CONNECTED COVERING NUMBERS

JONATHAN CHAPPELON*, KOLJA KNAUER, LUIS PEDRO MONTEJANO, AND JORGE LUIS RAMÍREZ ALFONSÍN

Dedicated to the memory of Michel Las Vergnas

Abstract. A connected covering is a design system in which the corresponding *block graph* is connected. The minimum size of such coverings are called *connected coverings numbers*. In this paper, we present various formulas and bounds for several parameter settings for these numbers. We also investigate results in connection with *Turán systems*. Finally, a new general upper bound, improving an earlier result, is given. The latter is used to improve upper bounds on a question concerning oriented matroid due to Las Vergnas.

1. Introduction

Let n, k, r be positive integers such that $n \geqslant k \geqslant r \geqslant 1$. A (n, k, r)-*covering* is a family $\mathcal{B}$ of k-subsets of $\{1, \ldots, n\}$, called *blocks*, such that each r-subset of $\{1, \ldots, n\}$ is contained in at least one of the blocks. The number of blocks is the covering's *size*. The minimum size of such a covering is called the *covering number* and is denoted by $\mathrm{C}(n, k, r)$. Given a (n, k, r)-covering $\mathcal{B}$, its graph $G(\mathcal{B})$ has $\mathcal{B}$ as vertices and two vertices are joined if they have one r-subset in common. We say that a (n, k, r)-covering is *connected* if the graph $G(\mathcal{B})$ is connected. The minimum size of a connected (n, k, r)-covering is called the *connected covering number* and is denoted by $\mathrm{CC}(n, k, r)$.

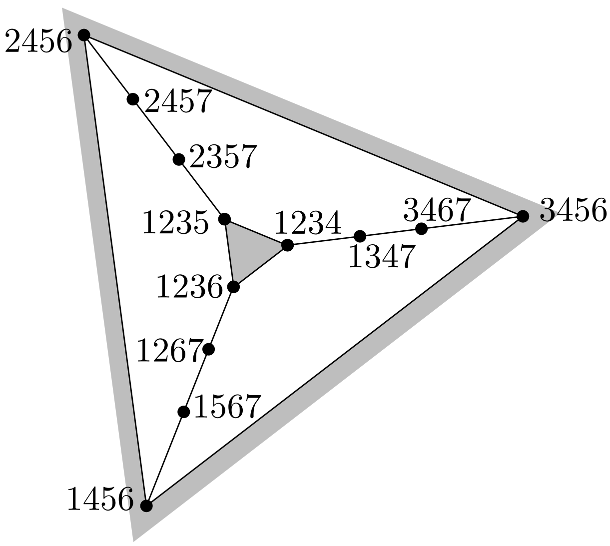

Figure 1. A connected $(7, 4, 3)$-covering with 12 blocks.

Date: January 11, 2015.
2010 *Mathematics Subject Classification.* 05B05, 5E105.
Key words and phrases. Covering design, Turán-system, uniform oriented matroid.
The second and forth authors were supported by the ANR TEOMATRO grant ANR-10-BLAN 0207.
The third author was supported by CONACYT.
* Corresponding Author: Phone: +33-467144166. Email: jonathan.chappelon@um2.fr.

The graph corresponding to a connected $(7,4,3)$-covering can be nicely illustrated as shown in Figure 1.

In this paper, we mainly pay our attention to coverings when $k=r+1$ and thus, we will denote $\mathrm{C}(n,r+1,r)$ (resp. $\mathrm{CC}(n,r+1,r)$) by $\mathrm{C}(n,r)$ (resp. by $\mathrm{CC}(n,r)$) for short. The original motivation to study $\mathrm{CC}(n,r)$ comes from the following question posed by Las Vergnas.

Question 1.1. Let $U_{r,n}$ be the rank r uniform matroid on n elements. What is the smallest number $s(n,r)$ of circuits of $U_{r,n}$, that uniquely determines all orientations of $U_{r,n}$? That is, whenever two uniform oriented matroids coincide on these circuits they must be equal.

In [3], Forge and Ramírez Alfonsín introduced the notion of connected coverings and proved that

$$s(n,r) \leqslant \mathrm{CC}(n,r). \tag{1}$$

The latter was then used to improve the best upper bound, $s(n,r) \leqslant \binom{n-1}{r}$, known at that time due to Hamidoune and Las Vergnas [7]; see also [4] for related results.

It turns out that $s(n,r)$ is also closely related to $\mathrm{C}(n,r)$. Indeed, by using results in [3, 4] it can be proved that

$$\mathrm{C}(n,r) \leqslant s(n,r). \tag{2}$$

A proof (needing some oriented matroid notions and thus lying slightly out of scope of this paper) of a more general version of the above inequality can be found in [10].

Covering designs have been the subject of an enormous amount of research papers (see [6] for many upper bounds and [19] for a survey in the dual setting of Turán-systems). Although the construction of block design is often very elusive and the proof of their existence is sometimes tough, here, we will be able to present explicit constructions yielding exact values and bounds for,$\mathrm{C}(n,r)$ and $\mathrm{CC}(n,r)$ for infinitely many cases. The study of $\mathrm{C}(n,r)$ and $\mathrm{CC}(n,r)$ seems to be interesting not only for Design Theory but also, in view of Equations (2) and (1), for the implications on the behavior of $s(n,r)$ in Oriented Matroid Theory. This relationship was already remarked in [3, Theorem 4.1] where it was proved that $\mathrm{CC}(n,r) \leqslant 2\mathrm{C}(n,r)$. The latter can be slightly improved as follows

$$\mathrm{CC}(n,r) \leqslant 2\mathrm{C}(n,r) - 1, \tag{3}$$

since the graph G associated to a covering with $\mathrm{C}(n,r)$ blocks (and thus with $|V(G)| = \mathrm{C}(n,r)$) can be made connected by adding at most $\mathrm{C}(n,r)-1$ extra vertices (blocks), obtaining a graph corresponding to a $(n,r{+}1,r)$-connected covering with at most $2\mathrm{C}(n,r){-}1$ blocks.

Many interesting variants of Question 1.1 can be investigated. For instance, for non-uniform (oriented) matroids (graphic, representable, etc.) and by varying the notion of what *determine* means (up to orientations, bijections, etc.). These (and other) variants are treated in another paper (see [10]).

This paper is organized as follows. In the next section, we recall some basic definitions and results concerning (connected) coverings and its connection with *Turán systems* needed for the rest of the paper. In Section 3, we investigate connected covering numbers when the value r is either small or close to n. Among other results, we give the exact value for $\mathrm{CC}(n,2)$ (Theorem 3.2), for $\mathrm{CC}(n,3)$ for $n \leqslant 12$ (Theorem 3.3) and for $\mathrm{CC}(n,n-3)$ (Theorem 3.6). A famous conjecture of Turán and its connection with our results is also discussed. In Section 4, we present a general upper bound for $\mathrm{CC}(n,r)$ (Theorem 4.8) allowing us to improve the best known upper bound for $s(n, r)$. We end the paper by discussing some asymptotic results in Section 5.

2. Basic results

Let n, m, p be positive integers such that $n \geqslant m \geqslant p$. A (n, m, p)-*Turán-system* is a family $\mathcal{D}$ of p-subsets of $\{1, \ldots, n\}$, called *blocks*, such that each m-subset of $\{1, \ldots, n\}$ contains at least one of the blocks. The number of blocks is the *size* of the Turán-system. The minimum size of such a covering is called the *Turán Number* and is denoted by $\mathrm{T}(n, m, p)$. Given a (n, m, p)-Turán-system $\mathcal{D}$, with $0 \leqslant 2p - m \leqslant p$, its graph $G(\mathcal{D})$ has as vertices $\mathcal{D}$ and two vertices are joined if they have one $2p - m$-subset in common. We say that a (n, m, p)-Turán-system with $0 \leqslant 2p - m \leqslant p$ is *connected* if $G(\mathcal{D})$ is connected.

The minimum size of a connected (n, m, p)-Turán-system is the *connected Turán Number* and is denoted by $\mathrm{CT}(n, m, p)$. By applying set complement to blocks, it can be obtained that

$$\mathrm{C}(n, k, r) = \mathrm{T}(n, n - r, n - k). \tag{4}$$

Moreover, if $0 \leqslant n - 2k + r \leqslant n - k$ then

$$\mathrm{CC}(n, k, r) = \mathrm{CT}(n, n - r, n - k). \tag{5}$$

Note that the precondition for (5) is fulfilled if $k = r + 1$.

Most of the papers on coverings consider n large compared with k and r, while for Turán numbers it has frequently been considered n large compared with m and p, and often focusing on the quantity $\lim_{n\to\infty} \mathrm{T}(n, m, p)/\binom{n}{p}$ for fixed m and p. Thus, for Turán-type problems, the value $\mathrm{C}(n, k, r)$ has usually been studied in the case when k and r are not too far from n.

Forge and Ramírez Alfonsín [3] proved that

$$\mathrm{CC}(n, r) \geqslant \frac{\binom{n}{r} - 1}{r} =: \mathrm{CC}_1^*(n, r). \tag{6}$$

Moreover, Sidorenko [18] proved that $\mathrm{T}(n, r + 1, r) \geqslant \left(\frac{n-r}{n-r+1}\right) \frac{\binom{n}{r}}{r}$. Together with (4), we obtain that

$$\mathrm{CC}(n, r) \geqslant \mathrm{C}(n, r) = \mathrm{T}(n, n - r, n - r - 1) \geqslant \left(\frac{r + 1}{r + 2}\right) \frac{\binom{n}{r+1}}{n - r - 1} =: \mathrm{CC}_2^*(n, r). \tag{7}$$

Combining (6) and (7), together with a straight forward computation we have

$$\mathrm{CC}(n, r) \geqslant \max\{\mathrm{CC}_1^*(n, r), \mathrm{CC}_2^*(n, r)\}, \tag{8}$$

where the maximum is attained by the second term if and only if $r \geqslant \frac{2}{3}(n - 1)$.

The following recursive lower bound for covering numbers was obtained by Schönheim [17] and, independently, by Katona, Nemetz and Simonovits [9]

$$\mathrm{C}(n, r) \geqslant \left\lceil \frac{n}{r + 1} \mathrm{C}(n - 1, r - 1) \right\rceil \tag{9}$$

which can be iterated yielding to

$$\mathrm{C}(n, r) \geqslant \left\lceil \frac{n}{r + 1} \left\lceil \frac{n - 1}{r} \left\lceil \ldots \left\lceil \frac{n - r + 1}{2} \right\rceil \ldots \right\rceil \right\rceil \right\rceil =: L(n, r). \tag{10}$$

Forge and Ramírez Alfonsín [3, Theorem 4.2] proved that $\mathrm{CC}(n, r) \leqslant \sum_{i=r+1}^{n-1} \mathrm{C}(i, r - 1)$. In this proof, they used the following recursive upper bound that will be useful for us later,

$$\mathrm{CC}(n,r) \leqslant \mathrm{CC}(n-1,r) + \mathrm{C}(n-1,r-1). \tag{11}$$

3. Results for small and large r

In this section, we investigate connected covering numbers for *small* and *large* r, that is, when r is very close to either 1 or n. Let us start with the following observations.

Remarks 3.1.
a) $\mathrm{CC}(n,0) = 1$ since any 1-element set contains the empty set.
b) $\mathrm{CC}(n,1) = n-1$ by taking the edges of a spanning tree of K_n.
c) $\mathrm{CC}(n,n-2) = n-1$ by taking all but one $(n-1)$-sets.
d) $\mathrm{CC}(n,n-1) = 1$ by taking the entire set.

All these values coincide with the corresponding covering numbers except in the case $r = 1$, where $\mathrm{C}(n,1) = \lceil \frac{n}{2} \rceil$.

3.1. **Results when r is small.**

For ordinary covering numbers, Fort and Hedlund [5] have shown that $\mathrm{C}(n,2) := \lceil \frac{n}{3} \lceil \frac{n}{2} \rceil \rceil$ that coincides with the lower bounds given in (10) when the case $r = 2$.

We also have the precise value for the connected case when $r = 2$.

Theorem 3.2. *Let n be a positive integer with $n \geqslant 3$. Then, we have*

$$\mathrm{CC}(n,2) = \left\lceil \frac{\binom{n}{2} - 1}{2} \right\rceil .$$

Proof. Note that the claimed value coincides with the lower bound $\mathrm{CC}_1^*(n,2)$. This lower bound comes from the fact that every connected covering has a construction sequence, where every new triangle shares at least one edge with an already constructed triangle. We present a construction sequence where indeed every new triangle (except possibly the last one) shares exactly one edge with the already constructed ones. Therefore, we attain the lower bound. Part of the construction is shown in Figure 2. We start presenting the

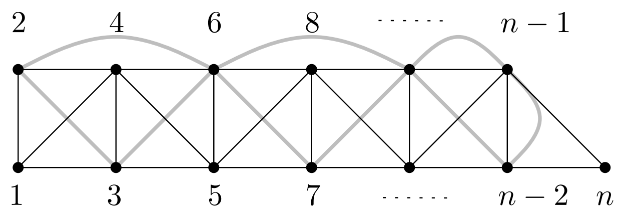

Figure 2. Part of the construction proving $\mathrm{CC}(n,2) \leqslant \mathrm{CC}_1^*(n,2)$.

black triangles from left to right. Then we present all triangles of the form $(2i-1, 2i, j)$ for $1 \leqslant i \leqslant \frac{n}{2}$ and $j \geqslant 2i+3$. (These are not depicted in the figure.) Now we present the gray triangles from left to right. A gray triangle of the form $(2i, 2i+1, 2i+4)$ is connected to the already presented ones via $(2i-1, 2i, 2i+4)$. Note (as in the figure) the last triangle may indeed share two edges of already presented triangles, depending on the parity of n. This accounts for the ceiling in the formula. It is easy to check that all edges are covered. □

The precise value of $C(n,3)$ remains unknown only for finitely many n, see [14, 15, 8]. The situation for connected coverings is worse.

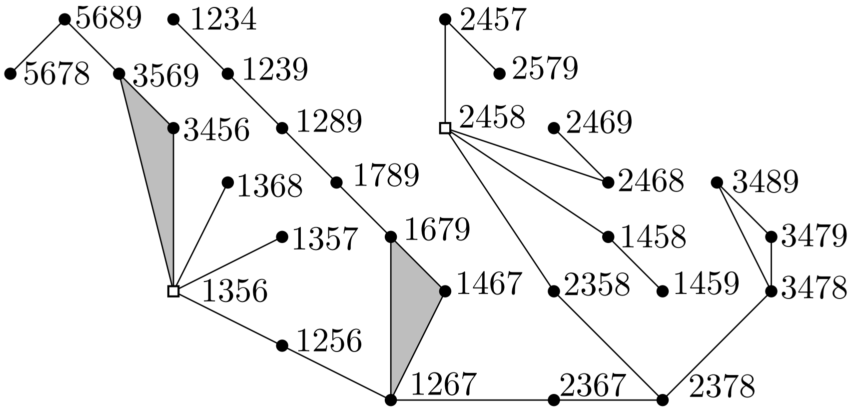

FIGURE 3. An example proving $\mathrm{CC}(9,3) \leqslant 28$. The circle-vertices are a covering.

Theorem 3.3. *Let n be a positive integer with $4 \leqslant n \leqslant 12$. Then, we have*

$$\mathrm{CC}(n,3) = \left\lceil \frac{\binom{n}{3} - 1}{3} \right\rceil .$$

Proof. Note that the claimed value coincides with the lower bound $\mathrm{CC}_1^*(n,3)$. For $n \leqslant 6$ this is already checked in [3]. Figure 1 proves $\mathrm{CC}(7,3) \leqslant 12 = \mathrm{CC}_1^*(7,3)$. By using (11), $\mathrm{C}(7,2) = 7$, and $\mathrm{CC}(7,3) = 12$, we obtain that $\mathrm{CC}(8,3) \leqslant 19 = \mathrm{CC}_1^*(8,3)$. Figure 3 proves $\mathrm{CC}(9,3) \leqslant 28 = \mathrm{CC}_1^*(9,3)$. From equation (11) and the fact that $\mathrm{CC}(9,3) = 28$ and $\mathrm{C}(9,2) = 12$, we conclude that $\mathrm{CC}(10,3) \leqslant 40 = \mathrm{CC}_1^*(10,3)$. Now, Figure 4 proves that $\mathrm{CC}(11,3) \leqslant 55 = \mathrm{CC}_1^*(11,3)$. Finally, to construct a connected covering witnessing $\mathrm{CC}_1^*(12,3)$ we delete the block $\{2,4,6,8\}$ from the covering in Figure 4. One can check that this still leaves a covering $\mathcal{B}$, whose graph now has three components. Now, we take the following (disconnected) $(11,3,2)$-covering:

$$\left\{ \begin{array}{l} \{1,3,11\},\{1,4,6\},\{1,2,8\},\{1,5,9\},\{1,7,10\},\{3,4,9\},\{2,3,10\},\{3,5,6\}, \\ \{3,7,8\},\{2,4,6\},\{4,5,7\},\{4,10,11\},\{4,6,8\},\{2,5,11\},\{2,7,9\},\{5,8,10\}, \\ \{6,7,11\},\{8,9,11\},\{6,9,10\} \end{array} \right\}.$$

We add to each of these block the element 12 and thus together with $\mathcal{B}$ obtain a $(12,4,3)$-covering $\mathcal{B}'$. To see that $\mathcal{B}'$ is connected, note that each of the blocks containing 12 is connected to a block from $\mathcal{B}$. Moreover, the blocks $\{1,4,6,12\},\{2,4,6,12\},\{4,6,8,12\}$ form a triangle and each of them has a neighbor in a different component of $G(\mathcal{B})$. Thus, $G(\mathcal{B}')$ is connected and $\mathcal{B}'$ has 73 blocks which coincides with $\mathrm{CC}_1^*(12,3)$. □

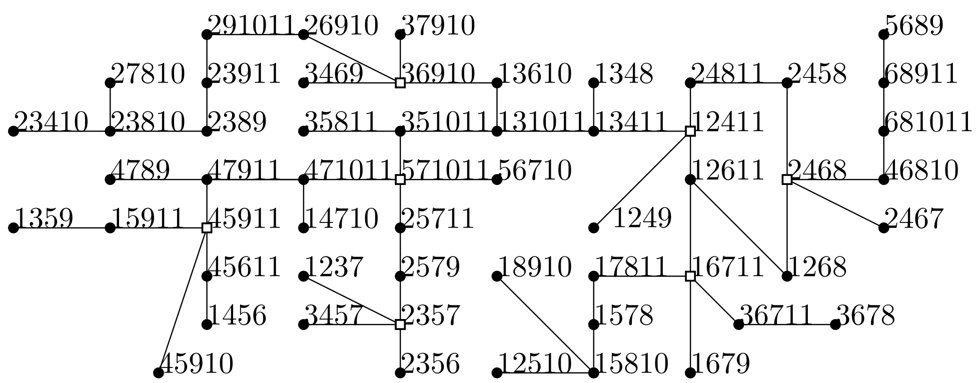

FIGURE 4. An example proving $\mathrm{CC}(11,3) \leqslant 55$. The circle-vertices are a covering.

Theorem 3.3 supports the following

Conjecture 3.4. For every positive integer $n \geqslant 4$, we have

$$\mathrm{CC}(n,3) = \mathrm{CC}_1^*(n,3).$$

Even, more ambitious,

Question 3.5. Let n and r be two positive integers such that $n \geqslant r+1 \geqslant 4$. Is it true that if $\mathrm{CC}(n,r) = \mathrm{CC}_1^*(n,r)$ then $\mathrm{CC}(n',r) = \mathrm{CC}_1^*(n',r)$ for every integer $n' \geqslant n$?

3.2. **Results when r is large.**

Theorem 3.6. *Let n be a positive integer with $n \geqslant 3$. Then, we have*

$$\mathrm{CC}(n,n-3) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} + 1.$$

Proof. The parameter $\mathrm{C}(n,n-3) = \mathrm{T}(n,3,2)$ was determined already by Mantel in 1907 [13] and is $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$. Turán proved that the *unique* minimal configuration of sets of size 2 hitting all sets of size 3 of an n-set are the edges of two vertex-disjoint complete graphs $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$, see [20].

Now, by (4) and (5), the covering corresponding to the Turán-system is connected if and only if the graph whose edges correspond to the blocks of the Turán-system is connected. Thus, since the unique optimal construction by Turán is not connected but can be made connected by adding a single edge connecting the two complete graphs, this is optimal with respect to connectivity. Therefore, $\mathrm{CC}(n,n-3) = \mathrm{T}(n,3,2)+1$, giving the result. □

Proposition 3.7. *Let $n \neq 5,6,8,9$ be a positive integer with $n \geqslant 4$. Then, we have*

$$\mathrm{CC}(n,n-4) \leqslant \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases}$$

If $n = 5,6,9$ the value of $\mathrm{CC}(n,n-4)$ is one larger than claimed in the formula. Further, $\mathrm{CC}(8,4) \in \{20,21\}$, i.e., it remains open if the above formula has to be increased by one or not in order to give the precise value.

Proof. We will show that a Turán-system $\mathcal{D}$ verifying the claimed bounds due to Kostochka [11] is connected. Indeed the construction of [11] is a parametrized family of Turán-systems, each of whose members attains the claimed bound. Our construction results from picking special parameters:

Assume that $n \geqslant 12$ and n is divisible by 3. Split $[n]$ into three sets A_1, A_2, A_3 of equal size. Pick special elements $x_i, y_i \in A_i$ and denote $B_i := A_i \setminus \{x_i, y_i\}$ for $i = 0,1,2$. The blocks of $\mathcal{D}$ consist of 3-element sets $\{a,b,c\}$ of the following forms:

- L_i: $a,b,c \in A_i$,
- $T1_i$: $a = x_i$ and $b,c \in A_{i+1}$,
- $T2_i$: $a = y_i$ and $b,c \in B_{i-1} \cup \{x_{i+1}, y_{i+1}\}$,
- $T3_i$: $a \in B_i$ and $b,c \in B_{i-1} \cup \{x_{i+1}, y_{i-1}\}$

where $i = 0,1,2$, and addition of indices is understood modulo 3.

Let us now show that $\mathcal{D}$ is connected. Clearly, all blocks in a given A_i are connected and all 2-element subsets in each A_i are covered by a block in this A_i. Thus, it suffices to verify that there are two 2-element sets $\{e,f\} \subseteq A_0$ and $\{e',f'\} \subseteq A_2$ which can be connected by a sequence of blocks of $\mathcal{D}$, because then any block in A_0 containing $\{e,f\}$ is connected to any block in A_2 containing $\{e',f'\}$. The connectivity of $\mathcal{D}$ then follows by the symmetry

of the construction. Let $\{e,f\} \subseteq B_0$. Take $\{e,f,y_1\} \in T2_1$, then $\{e,y_1,y_2\} \in T2_1$, and then $\{e,f',y_2\} \in T3_0$, where $f' \in B_2$, i.e, $\{y_2,f'\} \subseteq B_2$.
Now, following [11] deleting any element of such a system yields a Turán-system $\mathcal{D}'$ of the claimed size for $n' = n-1$. We can just delete any x_i, since these are not used for connectivity. Following [11], two elements can be removed from $\mathcal{D}$ to obtain a Turán-system $\mathcal{D}''$ of the claimed size for $n'' = n-2$, if the set formed by these two elements belongs to exactly $\frac{n}{3}-1$ blocks. This is the case for $\{x_i, x_{i+1}\}$, which belongs to exactly $\frac{n}{3}-1$ blocks from $T1_i$. Again, this preserves connectivity.
We are left with the cases $n \leqslant 9$. In [18] it is shown that the Turán-systems of the claimed size for $n=9$ are exactly the members of the family constructed in [11]. There are exactly two such systems:
In both cases [9] is split into three sets A_1, A_2, A_3 of size 3. In the first system we pick a $x_i \in A_i$ and denote $A_i \setminus \{x_i\}$ by B_i. The blocks then are the 3-element sets $\{a,b,c\}$ of the following forms:

L_i: $a,b,c \in A_i$,
$T1_i$: $a = x_i$ and $b,c \in A_{i+1}$,
$T2_i$: $a \in B_i$ and $b,c \in B_{i-1} \cup \{x_{i+1}\}$.

The second system coincides with an instance of a construction due to Turán [21]. It consists of the following 3-element sets:

L_i: $a,b,c \in A_i$,
$T1_i$: $a \in A_i$ and $b,c \in A_{i+1}$.

It is easy to check that both systems are not connected. On the other hand, the second one can be made connected adding a single block taking one element from each A_i. This proves the claim for $n=9$. Further, removing any vertex not contained in the added block, one obtains a connected Turán-system for $n=8$ with 21 blocks. While there are Turán-systems showing $\mathrm{T}(8,4,3) = 20$ we do not know if there is any such connected system.
See Figure 1 for proving our statement for $n=7$, Theorem 3.2 for $n=6$, and Remark 3.1 for $n=4,5$. ☐

A famous conjecture of Turán [21] states that the bounds in Proposition 3.7 are best possible for $\mathrm{C}(n,n-4)$. By combining (1) and Proposition 3.7, for $n \geqslant 10$ we have

$$\mathrm{C}(n,n-4) \leqslant \mathrm{CC}(n,n-4) \leqslant \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases} \tag{12}$$

Turán's conjecture has been verified for all $n \leqslant 13$ by [18] and so, by (12), the connected covering number can also be determined for these same values.

Towards proving Turán's conjecture, it would be of interest to investigate the following.

Question 3.8. Is it true that one of the inequalities in (12) is actually an equality ?

Bounds and precise values for all $\mathrm{CC}(n,r)$ with $n \leqslant 14$ are given in Table 1. All the exact values previously given in [3] for the same range have been improved by using our above results.

Table 1 led us to consider the following.

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2			1	3	$5^{e,t}$	7^e	10^e	14^e	18^e	22^e	27^e	33^e	39^e	45^e
3				1	4	$7^{p,t}$	$12^{p,u}$	19^p	28^p	40^p	55^p	73^p	$[95^l, 97^r]$	$[121^l, 123^r]$
4					1	5	10^t	$[20, 21^u]$	$[32^l, 35^r]$	$[53^l, 59^r]$	$[83^l, 89^r]$	$[124^l, 136^r]$	$[179^l, 193^r]$	$[250^l, 271^r]$
5						1	6	13^t	31^u	$[51^l, 61^r]$	$[96^a, 111^r]$	$[159^l, 177^r]$	$[258^l, 290^r]$	$[401^l, 447^r]$
6							1	7	17^t	45^u	$[84^a, 95^r]$	$[165^a, 195^r]$	$[286^l, 327^r]$	$[501^l, 572^r]$
7								1	8	21^t	63^u	$[126^a, 147^r]$	$[269^a, 323^r]$	$[491^l, 587^r]$
8									1	9	26^t	84^u	$[185^a, 210^r]$	$[419^a, 505^r]$
9										1	10	31^t	112^u	$[259^s, 297^r]$
10											1	11	37^t	$[143^s, 144^u]$
11												1	12	43^t
12													1	13
13														1

TABLE 1. Bounds and values of $\mathrm{CC}(n, r)$ for $n \leqslant 14$.

Key of Table 1 :

r — Upper bound for $\mathrm{CC}(n, r)$(from Equation (11))
e — Exact values for $\mathrm{CC}(n, 2)$ (Theorem 3.2)
t — Exact values for $\mathrm{CC}(n, n-3)$ (Theorem 3.6)
l — Lower bound $\mathrm{CC}_1^*(n, r)$
p — Some exact values for $\mathrm{CC}(n, 3)$ (Theorem 3.3)
u — Upper bound for $\mathrm{CC}(n, n-4)$ (Proposition 3.7)
s — Lower bound for $\mathrm{C}(n, r)$ (from Equation (9))
a — Lower bounds for $\mathrm{C}(n, r)$ (from [1])

Question 3.9. Is the sequence $(\mathrm{CC}(n, i))_{0 \leqslant i \leqslant n-1}$ *unimodal* for every n ? or perhaps *logarithmically concave*[1] ?

4. A GENERAL UPPER BOUND

Let n and r be positive integers such that $n \geqslant r + 1 \geqslant 3$. Forge and Ramírez Alfonsín [3] obtained the following general upper bound

$$\mathrm{S}(n, r) := \sum_{i=1}^{\left\lfloor \frac{n-r+1}{2} \right\rfloor} \binom{n-2i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor \geqslant \mathrm{CC}(n, r). \tag{13}$$

Let us notice that the upper bounds obtained by applying the recursive equation (11), that were used in Table 1, are better than the one given by (13). Moreover, by iterating (11) it can be obtained

$$\mathrm{CC}(n, r) \leqslant \sum_{i=r}^{n-1} \mathrm{C}(i, r-1). \tag{14}$$

Although (14) might be used to get an *explicit* upper bound for $s(n, r)$, it is not clear how good it would be since that would depend on the known exact values and the upper bounds of $\mathrm{C}(n, r)$ used in the recurrence (and thus intrinsically difficult to compute). On

[1]A finite sequence of real numbers $\{a_1, a_2, \ldots, a_n\}$ is said to be *unimodal* (resp. *logarithmically concave* or *log-concave*) if there exists a t such that $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_t$ and $s_t \geqslant s_{t+1} \geqslant \cdots \geqslant s_n$ (resp. if $a_i^2 \geqslant a_{i-1}a_{i+1}$ holds for every a_i with $1 \leqslant i \leqslant n-1$). Notice that a log-concave sequence is unimodal.

the contrary, in [3] Equation (13) was used to give the best known (to our knowledge) explicit upper bound for $s(n,r)$.

In this section, we will construct a connected $(n,r+1,r)$-covering giving an upper bound for $\mathrm{CC}(n,r)$ better than $\mathrm{S}(n,r)$ and so, yielding a better upper bound for $s(n,r)$ than that given in [3].

Theorem 4.1. *Let n and r be positive integers such that $n \geqslant r+1 \geqslant 3$. Then $\mathrm{CC}(n,r) \leqslant \mathrm{N}(n,r)$, where*

$$\mathrm{N}(n,r) := \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil -1} (n-r-2i)\binom{r-2+2i}{r-2} + \left\lceil \frac{n-r}{2} \right\rceil - 1 + \delta_0 \mathrm{C}(n-2,r-2), \tag{15}$$

and δ_0 is the parity function of $n-r$, that is, $\delta_0 = \begin{cases} 0 & \text{if } n-r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$

Proof. From this point on, for any positive integer s, we will denote $[s] := \{1,\dots,s\}$ and by $\binom{[s]}{t}$ the set of all t-subsets of $[s]$. Moreover, for any subset of integers $\{b_1,\dots,b_s\}$, we may suppose that $b_i < b_j$ for all integers i and j such that $1 \leqslant i < j \leqslant s$.

Case 1. Suppose that $n-r$ is odd and let m such that $n-r = 2m+1$. We will construct a connected $(r+2m+1, r+1, r)$-covering of size

$$m + \sum_{i=0}^{m} \binom{r-2+2i}{r-2}(2m+1-2i).$$

We consider a particular $(r+2m+1,r+1,r)$-covering, which is constituted by a large number of blocks but whose associated graph has a small number of connected components. For any $i \in \{0,\dots,m\}$, let $\mathcal{N}_i$ be the following subset of $(r+1)$-subsets of $[r+2m+1]$:

$$\mathcal{N}_i := \left\{ \{b_1,\dots,b_{r+1}\} \;\middle|\; \begin{array}{l} \{b_1,\dots,b_{r-2}\} \in \binom{[r+2i-2]}{r-2} \\ b_{r-1} = r+2i-1,\ b_r = r+2i \\ b_{r+1} \in \{r+2i+1,\dots,r+2m+1\} \end{array} \right\}.$$

Claim 4.2. *The set $\bigcup_{i=0}^{m} \mathcal{N}_i$ is a $(r+2m+1,r+1,r)$-covering.*

Let $b = \{b_1,\dots,b_r\} \in \binom{[r+2m+1]}{r}$. If $b_{r-1} = r-1+2i$ for some $i \in \{0,\dots,m\}$, then $b \subset B$ for some $B \in \mathcal{N}_i$. The same occurs if $b_{r-1} = r+2i$.

Claim 4.3. *The graph $G(\mathcal{N}_i)$ is connected, for any $i \in \{0,\dots,m\}$.*

Let $B = \{b_1,\dots,b_{r+1}\}$ and $C = \{c_1,\dots,c_{r+1}\}$ in $\mathcal{N}_i$. Clearly, B is adjacent to $\{b_1,\dots,b_r,c_{r+1}\}$ in $G(\mathcal{N}_i)$. Since $b_{r-1} = c_{r-1}$, $b_r = c_r$ and $\{d_1,\dots,d_{r-2},c_{r-1},c_r,c_{r+1}\} \in \mathcal{N}_i$ for all $\{d_1,\dots,d_{r-2}\} \subset [r-2+2i]$, then there exists a path from B to C.

Claim 4.4. *There exists a $(r+1)$-subset C_i such that $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$ is connected for any $i \in \{0,\dots,m-1\}$.*

Let $B_i = \{1,\dots,r-2,r-1+2i,r+2i,r+1+2i\} \in \mathcal{N}_i$ and $B_{i+1} = \{1,\dots,r-2,r+1+2i,r+2+2i,r+3+2i\} \in \mathcal{N}_{i+1}$. Then, the $(r+1)$-subset $C_i = \{1,\dots,r-2,r+2i,r+1+2i,r+2+2i\}$ is adjacent to B_i and B_{i+1} in $G(\mathcal{N}_i \cup C_i \cup \mathcal{N}_{i+1})$. This concludes the proof of Claim 3.

By Claims 4.2, 4.3 and 4.4, we obtain that $\left(\bigcup_{i=0}^{m} \mathcal{N}_i\right) \bigcup \left(\bigcup_{i=0}^{m-1} C_i\right)$ is a connected $(r+2m+1,r+1,r)$-covering. Finally, since $|\mathcal{N}_i| = \binom{r-2+2i}{r-2}(2m+1-2i)$ for any $i \in \{0,\dots,m\}$, the theorem holds in this case.

Case 2. Suppose $n-r$ is even and let m be such that $n-r=2m$. We are going to construct a $(r+2m,r+1,r)$-connected covering of size

$$m-1+\mathrm{C}(r-2+2m,r-2)+\sum_{i=0}^{m-1}\binom{r-2+2i}{r-2}(2m-2i).$$

As already defined in Case 1, we consider the collection $\mathcal{N}_i$ of $(r+1)$-subsets of $[r+2m]$ defined by $\mathcal{N}_i := \binom{[r+2i-2]}{r-2}\times\{r+2i-1\}\times\{r+2i\}\times\{r+2i+1,\ldots,r+2m\}$ for any $i\in\{0,\ldots,m-1\}$. Let $\mathcal{C}$ be a $(r+2m-2,r-1,r-2)$-covering of size $\mathrm{C}(r+2m-2,r-2)$ and consider the set $\mathcal{N}_m := \{B\cup\{r+2m-1,r+2m\}\mid B\in\mathcal{C}\}$. Then, one can check that $\bigcup_{i=0}^{m}\mathcal{N}_i$ is a $(r+2m,r+1,r)$-covering. Similarly as in the proofs of Claims 4.3 and 4.4, it follows that $G(\mathcal{N}_i)$ is connected for any $i\in\{0,\ldots,m-1\}$ and there exists a $(r+1)$-subset C_i such that $G(\mathcal{N}_i\cup C_i\cup\mathcal{N}_{i+1})$ is connected for any $i\in\{0,\ldots,m-2\}$.

Claim 4.5. . *For any $B\in\mathcal{N}_m$, there exist $i\in\{0,\ldots,m-1\}$ and $C\in\mathcal{N}_i$ such that B is adjacent to C in the graph $G(\mathcal{N}_i\cup\mathcal{N}_m)$.*

Let $B=\{b_1,\ldots,b_{r-1},r+2m-1,r+2m\}\in\mathcal{N}_m$. If $b_{r-1}=r+2i-1$ for some $i\in\{0,\ldots,m-1\}$, then $\{b_1,\ldots,b_{r-2}\}\in\binom{[r+2i-2]}{r-2}$. Let $C=\{b_1,\ldots,b_{r-2},r+2i-1,r+2i,r+2m\}$, by definition $C\in\mathcal{N}_i$ and moreover, since $\{b_1,\ldots,b_{r-2},r+2i-1,r+2m\}\subset B$ and $\{b_1,\ldots,b_{r-2},r+2i-1,r+2m\}\subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i\cup\mathcal{N}_m)$. Either, if $b_{r-1}=r+2i$ for some $i\in\{0,\ldots,m-1\}$, we have that $\{b_1,\ldots,b_{r-2}\}\in\binom{[r+2i-1]}{r-2}$. We distinguish two cases on the value of b_{r-2}. First, if $b_{r-2}<r+2i-1$, then $\{b_1,\ldots,b_{r-2}\}\in\binom{[r+2i-2]}{r-2}$. Consider now $C=\{b_1,\ldots,b_{r-2},r+2i-1,r+2i,r+2m\}$. As above, since $\{b_1,\ldots,b_{r-2},r+2i,r+2m\}\subset B$ and $\{b_1,\ldots,b_{r-2},r+2i,r+2m\}\subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i\cup\mathcal{N}_m)$. Finally, suppose that $b_{r-2}=r+2i-1$ and let $\alpha\in[r+2i-2]\setminus\{b_1,\ldots,b_{r-3}\}$ and $C=\{b_1,\ldots,b_{r-3},r+2i-1,r+2i,r+2m\}\cup\{\alpha\}\in\binom{[r+2m]}{r+1}$. Since $\{b_1,\ldots,b_{r-3},r+2i-1,r+2i,r+2m\}\subset B$ and $\{b_1,\ldots,b_{r-3},r+2i-1,r+2i,r+2m\}\subset C$, we deduce that B and C are adjacent in the graph $G(\mathcal{N}_i\cup\mathcal{N}_m)$. This concludes the proof of Claim 4.5.

Hence, $(\bigcup_{i=0}^{m}\mathcal{N}_i)\cup(\bigcup_{i=0}^{m-2}C_i)$ is a connected $(r+2m,r+1,r)$-covering. Since $|\mathcal{N}_i|=\binom{r-2+2i}{r-2}(2m-2i)$ for any $i\in\{0,\ldots,m-1\}$ and $|\mathcal{N}_m|=\mathrm{C}(n-2,r-2)$, the theorem holds. ☐

Let us illustrate the construction given in the above theorem.

Example 4.6. $\mathrm{N}(7,4)=10$. We consider

$$\mathcal{N}_0=\{12345,12346,12347\}\text{ and }\mathcal{N}_1=\{12567,13567,14567,23567,24567,34567\}.$$

It can be checked that $\mathcal{N}_0\cup\mathcal{N}_1$ is a $(7,5,4)$-covering and $G(\mathcal{N}_0)$ and $G(\mathcal{N}_1)$ are connected. Now, by taking $C_0=12456$, it follows that $G(\mathcal{N}_0\cup C_0\cup\mathcal{N}_1)$ is connected.

We may now show that $\mathrm{S}(n,r)>\mathrm{N}(n,r)$. For this we need first the following Theorem and Proposition.

Theorem 4.7. *Let r and n be positive integers such that $n\geqslant r+1\geqslant 3$. Then,*

$$\mathrm{S}(n,r)=\mathrm{N}(n,r)+\sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor-1}\left(\left\lfloor\frac{n-r}{2}\right\rfloor-i\right)\binom{r-2+2i}{r-3}+\delta_0\left(1-\mathrm{C}(n-2,r-2)\right),$$

where δ_0 is the parity function of $n-r$.

Proof. By induction on $n > r$. From (13) and (15), the identity is verified for $n = r+1$ and $n = r+2$. Suppose now that the identity is verified for a certain value of n and let D be the difference

$$D := (\mathrm{S}(n+2,r) - \mathrm{N}(n+2,r)) - (\mathrm{S}(n,r) - \mathrm{N}(n,r)).$$

Then

$$\mathrm{S}(n+2,r) = \mathrm{N}(n+2,r) + (\mathrm{S}(n,r) - \mathrm{N}(n,r)) + D.$$

By using (13), we obtain

$$\begin{aligned} \mathrm{S}(n{+}2,r){-}\mathrm{S}(n,r) &= \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor + 1} \binom{n+2-2i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor + 1 - \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-2i}{r-1} - \left\lfloor \frac{n-r}{2} \right\rfloor \\ &= \binom{n}{r-1} + 1. \end{aligned}$$

By using (15), we have

$$\begin{aligned} \mathrm{N}(n{+}2,r){-}\mathrm{N}(n,r) &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} (n+2-r-2i)\binom{r-2+2i}{r-2} + \left\lceil \frac{n-r}{2} \right\rceil + \delta_0 \mathrm{C}(n,r-2) \\ &\quad - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} (n-r-2i)\binom{r-2+2i}{r-2} - \left\lceil \frac{n-r}{2} \right\rceil + 1 - \delta_0 \mathrm{C}(n-2,r-2) \\ &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} 2\binom{r-2+2i}{r-2} + \left(n+2-r-2\left\lceil \frac{n-r}{2} \right\rceil\right)\binom{r-2+2\lceil \frac{n-r}{2} \rceil}{r-2} \\ &\quad + \delta_0 \left(\mathrm{C}(n,r-2) - \mathrm{C}(n-2,r-2)\right) + 1. \end{aligned}$$

Moreover, for $n-r$ odd, it follows that

$$\begin{aligned} \mathrm{N}(n+2,r) - \mathrm{N}(n,r) = &\sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2\binom{r-2+2i}{r-2} + (\delta_0 - 1)\binom{n-1}{r-2} \\ &+ \delta_0 \left(\mathrm{C}(n,r-2) - \mathrm{C}(n-2,r-2)\right) + 1. \end{aligned}$$

Therefore

$$D = \binom{n}{r-1} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2\binom{r-2+2i}{r-2} + (1-\delta_0)\binom{n-1}{r-2} + \delta_0 \left(\mathrm{C}(n-2,r-2) - \mathrm{C}(n,r-2)\right).$$

From the identity $\binom{r-2+2i}{r-2} = \binom{r-1+2i}{r-2} - \binom{r-2+2i}{r-3}$, we obtain that

$$\begin{aligned}\sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} 2\binom{r-2+2i}{r-2} &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-2} + \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-1+2i}{r-2} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &= \sum_{i=r-2}^{r-1+2\lceil \frac{n-r}{2} \rceil} \binom{i}{r-2} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &= \binom{r+2\left\lceil \frac{n-r}{2} \right\rceil}{r-1} - \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3}.\end{aligned}$$

Thus,

$$\begin{aligned}D = &\binom{n}{r-1} - \binom{r+2\left\lceil \frac{n-r}{2} \right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} + \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} \\ &+\delta_0 \left(\mathrm{C}(n-2, r-2) - \mathrm{C}(n, r-2)\right).\end{aligned}$$

If $n-r$ is even, then $\delta_0 = 1$ and

$$\binom{n}{r-1} - \binom{r+2\left\lceil \frac{n-r}{2} \right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} = \binom{n}{r-1} - \binom{n}{r-1} = 0.$$

Either, if $n-r$ is odd, then $\delta_0 = 0$ and

$$\begin{aligned}\binom{n}{r-1} - \binom{r+2\left\lceil \frac{n-r}{2} \right\rceil}{r-1} + (1-\delta_0)\binom{n-1}{r-2} &= \binom{n}{r-1} - \binom{n+1}{r-1} + \binom{n-1}{r-2} \\ &= -\binom{n}{r-2} + \binom{n-1}{r-2} \\ &= -\binom{n-1}{r-3}.\end{aligned}$$

It follows that

$$\begin{aligned}D &= \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil} \binom{r-2+2i}{r-3} + (\delta_0 - 1)\binom{n-1}{r-3} + \delta_0 \left(\mathrm{C}(n-2, r-2) - \mathrm{C}(n, r-2)\right) \\ &= \sum_{i=0}^{\lfloor \frac{n-r}{2} \rfloor} \binom{r-2+2i}{r-3} + \delta_0 \left(\mathrm{C}(n-2, r-2) - \mathrm{C}(n, r-2)\right).\end{aligned}$$

Now, with the induction hypothesis, we obtain

$$\begin{aligned}
\mathrm{S}(n{+}2,r){-}\mathrm{N}(n{+}2,r) &= (\mathrm{S}(n,r)-\mathrm{N}(n,r)) + D \\
&= \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor-1}\left(\left\lfloor\frac{n-r}{2}\right\rfloor - i\right)\binom{r-2+2i}{r-3} + \delta_0\left(1-\mathrm{C}(n-2,r-2)\right) \\
&\quad + \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor}\binom{r-2+2i}{r-3} + \delta_0\left(\mathrm{C}(n-2,r-2)-\mathrm{C}(n,r-2)\right) \\
&= \sum_{i=0}^{\lfloor\frac{n-r}{2}\rfloor}\left(\left\lfloor\frac{n-r}{2}\right\rfloor + 1 - i\right)\binom{r-2+2i}{r-3} + \delta_0\left(1-\mathrm{C}(n,r-2)\right).
\end{aligned}$$

□

Theorem 4.8. *Let r and n be positive integers such that $n-r$ is an even number. Then,*

$$\mathrm{S}(n,r) \geqslant \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-2}\left(\frac{n-r}{2} - i - 1\right)\binom{r-2+2i}{r-3}.$$

Proof. It is known [6, page 7] that $\mathrm{C}(n,r) \leqslant \binom{n-2}{r-1} + \mathrm{C}(n-2,r)$. By applying this inequality repeatedly we have

$$\mathrm{C}(n-2,r-2) \leqslant \sum_{i=0}^{\frac{n-r}{2}-1}\binom{r-2+2i}{r-3} + 1.$$

Then, we deduce from Theorem 4.7 that

$$\begin{aligned}
\mathrm{S}(n,r) &= \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-1}\left(\frac{n-r}{2} - i\right)\binom{r-2+2i}{r-3} + 1 - \mathrm{C}(n-2,r-2) \\
&\geqslant \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-1}\left(\frac{n-r}{2} - i\right)\binom{r-2+2i}{r-3} - \sum_{i=0}^{\frac{n-r}{2}-1}\binom{r-2+2i}{r-3} \\
&= \mathrm{N}(n,r) + \sum_{i=0}^{\frac{n-r}{2}-2}\left(\frac{n-r}{2} - i - 1\right)\binom{r-2+2i}{r-3}.
\end{aligned}$$

□

5. Asymptotics

In [16] Rödl uses the probabilistic method to show the existence of *asymptotically good coverings*. Restricted to our case this means that

$$\frac{\mathrm{C}(n,r)}{\binom{n}{r}} \to \frac{1}{r+1} \text{ as } n \to \infty.$$

Since $\mathrm{CC}(n,r) \leqslant 2\mathrm{C}(n,r)$ (see [3]) we immediately obtain:

$$\frac{\mathrm{CC}(n,r)}{\binom{n}{r}} \to a \leqslant \frac{2}{r+1} \text{ as } n \to \infty.$$

In [3] it was shown that

$$\frac{\mathrm{S}(n,r)}{\binom{n}{r}} \to \frac{1}{2} \text{ as } n \to \infty$$

and since by Theorem 4.8 the difference $\mathrm{N}(n,r) - \mathrm{S}(n,r)$ is in $\mathcal{O}(n^{r-1})$ we have the same asymptotic behavior for $\mathrm{N}(n,r)$.
It is however still a topic of research to find explicit constructions witnessing the bound of Rödl, see [12].

Acknowledgments

Much of this work in particular for the construction of the connected covering designs in Figures 1, 3, 4 strongly benefited from the La Jolla Covering Repository (`http://www.ccrwest.org/cover.html`) maintained by Dan Gordon.

References

[1] D. Applegate, E.M. Rains, N.J.A. Sloane, On asymmetric coverings and covering numbers, J. Combin. Des. 11 (2003), 218–228.
[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G.M. Ziegler, Oriented matroids, Cambridge University Press, Cambridge, England, second edition, 1999.
[3] D. Forge, J.L. Ramírez Alfonsín, Connected coverings and an application to oriented matroids, Discrete Math. 187 (1998), 109–121.
[4] D. Forge, J.L. Ramírez Alfonsín, H. Yeun, Disconnected coverings for oriented matroids via simultaneous mutations, Discrete Math. 258 (2002), 353–359.
[5] M.K. Fort Jr., G.A. Hedlund, Minimal coverings of pairs by triples, Pacific J. Math. 8 (1958), 709–719.
[6] D.M. Gordon, O. Patashnik, G. Kuperberg, New constructions for covering designs, J. Combin. Des. 3 (1995), 269–284.
[7] Y.O. Hamidoune, M. Las Vergnas, Directed switching games on graphs and matroids, J. Combin. Theory Ser. B 40 (1986), 237–269.
[8] L. Ji, An improvement on covering triples by quadruples, J. Combin. Des. 16 (2008), 231–243.
[9] G. Katona, T. Nemetz, M. Simonovits, On a problem of Turán in the theory of graphs, Mat. Lapok 15 (1964), 228–238.
[10] K. Knauer, L.P. Montejano, J.L. Ramírez Alfonsín, How many circuits determine an oriented matroid?, `http://arxiv.org/abs/1407.7388`, 17 pages.
[11] A.V. Kostochka, A class of constructions for Turán's (3, 4)-problem, Combinatorica 2 (1982), 187–192.
[12] N.N. Kuzjurin, Explicit constructions of Rödl's asymptotically good packings and coverings, Combin. Probab. Comput. 9 (2000), 265–276.
[13] W. Mantel, Problem 28, Wiskundige Opgaven 10 (1907), 320.
[14] W.H. Mills, On the covering of triples by quadruples, Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974), Congr. Numer. X (1974), 563–581.
[15] W.H. Mills, A covering of triples by quadruples, Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981), Congr. Numer. 33 (1981), 253–260.
[16] V. Rödl, On a packing and covering problem, European J. Combin. 6 (1985), 69–78.
[17] J. Schönheim, On coverings, Pacific J. Math. 14 (1964), 1405–1411.
[18] A.F. Sidorenko, The method of quadratic forms in a combinatorial problem of Turán, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 76 (1982), 3–6.
[19] A.F. Sidorenko, What we know and what we do not know about Turán numbers, Graphs Combin. 11 (1995), 179–199.
[20] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, Mat. Fiz. Lapok 48 (1941), 436–452.
[21] P. Turán, Research problems, Magyar Tud. Akad. Mat. Kut. Int. Közl. 6 (1961), 417–423.

Université Montpellier 2, Institut de Mathématiques et de Modélisation de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France.
E-mail address: jonathan.chappelon@um2.fr
E-mail address: kolja.knauer@googlemail.com
E-mail address: lpmontejano@gmail.com
E-mail address: jramirez@um2.fr