Matroid base polytope decomposition

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2[1]

(join work with V. Chatelain)

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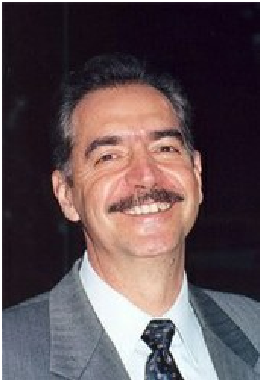
Convexity, Topology, Combinatorics and Beyond :

A workshop in the honor of Montejano's 60th birthday



Definitions

A matroid $M = (E, \mathcal{I})$ is a finite ground set $E = \{1, \ldots, n\}$ together with a collection $\mathcal{I} \subseteq 2^E$, known as independent sets satysfying

- if $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$
- if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exist an element $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$.

A base is any maximal independent set. The collection of bases $\mathcal{B}$ satisfy the base exchange axiom
if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exist $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

Remark : All bases have the same cardinality, say r. We say that matroid $M = (E, \mathcal{B})$ has rank $r = r(M)$.

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Examples

• Uniform matroids $U_{n,r}$ given by $E = \{1, \ldots, n\}$ and $\mathcal{I} = \{I \subseteq E : |I| \leq r\}$.

• Linear matroids Let $\mathbb{F}$ be a field, $A \in \mathbb{F}^{m \times n}$ an $(m \times n)$-matrix over $\mathbb{F}$. Let $E = \{1, \ldots, n\}$ be the index set of the columns of A. $I \subseteq E$ is independent if the columns indexed by I are linearly independent.

A matroid is said to be representable over $\mathbb{F}$ if it can be expressed as linear matroid with matrix A and linear independence taken over $\mathbb{F}$.

• Graphic matroid Let $G = (V, E)$ be an undirected graph. Matroid $M = (E, \mathcal{I})$ where $\mathcal{I} = \{F \subseteq E : F \text{ is acyclic}\}$.

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Applications

- Graph theory
- Combinatorial optimisation (via greedy characterization)
- Knot theory (Jone's polynomial)
- Hyperplane arrangements (via oriented matroids)
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- Rigidity
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Let $P(M)$ the base polytope of M defined as the convex hull of the incidence vectors of bases of M, that is,

$$P(M) := conv\left\{\sum_{i \in B} e_i : B \in \mathcal{B}(M)\right\}$$

where e_i denote the standard unit vector of $\mathbb{R}^n$

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Remark

(a) $P(M)$ is a polytope of dimension at most $n-1$.

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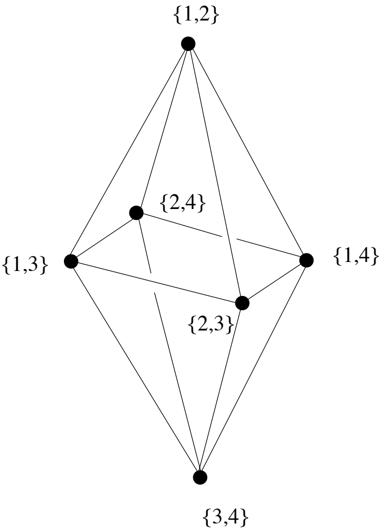
Exemple : $P(U_{4,2})$



A decomposition of $P(M)$ is a decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is also a base matroid polytope for some M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is a face of both $P(M_i)$ and $P(M_j)$.

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(L. Lafforgue) General method of compactification of the fine Schubert cell of the Grassmannian. It is proved that such compactification exists if the $P(M)$ is indecomposable.
Remark : Lafforgue's work implies that for a matroid M represented by vectors in $\mathbb{F}^r$, if $P(M)$ is indecomposable, then M will be rigid, that is, M will have only finitely many realizations, up to scaling and the action of $GL(r, \mathbb{F})$.

(Hacking, Keel and Tevelev) Compactification of the moduli space of hyperplane arrangements

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Known results

Theorem (Kapranov 1993) Any decomposition of a rank 2 matroid can be achieved by a sequence of hyperplane splits.

Theorem (Billera, Jia and Reiner 2009)

• Found five rank 3 matroids on 6 elements for which the corresponding polytopes are indecomposable.

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Combinatorial decomposition

A base decomposition of a matroid M is a decomposition

$$\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$$

where $\mathcal{B}(M_k)$, $1 \leq k \leq t$ and $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$, $1 \leq i \neq j \leq t$ are collections of bases of matroide.

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• If $P(M)$ is decomposable then M is clearly combinatorial decomposable.

• A combinatorial decomposition of M could not yield to a decomposition of $P(M)$.

Example

$\mathcal{B}(M) = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ has the following combinatorial decomposition

$\mathcal{B}(M_1) = \{\{1,2\},\{2,3\},\{2,4\}\}$ and
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We verify that $\mathcal{B}(M_1), \mathcal{B}(M_2)$ and $\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{2,3\}$ are collections of bases of matroids.

However, $P(M_1)$ and $P(M_2)$ do not decompose $P(M)$.

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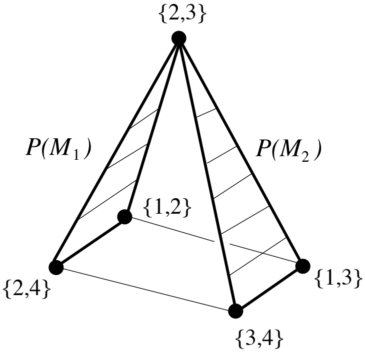
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Proposition Let P be d-polytope with set of vertices X. Let H be a hyperplane non-supporting P and $H \cap P \neq \emptyset$ (and so H partition X into X_1 and X_2 with $X_1 \cap X_2 = W$). Then, for each edge $[u, v]$ of P we have that $\{u, v\} \subset X_i$ for either $i = 1$ or 2 if and only if $P = P_1 \cup P_2$ with $P_i = conv(X_i)$, $i = 1, 2$.

Remark : P_i, $i = 1, 2$ is a polytope of the same dimension as P and they share the facet $conv(W)$.

Let (E_1, E_2) be a partition of E and et $r_i > 1$, $i = 1, 2$ be the rank of $M|_{E_i}$. We say that (E_1, E_2) is a good partition if there exist integers $0 < a_1 < r_1$ et $0 < a_2 < r_2$ such that :

(P1) $r_1 + r_2 = r + a_1 + a_2$ and

(P2) for any $X \in \mathcal{I}(M|_{E_1})$ with $|X| \leq r_1 - a_1$ and
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Lemma Let (E_1, E_2) be a good partition of E. Let
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Then, $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ are collections of bases of matroids, say M_1 et M_2.

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Theorem (Chatelain and R.A. 2011) Let $M = (E, \mathcal{B})$ be a matroid and let (E_1, E_2) a good partition of E. Then, $P(M) = P(M_1) \cup P(M_2)$ is a nontrivial hyperplane split where M_1 et M_2 are the matroids of lemma above.

We say that two hyperplane splits $P(M_1) \cup P(M_2)$ and $P(M_1') \cup P(M_2')$ of $P(M)$ are equivalent if $P(M_i)$ is combinatorially equivalent to $P(M_i')$, $i = 1, 2$. They are different otherwise.

Corollary (Chatelain and R.A. 2011) Let $n \geq r + 2 \geq 4$ be integers and let $h(U_{n,r})$ be the number of different hyperplane splits of $P(U_{n,r})$. Then,

$$h(U_{n,r}) \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

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Example Let us consider $U_{4,2}$. Then, $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$ is a good partition (and thus $r_1 = r_2 = 2$) with $a_1 = a_2 = 1$.

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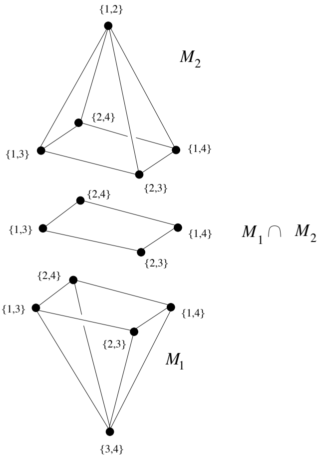

Lattice path matroids

Let $m = 3$ and $r = 4$ and let $M[Q, P]$ be the transversal matroid on $\{1, \ldots, 7\}$ with presentation $(N_i : i \in \{1, \ldots, 4\})$ where $N_1 = [1, 2, 3, 4]$, $N_2 = [3, 4, 5]$, $N_3 = [5, 6]$ and $N_4 = [7]$.

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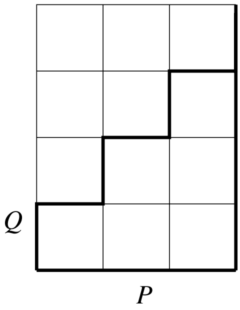
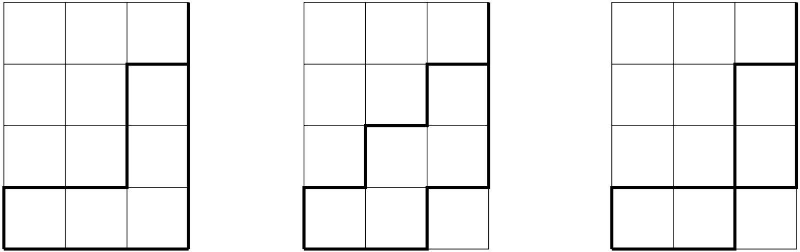


Example Transversal matroids (a) M_1, (b) M_2 et (c) $M_1 \cap M_2$.



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Theorem (Chatelain and R.A. 2011) Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be two matroids of ranks r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a nontrivial hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a nontrivial hyperplane split.

Remark The class of lattice path matroids are closed under direct sum.

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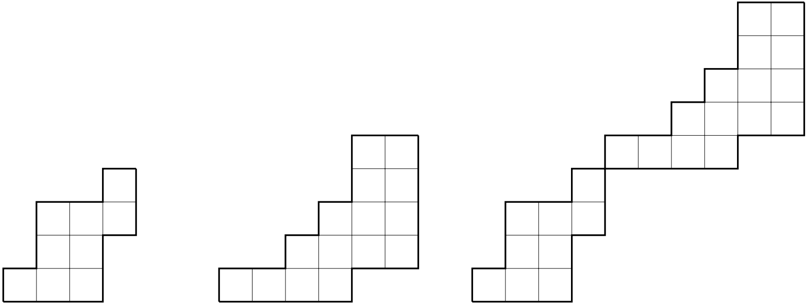

Binary matroids

A matroid is called binary if it is representable over $\mathbb{F}_2$.
Let $G(M)$ be the base graph of a matroid M ($G(M)$ is the 1-squeleton of $P(M)$).

Theorem (Maurer 1976) If x, y are two vertices at distance two then the neighbours of x and y form either a square or a pyramid or an octahedron.

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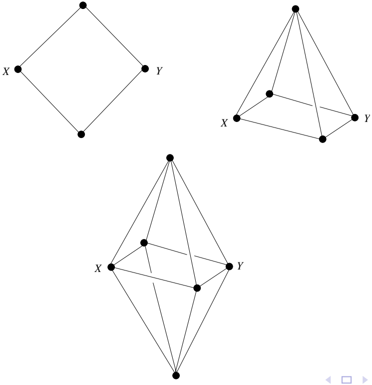

Lemma Let $M = (E, \mathcal{B})$ a binary matroid and let $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ is the collection of bases of a matroid, says M_1. If $X \in \mathcal{B}_1$ and all the neighbours of X are elements of $\mathcal{B}_1$ then $\mathcal{B}_1 = \mathcal{B}$.

Theorem (Chatelain and R.A. 2011) Let M be a binary matroid. Then, $P(M)$ do not have a nontrivial hyperplane split.

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