leqslant r(K_3-e, K_7-K_{1,3})+r(K_4-e, K_6-K_{1,3})=11+16=27$$

and

$$r(K_4-e, K_7-P_3) \leqslant r(K_3-e, K_7-P_3)+r(K_4-e, K_6-P_3)=11+16=27.$$

We are able to improve the above upper bounds.

Corollary 2. $21 \leqslant r(K_4-e, K_7-K_{1,3}) \leqslant 22$.

Proof. It is clear that $r(K_4-e, K_6) \leqslant r(K_4-e, K_7-K_{1,3})$. Since $r(K_4-e, K_6)=21$ (see [10]), then $21 \leqslant r(K_4-e, K_7-K_{1,3})$. We will now show that $r(K_4-e, K_7-K_{1,3}) \leqslant 22$. By taking $N=22$, $s=3$ and $n=6$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4\times 16}{6} \right\rceil = 11 = r(K_3-e, K_7-K_{1,3})$ and so, by Theorem 1, we have that $r(K_4-e, K_7-K_{1,3}) \leqslant 22$, and the result follows. □

The above upper bound improves the previously best known one, given by $r(K_4-e, K_7-K_{1,3}) \leqslant 27$.

3.2. **Case** $m=5$**.** The following equality is claimed in [8] without a proof.

Corollary 3. $r(K_5-e, K_5-K_{1,3})=19$.

Proof. It is clear that $r(K_5-e, K_4) \leqslant r(K_5-e, K_5-K_{1,3})$. It is known that $r(K_5-e, K_4)=19$ (see [10]), then $19 \leqslant r(K_5-e, K_5-K_{1,3})$. We will now show that $r(K_5-e, K_5-K_{1,3}) \leqslant 19$. By Corollary 1, we have that $r(K_4-e, K_5-K_{1,3})=11$. Then, by taking $N=19$, $s=3$ and $n=4$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4\times 15}{4} \right\rceil = 15 > r(K_4-e, K_5-K_{1,3})=11$ and so, by Theorem 1, we have $r(K_5-e, K_5-K_{1,3}) \leqslant 19$, and the result follows. □

Corollary 4. $r(K_5-e, K_6-K_{1,s})=r(K_5-e, K_5)$ *for* $s=3,4$.

Proof. It is clear that $r(K_5-e, K_5) \leqslant r(K_5-e, K_6-K_{1,s})$ for all $s \geqslant 1$. Let us now prove that $r(K_5-e, K_5) \geqslant r(K_5-e, K_6-K_{1,s})$ for $s=3,4$. Since $r(K_5-e, K_6-K_{1,4}) \leqslant r(K_5-e, K_6-K_{1,3})$ then it is sufficient to prove that $r(K_5-e, K_6-K_{1,3}) \leqslant r(K_5-e, K_5)$. For, let $N=r(K_5-e, K_5) \geqslant 30$ (this lower bound was proved by Exoo [6]). Since $N \geqslant 30$ then if $s=3$ and $n=5$ we obtain that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant \left\lceil \frac{4\times 25}{5} \right\rceil = 20 > 17 \geqslant r(K_4-e, K_6-K_{1,3})$ (see [10] or Corollary 1(b) for the last inequality). So, by Theorem 1, we obtain that $r(K_5-e, K_6-K_{1,3}) \leqslant N = r(K_5-e, K_5)$. □

We notice that in the case $s=2$, if $r(K_5-e, K_5) \geqslant 32$ then we may obtain that $r(K_5-e, K_6-K_{1,2})=r(K_5-e, K_5)$ (by using the same arguments as above). It is known that $r(K_5-e, K_5) \geqslant 30$.

4. New upper bounds for $r(K_m, K_n - P_3)$

In this section we will apply our main result to give new upper bounds for $r(K_m, K_n - P_3)$ in numerous cases. The value of $r(K_n, K_m - P_3)$ have already been studied in some cases. In [1, 4], it is proved that $r(K_5, K_5 - P_3) = 25$ and in [5] it is shown that $r(K_4, K_5 - P_3) = r(K_4, K_4) = 18$.

Let us first notice that, by taking $G_1 = K_m$ in Theorem 1, we obtain

Corollary 5. *Let N be an integer such that $N \geqslant r(K_{m+1}, K_n)$. If $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant r(K_m, K_{n+1} - K_{1,s})$ then $r(K_{m+1}, K_{n+1} - K_{1,s}) \leqslant N$.*

The case when $m = 3$ has already been studied in [2] where it is proved that

$$r(K_3, K_{n+1} - K_{1,s}) = r(K_3, K_n) \text{ if } n \geqslant s + 1 > (n-1)(n-2)/(r(3,n) - n).$$

As a consequence, we have

$$(2)\qquad \begin{array}{ll} r(K_3, K_6 - P_3) = r(K_3, K_5) & (\text{with } n = 5 \text{ and } s = 2), \\ r(K_3, K_7 - K_{1,3}) = r(K_3, K_6) & (\text{with } n = 6 \text{ and } s = 3), \\ r(K_3, K_{10} - K_{1,s}) = r(K_3, K_9) & (\text{with } n = 9 \text{ for any } 2 \leqslant s \leqslant 9), \\ r(K_3, K_{11} - K_{1,s}) = r(K_3, K_{10}) & (\text{with } n = 10 \text{ for any } 3 \leqslant s \leqslant 10). \end{array}$$

4.1. **Results on $r(K_m, K_5 - P_3)$.** In [3, Theorem 4], it was shown that if $n \geqslant m \geqslant 3$ and $m + n \geqslant 8$, then

$$(3)\quad r(K_{m+1} - K_{1,m-p}, K_{n+1} - K_{1,n-q}) = r(K_m, K_n) \text{ where } p = \left\lceil \tfrac{m}{n-1} \right\rceil \text{ and } q = \left\lceil \tfrac{n}{m-1} \right\rceil.$$

This result implies the following

Corollary 6. *Let $n \geqslant m \geqslant 3$ and $m + n \geqslant 8$ and let $p = \left\lceil \frac{m}{n-1} \right\rceil$ and $q = \left\lceil \frac{n}{m-1} \right\rceil$. Then,*

$$r(K_m, K_{n+1} - K_{1,n-q}) = r(K_{m+1} - K_{1,m-p}, K_n) = r(K_m, K_n).$$

Proof. We clearly have

$$r(K_m, K_n) \leqslant r(K_m, K_{n+1} - K_{1,n-q}) \leqslant r(K_{m+1} - K_{1,m-p}, K_{n+1} - K_{1,n-q}) \overset{(3)}{=} r(K_m, K_n)$$

and thus $r(K_m, K_{n+1} - K_{1,n-q}) = r(K_m, K_n)$ (the proof for $r(K_{m+1} - K_{1,m-p}, K_n) = r(K_m, K_n)$ is similar). □

By taking $m = n = 4$ (and thus $q = 2$) in Corollary 6 we have that

$$r(K_4, K_5 - P_3) = r(K_4, K_4) = 18.$$

It is also known [1] that

$$r(K_5, K_5 - P_3) = r(K_5, K_4) = 25,$$

and, by Corollary 6, we have

$$(4)\qquad \begin{array}{ll} r(K_6, K_4 - P_3) = r(K_6, K_3) = 18 & (\text{with } m = 5 \text{ and } n = 3), \\ r(K_7, K_4 - P_3) = r(K_7, K_3) = 23 & (\text{with } m = 6 \text{ and } n = 3), \\ r(K_8, K_4 - P_3) = r(K_8, K_3) = 28 & (\text{with } m = 7 \text{ and } n = 3), \\ r(K_9, K_4 - P_3) = r(K_9, K_3) = 36 & (\text{with } m = 8 \text{ and } n = 3), \\ r(K_{10}, K_4 - P_3) = r(K_{10}, K_3) \leqslant 43 & (\text{with } m = 9 \text{ and } n = 3). \end{array}$$

The best known upper bounds of $r(K_n, K_5 - P_3)$ for $n \geqslant 6$ are obtained by applying the following classical recursive formula :

$$(5)\qquad r(K_m, K_n - K_{1,s}) \leqslant r(K_{m-1}, K_n - K_{1,s}) + r(K_m, K_{n-1} - K_{1,s}).$$

By using (4), we obtain

$$\begin{aligned}
r(K_6, K_5 - P_3) &\leqslant r(K_5, K_5 - P_3) + r(K_6, K_4 - P_3) = 25 + r(K_6, K_3) = 25 + 18 = 43,\\
r(K_7, K_5 - P_3) &\leqslant r(K_6, K_5 - P_3) + r(K_7, K_4 - P_3) = 43 + 23 = 66,\\
r(K_8, K_5 - P_3) &\leqslant r(K_7, K_5 - P_3) + r(K_8, K_4 - P_3)\\
&\leqslant r(K_6, K_5 - P_3) + r(K_7, K_4 - P_3) + 28 = 43 + 23 + 28 = 94,\\
r(K_9, K_5 - P_3) &\leqslant r(K_8, K_5 - P_3) + r(K_9, K_4 - P_3) = 94 + 36 = 130,\\
r(K_{10}, K_5 - P_3) &\leqslant r(K_9, K_5 - P_3) + r(K_{10}, K_4 - P_3)\\
&\leqslant r(K_8, K_5 - P_3) + r(K_9, K_4 - P_3) + 43 = 94 + 36 + 43 = 173.
\end{aligned}$$

We are able to improve all the above upper bounds.

Corollary 7.
(a) $r(K_6, K_5 - P_3) \leqslant 41$.
(b) $r(K_7, K_5 - P_3) \leqslant 61$.
(c) $r(K_8, K_5 - P_3) \leqslant 85$.
(d) $r(K_9, K_5 - P_3) \leqslant 117$.
(e) $r(K_{10}, K_5 - P_3) \leqslant 159$.

Proof. (a) It is known that $r(K_6, K_4) \leqslant 41$. Then, by taking $N = 41$, $s = 2$ and $n = 4$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{3\times 37}{4} \right\rceil = 28 > r(K_5, K_5 - P_3) = 25$ and so, by Corollary 5, the result follows.

The proofs for the rest of the cases are analogues. We just need to check that conditions are satisfied by taking: $N = 61 \geqslant r(K_7, K_4)$ for (b), $N = 85 > 84 \geqslant r(K_8, K_4)$ for (c), $N = 117 > 115 \geqslant r(K_9, K_4)$ for (d) and $N = 159 > 149 \geqslant r(K_{10}, K_4)$ for (e). □

By applying recursion (5) to $r(K_{11}, K_5 - P_3)$ one may obtain that $r(K_{11}, K_5 - P_3) \leqslant 224$ if the old known values are used in the recursion, and it can be improved to $r(K_{11}, K_5 - P_3) \leqslant 210$ by using the new values given in Corollary 7. The latter beats the upper bound $r(K_{11}, K_5 - P_3) \leqslant 215$ obtained via Corollary 5.

We can also use Corollary 5 to give the following equality.

Corollary 8. *If* $37 \leqslant r(K_6, K_4)$ *then* $r(K_6, K_5 - P_3) = r(K_6, K_4)$.

Proof. It is clear that $r(K_6, K_4) \leqslant r(K_6, K_5 - P_3)$. We show that $r(K_6, K_5 - P_3) \leqslant r(K_6, K_4)$. Let $N = r(K_6, K_4) \geqslant 37$. Since $N \geqslant 37$ and by taking $s = 2$ and $n = 4$ we have $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant \left\lceil \frac{3\times 33}{4} \right\rceil = 25 = r(K_5, K_5 - P_3)$, and so, by Corollary 5, $r(K_6, K_5 - P_3) \leqslant N = r(K_6, K_4)$. □

It is known that $36 \leqslant r(K_6, K_4)$. In the case when $r(K_6, K_4) = 36$ the above result might not hold.

4.2. **Results on** $r(K_m, K_6 - P_3)$**.** Since $r(K_3, K_5) = 14$ then, by (2) we have $r(K_3, K_6 - P_3) = 14$ [7]. So, by (5), we have

$$r(K_4, K_6 - P_3) \leqslant r(K_3, K_6 - P_3) + r(K_4, K_5 - P_3) = 14 + 18 = 32.$$

Moreover, it is known that the upper bound is strict if the terms of the right side are even, which is our case, and so, $r(K_4, K_6 - P_3) \leqslant 31$.

Corollary 9.
(a) $25 \leqslant r(K_4, K_6 - P_3) \leqslant 27$.
(b) $r(K_5, K_6 - P_3) \leqslant 49$.
(c) $r(K_6, K_6 - P_3) \leqslant 87$.

Proof. (a) We clearly have that $25 = r(K_4, K_5) \leqslant r(K_4, K_6 - P_3)$. It is known that $r(K_4, K_5) = 25$. We take $N = 27 > r(K_4, K_5)$, $s = 2$ and $n = 5$. So, $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{3\times 22}{5} \right\rceil = 14 = r(K_3, K_6 - P_3)$ and so, by Corollary 5, $r(K_4, K_6 - P_3) \leqslant 27$.
The proofs for (b) and (c) are analogues. We just need to check that conditions of Corollary 5 are satisfied by taking: $N = 49 \geqslant r(K_5, K_5)$ for (b) and $N = 87 \geqslant r(K_6, K_5)$ for (c). □

The recursive formula (5) gives now (by using the new above values) $r(K_7, K_6 - P_3) \leqslant 148$ (before, by using the old values, it gave 158). This new upper bound beats the upper bound $r(K_7, K_6 - P_3) \leqslant 149$ obtained by Corollary 5.

4.3. **Results on $r(K_m, K_n - P_3)$ for a variety of m and n.**

Corollary 10. *For each $3 \leqslant m \leqslant 5$ and each $7 \leqslant n \leqslant 16$, we have that $r(K_m, K_n - P_3) \leqslant u(m, n)$, where the value of $u(m, n)$ is given in the (m, n) entry of the below table (the value between parentheses is the best previously known upper bound).*

$m \setminus n$	7	8	9	10	11	12	13	14	15	16
3					44(47)	52(59)	61(72)	70(86)	80(101)	91(117)
4	41(49)	61(72)		115(136)	154(183)	199(242)	253(319)	313(405)	383(506)	466(623)
5	87(105)	143(177)	222(277)							

Proof. We just need to check that conditions of Corollary 5 are satisfied by taking: $N = 41 \geqslant r(K_4, K_6)$ for $u(4, 7)$, $N = 87 \geqslant r(K_5, K_6)$ for $u(5, 7)$, $N = 61 \geqslant r(K_4, K_7)$ for $u(4, 8)$, $N = 143 \geqslant r(K_5, K_7)$ for $u(5, 8)$, $N = 222 > 216 \geqslant r(K_5, K_8)$ for $u(5, 9)$, $N = 115 \geqslant r(K_4, K_9)$ for $u(4, 10)$, $N = 47 > 42 \geqslant r(K_3, K_{10})$ for $u(3, 11)$, $N = 154 > 149 \geqslant r(K_4, K_{10})$ for $u(4, 11)$, $N = 52 > 51 \geqslant r(K_3, K_{11})$ for $u(3, 12)$, $N = 199 > 191 \geqslant r(K_4, K_{11})$ for $u(4, 12)$, $N = 61 > 59 \geqslant r(K_3, K_{12})$ for $u(3, 13)$, $N = 253 > 238 \geqslant r(K_4, K_{12})$ for $u(4, 13)$, $N = 70 > 69 \geqslant r(K_3, K_{13})$ for $u(3, 14)$, $N = 313 > 291 \geqslant r(K_4, K_{13})$ for $u(4, 14)$, $N = 80 > 78 \geqslant r(K_3, K_{14})$ for $u(3, 15)$, $N = 383 > 349 \geqslant r(K_4, K_{14})$ for $u(4, 15)$, $N = 91 > 88 \geqslant r(K_3, K_{15})$ for $u(3, 16)$, $N = 466 > 417 \geqslant r(K_4, K_{15})$ for $u(4, 16)$. □

5. Some bounds for $r(K_m, K_n - K_{1,s})$ when $s \geqslant 3$

Here, we will focus our attention to upper bounds for $r(K_m, K_n - K_{1,3})$ that yields to upper bounds for $r(K_m, K_n - K_{1,s})$ when $s \geqslant 4$ since

$$r(K_m, K_n - K_{1,s}) \leqslant r(K_m, K_n - K_{1,3}) \text{ for all } s \geqslant 4.$$

5.1. **Results on $r(K_m, K_6 - K_{1,3})$.** In [3] it was proved that $r(K_5, K_6 - K_{1,3}) = r(K_5, K_5) \leqslant 49$. So by (5) we have

$$r(K_6, K_6 - K_{1,3}) \leqslant r(K_5, K_6 - K_{1,3}) + r(K_6, K_5 - K_{1,3}) = 49 + 41 = 90.$$

Corollary 11. *For each $6 \leqslant m \leqslant 15$, we have that $r(K_m, K_6 - K_{1,3}) \leqslant u(m)$, where the value of $u(m)$ is given in the below table (the value between parentheses is the best previously known upper bound).*

m	6	7	8	9	10	11	12	13	14	15
b_u	87(90)	143(151)	216(235)	316(350)	442(499)	633(690)	848(928)	1139(1219)	1461(1568)	1878(1568)

Proof. It follows by Corollary 5 and by taking N as the best known upper bound of $r(K_n, K_5)$ for each $n = 6, \ldots, 15$. □

We notice that in case (1), by using similar arguments as above, we could prove that $r(K_6, K_6 - K_{1,3}) = r(K_6, K_5)$ if $66 \leqslant r(K_6, K_5)$.

5.2. **Results on $r(K_m, K_7 - K_{1,3})$.** In [2] it was proved that $r(K_3, K_7 - K_{1,3}) = 18$. Since $r(K_3, K_6) = 18$ then, by (2) we have $r(K_3, K_7 - K_{1,3}) = 18$. So, by (5), we have

$$r(K_4, K_7 - K_{1,3}) \leqslant r(K_3, K_7 - K_{1,3}) + r(K_4, K_6 - K_{1,3}) = 18 + 25 = 43.$$

Corollary 12. *For each $4 \leqslant m \leqslant 11$, we have that $r(K_m, K_7 - K_{1,3}) \leqslant u(m)$, where the value of $u(m)$ is given in the below table (the value between parentheses is the best previously known upper bound).*

m	4	5	6	7	8	9	10	11
b_u	41(43)	87(90)	165(180)	298(331)	495(566)	780(916)	1175(1415)	1804(2105)

Proof. It follows by Corollary 5, by taking $s = 3$ and N equals to the best known upper bound for $r(K_n, K_6)$ when $n = 5, 6, 7, 8, 9, 11$ and $N = 1175 > 1171 \geqslant r(K_{10}, K_6)$ when $n = 10$. For instance, for (1) we take $N = 41 \geqslant r(K_4, K_6)$, $s = 3$ and $n = 6$. Then, $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil = \left\lceil \frac{4\times 35}{6} \right\rceil = 24 > r(K_3, K_7 - K_{1,3})$ and, by Corollary 5, $r(K_4, K_7 - K_{1,3}) \leqslant 41$. □

6. More equalities

From (3) we have that $r(K_4, K_{n+1} - K_{1,s}) = r(K_4, K_n)$ if $s \geqslant n - \left\lceil \frac{n}{3} \right\rceil$. The latter yields to the following equalities.

$r(K_4, K_7 - K_{1,s}) = r(K_4, K_6)$ if $s \geqslant 4$, $r(K_4, K_8 - K_{1,s}) = r(K_4, K_7)$ if $s \geqslant 5$,
$r(K_4, K_9 - K_{1,s}) = r(K_4, K_8)$ if $s \geqslant 5$, $r(K_4, K_{10} - K_{1,s}) = r(K_4, K_9)$ if $s \geqslant 6$,
$r(K_4, K_{11} - K_{1,s}) = r(K_4, K_{10})$ if $s \geqslant 6$, $r(K_4, K_{12} - K_{1,s}) = r(K_4, K_{11})$ if $s \geqslant 7$,
$r(K_4, K_{13} - K_{1,s}) = r(K_4, K_{12})$ if $s \geqslant 8$, $r(K_4, K_{14} - K_{1,s}) = r(K_4, K_{13})$ if $s \geqslant 8$,
$r(K_4, K_{15} - K_{1,s}) = r(K_4, K_{14})$ if $s \geqslant 9$, $r(K_4, K_{16} - K_{1,s}) = r(K_4, K_{15})$ if $s \geqslant 10$.

We are able to extend all these equalities for further values of s.

Corollary 13.
(a) $r(K_4, K_7 - K_{1,s}) = r(K_4, K_6)$ for $s = 3$. *(b) $r(K_4, K_8 - K_{1,s}) = r(K_4, K_7)$ for $s = 3, 4$.*
(c) $r(K_4, K_9 - K_{1,s}) = r(K_4, K_8)$ for $s = 4$. *(d) $r(K_4, K_{10} - K_{1,s}) = r(K_4, K_9)$ for $s = 4, 5$.*
(e) $r(K_4, K_{11} - K_{1,s}) = r(K_4, K_{10})$ for $s = 5$. *(f) $r(K_4, K_{12} - K_{1,s}) = r(K_4, K_{11})$ for $s = 6$.*
(g) $r(K_4, K_{13} - K_{1,s}) = r(K_4, K_{12})$ for $s = 6, 7$. *(h) $r(K_4, K_{14} - K_{1,s}) = r(K_4, K_{13})$ for $s = 7$.*
(i) $r(K_4, K_{15} - K_{1,s}) = r(K_4, K_{14})$ for $s = 8$. *(j) $r(K_4, K_{16} - K_{1,s}) = r(K_4, K_{15})$ for $s = 9$.*

Proof. (1) Since $r(K_4, K_6) \geqslant 36$ it follows that $r(K_4, K_7 - K_{1,3}) \geqslant 36$ and by (2), we have $r(K_3, K_7 - K_{1,3}) = r(K_3, K_6) = 18$. Let us take $N = r(K_4, K_6) \geqslant 36$, $s = 3$ and $n = 6$. So, $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant \left\lceil \frac{4\times 30}{6} \right\rceil = 20 > r(K_3, K_7 - K_{1,3}) = 18$ and the result follows by Corollary 5.
The proofs for the rest of the cases are analogues. We just need to check that conditions of Corollary 13 are satisfied by taking: $N = r(K_4, K_7) \geqslant 49$ and checking that $r(K_3, K_8 - K_{1,3}) = r(K_3, K_7) = 23$ for (2), $N = r(K_4, K_8) \geqslant 58$ and checking that $r(K_3, K_9 - K_{1,4}) = r(K_3, K_8) = 28$ for (3) and so on. □

We notice that, by using the same arguments as above, we could improve cases (5) and (7) by showing that $r(K_4, K_{11} - K_{1,4}) = r(K_4, K_{10})$ when $r(K_4, K_{10}) \neq 92$ and $r(K_4, K_{13} - K_{1,5}) = r(K_4, K_{12})$ when $r(K_4, K_{12}) \neq 128$.
In view of Corollary 13, we may pose the following question,

Question 1. Let $n \geqslant 7$ be an integer. For which integer s the equality $r(K_4, K_n - K_{1,s}) = r(K_4, K_{n-1})$ holds?

Or more ambitious, in view of [3, Theorem 4], we may pose the following,

Question 2. Let $m \geqslant 4$ and $n \geqslant 7$ be integers. For which integer $s \leqslant n-1$ the equality $r(K_m, K_n - K_{1,s}) = r(K_m, K_{n-1})$ holds?

7. Wheels versus $K_n - K_{1,s}$

In this section we obtain further relating results by applying Theorem 1 to other graphs. Indeed, we may consider G_1 as the cycle on $n-1$ vertices C_{n-1}, and thus G_1^v will be the wheel W_n by taking the new vertex v incident to all the vertices of C_{n-1}.

Corollary 14.
(a) $r(W_5, K_6 - K_{1,s}) = 27$ *for* $s = 3, 4, 5$.
(b) $r(W_5, K_7 - K_{1,s}) = r(W_5, K_6)$ *for* $s = 4, 5, 6$.
(c) $r(W_5, K_8 - K_{1,s}) = r(W_5, K_7)$ *for* $s = 4, 5, 6, 7$.

Proof. (a) It is clear that $r(W_5, K_5) \leqslant r(W_5, K_6 - K_{1,s})$ for any $1 \leqslant s \leqslant 5$. Since $r(W_5, K_5) = 27$ (see [10]), then $27 \leqslant r(W_5, K_6 - K_{1,s})$. We will now show that $r(W_5, K_6 - K_{1,s}) \leqslant 27$ for $3 \leqslant s \leqslant 5$. By taking $N = 27$, $s \geqslant 3$ and $n = 5$, we have that $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geqslant \left\lceil \frac{4\times 22}{5} \right\rceil = 18 = r(C_4, K_6) \geqslant r(C_4, K_6 - K_{1,s})$ and so, by Theorem 1, we have $r(W_5, K_6 - K_{1,s}) \leqslant 27$, and the result follows.

The proofs for (b) and (c) are analogues. We just need to check that conditions of Theorem 1 are satisfied by taking: $N = r(W_5, K_6) \geqslant 33$ for (b) and $N = r(W_5, K_7) \geqslant 43$ for (c) (see [10] for the lower bounds of $r(W_5, K_6)$ and $r(W_5, K_7)$). $\square$

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8. Appendix

The following table was obtained from [10].

	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
K_3	6	9	14	18	23	28	36	[40,42]
K_4		18	25	[36,41]	[49, 61]	[58,84]	73,115]	[92,149]
K_5			[43,49]	[58,87]	[80,143]	[101,216]	[126,316]	[144,442]
K_6				[102,165]	[113,298]	[132,495]	[169,780]	[179,1171]

TABLE 1. Some known bounds and values of $r(K_m, K_n)$.