Pseudo-symmetry for Almost Arithmetic Semigroups

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(Join work with Ø.J. Rødseth)

Some notation

Let $a_1, \ldots, a_n$ be positive integers with $\gcd(a_1, \ldots, a_n) = 1$.
We denote by
- $S = \langle a_1, \ldots, a_n \rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$
- $g(S)$ the Frobenius number of S (that is, the largest integer not representable as a nonnegative integer combination of $a_1, \ldots, a_n$)
- $N(S)$ the number of gaps in S.

The Apéry set of S for $m \in S$ is defined as

$$Ap(S; m) = \{s \in S \mid s - m \notin S\}$$

$$N(S) = \frac{1}{m} \sum_{w \in Ap(S;m)} w - \frac{1}{2}(m-1)$$

$$g(S) = \max Ap(S;m) - m$$

Example : If $a_1 = 3$ and $a_2 = 8$ then

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ······

So, $Ap(\langle 3,8\rangle;3) = \{0,8,16\}$,
$g(\langle 3,8\rangle) = 16-3 = 13$,
$N(\langle 3,8\rangle) = \frac{1}{3}(0+8+16) - \frac{1}{2}(3-1) = \frac{24}{3} - 1 = 7$.

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Almost Arithmetic semigroup

An almost arithmetic semigroup (AA-semigroup for short) is generated by an almost arithmetic progression, $S = \langle a, a+d, a+2d, \ldots, a+kd, c\rangle$.

Let $s_{-1} = a$ and determine s_0 by $ds_0 \equiv c \pmod{s_{-1}}, 0 \leq s_0 < s_{-1}$. If $s_0 \neq 0$, we use the Euclidean algorithm with negative division remainders,

$$\begin{aligned}
s_{-1} &= q_1 s_0 - s_1, & 0 &\leq s_1 < s_0;\\
s_0 &= q_2 s_1 - s_2, & 0 &\leq s_2 < s_1;\\
s_1 &= q_3 s_2 - s_3, & 0 &\leq s_3 < s_2;\\
&\cdots\\
s_{m-2} &= q_m s_{m-1} - s_m, & 0 &\leq s_m < s_{m-1};\\
s_{m-1} &= q_{m+1} s_m, & 0 &= s_{m+1} < s_m.
\end{aligned}$$

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s_{-1} &= q_1 s_0 - s_1, && 0 \leq s_1 < s_0;\\
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s_{m-2} &= q_m s_{m-1} - s_m, && 0 \leq s_m < s_{m-1};\\
s_{m-1} &= q_{m+1} s_m, && 0 = s_{m+1} < s_m.
\end{aligned}
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If $s_0 = 0$, we put $m = -1$. If $m \geq 0$, we have

$$\frac{s_{-1}}{s_0} = q_1 - \cfrac{1}{q_2 - \cfrac{1}{q_3 - \cfrac{1}{\ddots \, q_m - \cfrac{1}{q_{m+1}}}}}$$

which is known as the Jung-Hirzebruch continued fraction of s_{-1}/s_0.

We have $s_m = \gcd(a, c)$. We define integers P_i by $P_{-1} = 0$, $P_0 = 1$, and (if $m \geq 0$),

$$P_{i+1} = q_{i+1} P_i - P_{i-1}, \quad i = 0, \dots, m.$$

Then, by induction on i,

$$s_i P_{i+1} - s_{i+1} P_i = a, \qquad i = -1, 0 \dots, m,$$

and

$$-1 = P_{-1} < 0 = P_0 < \cdots < P_{m+1} = \frac{a}{s_m}.$$

In addition we have,

$$d s_i \equiv c P_i \pmod{a}, \quad i = -1, \dots, m+1.$$

Putting

$$R_i = \frac{1}{a}\left((a+kd)s_i - kcP_i\right),$$

we then see that all the R_i are integers. Moreover, we have $R_{-1} = a + kd$, $R_0 = \frac{1}{a}((a+kd)s_0 - kc)$, and

$$R_{i+1} = q_{i+1}R_i - R_{i-1}, \quad i = 0, \ldots, m,$$

and again we see that all the R_i are integers. Furthermore,

$$-\frac{c}{s_m} = R_{m+1} < R_m < \cdots < R_0 < R_{-1} = a + kd,$$

so there is a unique integer v such that

$$R_{v+1} \leq 0 < R_v.$$

Theorem (Rødseth 1979) If $S = \langle a, a+d, a+2d, \ldots, a+kd, c\rangle$ then

$$Ap(S;a) = \left\{ a\left\lceil \frac{y}{k} \right\rceil + dy + cz \mid (y,z) \in A \cup B \right\}$$

where

$$A = \{(y,z) \in \mathbb{Z}^2 \mid 0 \le y < s_v - s_{v+1}, 0 \le z < P_{v+1}\},$$

$$B = \{(y,z) \in \mathbb{Z}^2 \mid 0 \le y < s_v, 0 \le z < P_{v+1} - P_v\}.$$

Algorithm

Input : a, d, c, k, s_0 Output : $s_v, s_{v+1}, P_v, P_{v+1}$

$r_{-1} = a, r_0 = s_0$

$r_{i-1} = \kappa_{i+1} r_i + r_{i+1}, \kappa_{i+1} = \lfloor r_{i-1}/r_i \rfloor, 0 = r_{\mu+1} < r_\mu < \cdots < r_{-1}$

$p_{i+1} = \kappa_{i+1} p_i + p_{i-1}, \quad p_{-1} = 0, \quad p_0 = 1$

$T_{i+1} = -\kappa_{i+1} T_i + T_{i-1}, \quad T_{-1} = a + kd, T_0 = \frac{1}{a}((a+kd)r_0 - kc)$

IF there is a minimal u such that $T_{2u+2} \leq 0$, THEN

$$\begin{pmatrix} s_v & P_v \\ s_{v+1} & P_{v+1} \end{pmatrix} = \begin{pmatrix} \gamma & 1 \\ \gamma - 1 & 1 \end{pmatrix} \begin{pmatrix} r_{2u+1} & -p_{2u+1} \\ r_{2u+2} & p_{2u+2} \end{pmatrix}, \gamma = \left\lfloor \frac{-T_{2u+2}}{T_{2u+1}} \right\rfloor + 1$$

ELSE $s_v = r_\mu, s_{v+1} = 0, P_v = p_\mu, P_{v+1} = p_{\mu+1}$.

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Symmetry

Let $g_S = \{g(s_1, \ldots, s_n) - s | s \in S\}$.

Ramark : S and g_S are disjoint sets (otherwise, $x = g(S) - s$ for some $s \in S$ and since $x \in S$ then $g(S) - s + s = g(S) \in S$!)

A semigroup S is called symmetric if $S \cup g_S = \mathbb{Z}$.

(Bresinsky, 1979) Monomial curves
(Kunz, 1979, Herzog, 1970) Gorestein rings
(Apéry, 1945) Classification plane of algebraic branches
(Buchweitz, 1981) Weierstrass semigroups
(Pellikaan and Torres, 1999) Algebraic codes

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Theorem (Sylvester) $\langle p, q\rangle$ is always symmetric.

Theorem (R.A. and Rødseth 2009)
Let $S = \langle a, a+d, \ldots, a+kd, c\rangle$ with $\gcd(a, d) = 1$. Then, S is symmetric if and only if one of the following conditions is satisfied.

(i) $s_v = 1$,

(ii) $s_v \equiv 2 \pmod{k}$ and $s_{v+1} = 0$,

(iii) $s_v \equiv 2 \pmod{k}$ and $s_v = a$,

(iv) $s_v - s_{v+1} = 1$ and $R_{v+1} = 1 - k$,

(v) $s_v \equiv 2 \pmod{k}$ and $s_v - s_{v+1} > 1$ and $R_{v+1} = 0$,

(vi) $k \geq 2, s_{v+1} = k - 1$ and $R_v = 1$.

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Lemma (Folklore) S is symmetric if and only if there is an i_0 such that

$$w(i_0) - w(i) = w(i_0 - i) \text{ for all } i$$

where $w(i)$ denote the unique $w \in Ap(S; m)$ satisfying $w \equiv i \pmod{m}$.

Remark If the above holds for some (fixed) i_0 and all i then $w(i_0) = \max Ap(S; m)$.

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Pseudo-symmetry

A semigroup S is pseudo-symmetric if and only if $g(S)$ is even and $S \cup g_S = \mathbb{Z} \setminus \{g/2\}$.

Lemma (Folklore) A semigroup S is pseudo-symmetric if and only if $g(S) = 2N(S) - 2$.

Proof (idea) :

$$w(g/2+i) + w(g/2-i) = w(g) + \begin{cases} m & \text{if } i \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

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It is known that S is symmetric if and only if $g(S) = 2N(S) - 1$.
So, by setting $\Delta(S) = 2N(S) - 1 - g(S)$ we have

- S is symmetric if and only if $\Delta(S) = 0$
- S is pseudo-symmetric if and only if $\Delta(S) = 1$

S is irreducible if it is not the intersection of two strictly larger numerical semigroups.

By results due to Rosales, Branco and Fröberg, Gottlieb, Häggvist we have that

- S is irreducible if and only if $\Delta(S) \leq 1$.

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Generalized Arithmetic Semigroup

A generalized arithmetic semigroup is generated by a generalized arithmetic progression, $S = \langle a, ha, ha+d, ha+2d, \ldots, ha+kd \rangle$ with $\gcd(a,d) = 1$.

We know that $Ap(S;a) = \{ha\left\lceil \frac{r}{k} \right\rceil + dr | 0 \leq r < a\}$.
So

$$g(S) = ha\left\lceil \frac{a-1}{k} \right\rceil + d(a-1) - a,$$

and

$$N(S) = \frac{1}{2}\left(h\left\lceil \frac{a-1}{k} \right\rceil (2a - 2 - k\left\lceil \frac{a-1}{k} \right\rceil + k) + (a-1)(d-1)\right).$$

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Obtaining,

$$\Delta(S) = ha\left\lceil \frac{a-1}{k} \right\rceil \left(a - 2 - k\left\lceil \frac{a-1}{k} \right\rceil + k \right)$$

Theorem (Estrada and Lopez 1994)
S is symmetric (i.e., $\Delta(S) = 0$) if and only if $a = 1$ or $a \equiv 2 \pmod{k}$.

Theorem (Matthews 2004)
S is pseudo-symmetric (i.e., $\Delta(S) = 1$) if and only if $h = 1$, $k \geq 2$ and $a = 3$, that is, if and only if $S = \langle 3, 3+d, 3+2d \rangle$.

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Let us call the AA-semigroup proper if $S \neq \langle a, a+d, \ldots, a+kd \rangle$ and $S \neq \langle a/e, c/e \rangle$ where $e = \gcd(a, c)$.

Theorem (R.A. and Rødseth 2012)

Let S be a proper AA-semigroup with $\gcd(a, d) = 1$ and $k \geq 2$. Then, S is pseudo-symmetric if and only if one of the following conditions holds :

(i) $2(c+d) = a(c-1)$ for odd $c \geq 5$,

(ii) $s_0 - s_1 = 1,\ R_1 = 1 - 2k, R_0 \geq \max\{1, 2-k+\lfloor kd/a \rfloor\}$.

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Proof (idea) : Evaluation of $\Delta(S) = 2N(S) - 1 - g(S)$ by using

$$g(S) = \max\{a\lfloor \tfrac{s_v - s_{v+1} - 1}{k} \rfloor - ds_{v+1}, a\lfloor \tfrac{s_v - 1}{k} \rfloor - cp_v\} - c(p_{v+1} - 1) + d(s_v - 1) - a$$

and

$$N(S) = \sum_{y=0}^{s_v - 1} \sum_{z=0}^{P_{v+1} - P_v - 1} L(y,z) + \sum_{y=0}^{s_v - s_{v+1} - 1} \sum_{z = P_{v+1} - P_v}^{P_{v+1} - 1} L(y,z)$$

with $L(y,z) = a\left\lceil \frac{y}{k} \right\rceil + dy + cz.$

Case $k = 1$

Proposition (R.A. and Rødseth 2012)
Let $S = \langle a, b, c \rangle$ with $\gcd(a, b) = 1$ and $b = a + d$. Then, S is symmetric if and only if $v = -1$ or $R_{v+1} = 0$ or $v = m$.

Proof (idea) : By computing
$\Delta(S) = s_{v+1}(P_{v+1} - P_v)R_v$ if $ds_{v+1} \leq cP_v$ and
$\Delta(S) = -(s_v - s_{v+1})P_v R_v$ if $ds_{v+1} \geq cP_v$

Corollary (Fröberg, Gottlieb,Häggvist 1987)
Let a, b, c be positive integers relatively prime in pairs, and let $S = \langle a, b, c \rangle$. Then, S is symmetric if and only if S is generated by one or two of a, b, c.

Proof : If $v = -1$ then $S = \langle a, b \rangle$, if $R_{v+1} = 0$ then $c|a$ and $S = \langle b, c \rangle$ and if $v = m$ then $S = \langle a, c \rangle$.

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Proposition (R.A. and Rødseth 2012)
If $bs_{v+1} \leq cP_v$, then S is pseudo-symmetric if and only if

$$s_{v+1} = 1 \text{ and } P_{v+1} - P_v = 1 \text{ and } R_v = 1$$

If $bs_{v+1} \geq cP_v$, then S is pseudo-symmetric if and only if

$$v = 0 \text{ and } s_0 - s_1 = 1 \text{ and } R_1 = -1.$$

Corollary (Rosales,García-Sánchez 2005)
Let $q_1 \geq 2, s_0 \geq 1$ and $R_0 \geq 1$ be integers. Set

$$a = (q_1 - 1)s_0 + 1, \ b = q_1 R_0 + 1, \ c = (R_0 + 1)(s_0 - 1) + 1.$$

Then, S is a pseudo-symmetric semigroup on three generators, coprime in pairs, if and only if $S = \langle a, b, c \rangle$ for some choice of q_1, s_0, R_0 such that $\gcd(a, b, c) = 1$.

Proposition (R.A. and Rødseth 2012)

If $bs_{v+1} \leq cP_v$, then S is pseudo-symmetric if and only if

$$s_{v+1} = 1 \text{ and } P_{v+1} - P_v = 1 \text{ and } R_v = 1$$

If $bs_{v+1} \geq cP_v$, then S is pseudo-symmetric if and only if

$$v = 0 \text{ and } s_0 - s_1 = 1 \text{ and } R_1 = -1.$$

Corollary (Rosales,García-Sánchez 2005)

Let $q_1 \geq 2, s_0 \geq 1$ and $R_0 \geq 1$ be integers. Set

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