A Problem of Optimal Portfolio Allocation with Transaction Costs

Benoîte de Saporta

INRIA Sophia Antipolis
Team OMEGA

CHRISTOPHETTE BLANCHET (University of Nice)
RAJNA GIBSON (University of Zurich)
ETIENNE TANRÉ and DENIS TALAY (INRIA Sophia Antipolis)

Groupe de Travail Finance, march 2 2006

Outline

Outline

Stochastic Control Problem

- State of the system W_t subject to stochastic dynamics
- Control process π_t
  - value chosen at each time depending on the available information: adapted
  - influence the dynamics of W_t
- Performance Criterion $J(W, \pi)$ to be maximised

Value Function

$$V = \sup_{\pi} J(W, \pi)$$

Aim

- Compute or identify the value function
- Find an optimal strategy (if it exists)

Example: Merton's Problem

Statement I

Market: risk-free asset $dS_t^0 = S_t^0 r dt$

risky asset $dS_t = \mu S_t dt + \sigma S_t dB_t$

- Control process $\pi_t \in [0;1]$: proportion of the wealth invested in the risky asset
- State W_t: wealth process

$$\frac{dW_t^\pi}{W_t^\pi} = \pi_t \frac{dS_t}{S_t} + (1-\pi_t)\frac{dS_t^0}{S_t^0}$$

$$= \big(\pi_t\mu + (1-\pi_t)r\big)dt + \pi_t \sigma dB_t$$

Example: Merton's Problem

Statement II

- Criterion: expected utility of the terminal wealth

$$U(x) = x^{\alpha}, \qquad 0 < \alpha < 1$$

Value Function

$$V(t,x) = \sup_{\pi} \mathbb{E}[U(W_T^{t,x,\pi})]$$

Aim

- Compute or identify the value function
- Find an optimal strategy (if it exists)

Hamilton-Jacobi-Bellman (HJB) Equation

$$\begin{aligned} \frac{\partial \Phi}{\partial t}(t,x) + \sup_{p\in[0;1]} \mathcal{L}^p\Phi(t,x) &= 0 \\ \Phi(T,x) = U(x) &= x^\alpha \end{aligned}$$

$$\mathcal{L}^p\Phi(t,x) = x\left(p\mu + (1-p)r\right)\frac{\partial \Phi}{\partial x}(t,x) + \frac{1}{2}\sigma^2 p^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}(t,x)$$

As $U(W_T^{t,x,\pi}) = U(xW_T^{t,1,\pi}) = x^\alpha U(W_T^{t,1,\pi})$, we're searching for a factorized solution $\Phi(t,x) = x^\alpha \varphi(t)$

$$\begin{aligned} 0 &= \varphi'(t) + \varphi(t) \sup_{p\in[0;1]} \{\alpha\left(p\mu + (1-p)r\right) + \frac{\alpha(\alpha-1)}{2}p^2\sigma^2\} \\ 1 &= \varphi(T) \end{aligned}$$

Example: Merton's Problem
Solution of HJB Equation

Solution

$$\Phi(t,x) = x^{\alpha} e^{\beta(T-t)}$$

with

$$\beta = \sup_{p\in[0;1]} \{\alpha(p\mu + (1-p)r) + \frac{\alpha(\alpha-1)}{2} p^2\sigma^2\}$$
$$= \alpha r + \frac{\alpha(\mu - r)^2}{2(1-\alpha)\sigma^2}$$

reached at $p^* = \frac{\mu - r}{(1-\alpha)\sigma^2}$

Optimal Portfolio Allocation
9/39
Benoîte de Saporta
Introduction
Motivation
Framework
Value Function
Dynamic Programming Principle
Hamilton Jacobi Bellman Equations
Numerical Results
Value Function
Efficient Strategy
Comparison of Strategies

Itô formula for Φ between t and T:

$$\begin{aligned}
\Phi(T, W_T^{t,x,\pi}) &= \Phi(t,x) + \int_t^T \left(\frac{\partial \Phi}{\partial u} + \mathcal{L}^{\pi_u}\Phi\right)(u, W_u^{t,x,\pi})du \\
&\quad + \text{martingale} \\
U(W_T^{t,x,\pi}) &\leq \Phi(t,x) + \int_t^T \left(\frac{\partial \Phi}{\partial u} + \sup_{p\in[0;1]} \mathcal{L}^{p}\Phi\right)(u, W_u^{t,x,\pi})du \\
&\quad + \text{martingale} \\
&\leq \Phi(t,x) + \text{ martingale}
\end{aligned}$$

Hence

$$\Phi(t,x) \geq \mathbb{E}[U(W_T^{t,x,\pi})]$$

with equality when $\pi_t = p^*$.

Optimal Portfolio Allocation
10/39

Benoîte de Saporta

Introduction
Motivation
Framework
Value Function
Dynamic Programming Principle
Hamilton Jacobi Bellman Equations
Numerical Results
Value Function
Efficient Strategy
Comparison of Strategies

Conclusion

- $V(t,x) = x^{\alpha} e^{\beta(T-t)}$
- Constant optimal strategy $\pi_t = \frac{\mu - r}{(1-\alpha)\sigma^2}$
- V is a solution of Hamilton Jacobi Bellman equation

Example: Merton's Problem

Conclusion

Conclusion

- $V(t,x) = x^{\alpha} e^{\beta(T-t)}$
- Constant optimal strategy $\pi_t = \frac{\mu - r}{(1-\alpha)\sigma^2}$
- V is a solution of Hamilton Jacobi Bellman equation

Our Problem

Market:

risk-free asset	$dS_t^0 = S_t^0 r dt$
risky asset	$dS_t = \mu(t) S_t dt + \sigma S_t dB_t$

- the drift alternately takes two different values $\mu_1 < 0$ and $\mu_2 > 0$
- transaction costs

Outline

Motivation

Main approaches to investment

- Fundamental approach
  - fundamental economic principles
- Technical Analysis approach
  - past prices behaviour
- Mathematical Approach
  - mathematical models

Aim

Compare the performance of technical analysis and miscalibrated mathematical models

Framework

Market:		
	risk-free asset	$dS^0_t = S^0_t r dt$
	risky asset	$dS_t = \mu(t) S_t dt + \sigma S_t dB_t$

- B standard Brownian motion,
- $\mu(t) \in \{\mu_1, \mu_2\}$ independent of B,
- control process $\pi_t \in \{0, 1\}$ proportion of the wealth invested in the risky asset
- state W^π_t wealth when strategy π is applied
- criterion expected utility of the terminal wealth

Aim

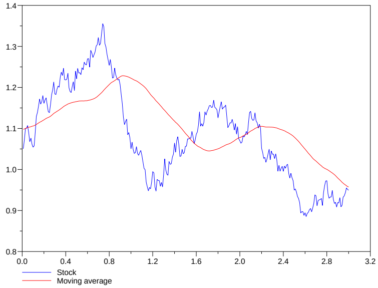
Maximise the expected utility of the terminal wealth

Technical Analyst strategy

Moving average

$$M_t^\delta = \frac{1}{\delta}\int_{t-\delta}^{t} S_u du$$

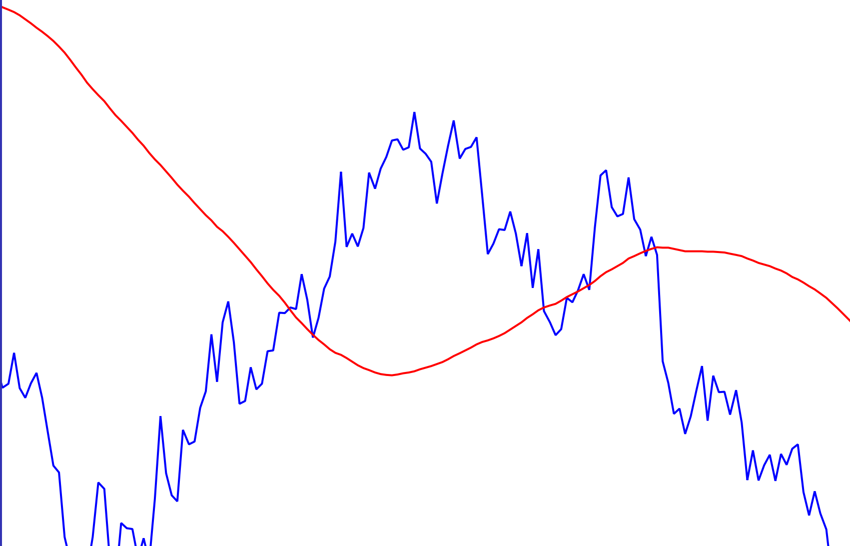
- If $S_t > M_t^\delta$ buy
- If $S_t < M_t^\delta$ sell



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\delta = 0.8$.

![INRIA Sophia Antipolis logo]

Technical Analyst strategy



Previous Work

Framework

BLANCHET, DIOP, GIBSON, KAMINSKI, TALAY, TANRÉ (2005)

risk-free asset	$dS^0_t = S^0_t r dt,$
risky asset	$dS_t = \mu(t) S_t dt + \sigma S_t dB_t,$

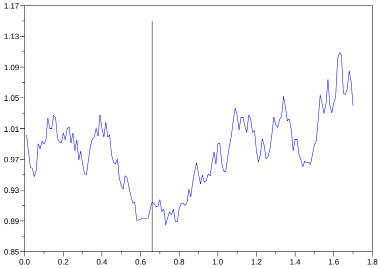
One change of drift

- $\mu(t) = \mu_1$ if $t < \tau$
- $\mu(t) = \mu_2$ if $t \geq \tau$

with $\mathbb{P}(\tau > t) = e^{-\lambda t}$

Strategy

detect τ



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda = 2$.

Previous Work

Results

- theoretical study of the value function
- theoretical study of detecting the change of drift
- numerical comparisons of strategies
  - well calibrated detection
  - miscalibrated detection
  - moving average

Conclusion

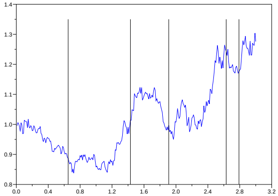
- Moving average strategy can overperform miscalibrated mathematical strategies
- Range of misspecifications for which this is true

Outline

New Model

- Several changes of drift
  (ξ_{2n+1}) iid Exp(λ_1)
  (ξ_{2n}) iid Exp(λ_2)
  $\tau_0 = 0$, $\tau_n = \xi_1 + \cdots + \xi_n$

$$\mu(t) = \begin{cases} \mu_1 & \text{if } \tau_{2n} \leq t < \tau_{2n+1} \\ \mu_2 & \text{if } \tau_{2n+1} \leq t < \tau_{2n+2} \end{cases}$$



$\mu_1 = -0.2$, $\mu_2 = 0.2$,
$\sigma = 0.15$, $\lambda_1 = \lambda_2 = 2$.

- Transaction costs
  - g_{01} buying cost
  - g_{10} selling cost

Admissible strategies

Control process: $\pi_t \in \{0, 1\}$ proportion of the wealth invested in the risky asset

$$\mathcal{F}_t^S = \sigma(S_u, u \leq t)$$

π_t must be $\mathcal{F}_t^S$-adapted

Problem

$$\mathcal{F}_t^S \neq \mathcal{F}_t^B = \sigma(B_u, u \leq t)$$

$\Longrightarrow$ Change of framework

Filtering Theory

Optional Projection: $F_t = \mathbb{P}(\mu(t) = \mu_1 \mid \mathcal{F}_t^S)$

$$\overline{B}_t = \frac{1}{\sigma}\Big(\log\frac{S_t}{S_0} - \int_0^t \big(\mu_1 F_s + \mu_2(1-F_s) - \frac{\sigma^2}{2}\big)ds\Big)$$

Martinez, Rubenthaler, Tanré 2005

- $\overline{B}$ is a $(\mathcal{F}^S)$ Brownian motion
- $\mathcal{F}^S = \mathcal{F}^{\overline{B}}$

$$\frac{dS_t}{S_t} = \big(\mu_1 F_t + \mu_2(1-F_t)\big)dt + \sigma d\overline{B}_t$$

Kurtz, Ocone 1988

$$dF_t = \big(-\lambda_1 F_t + \lambda_2(1-F_t)\big)dt + \frac{\mu_1-\mu_2}{\sigma}F_t(1-F_t)d\overline{B}_t$$

New Framework

Control process: π_t

State: pair (W_t, F_t)
Dynamics:

$$\frac{dW_t^\pi}{W_{t-}^\pi} = \big(\pi_t(\mu_1 F_t + \mu_2(1-F_t)) + (1-\pi_t)r\big)dt + \pi_t \sigma d\bar{B}_t$$
$$-g_{01}\delta(\Delta\pi_t = 1) - g_{10}\delta(\Delta\pi_t = -1)$$

$$dF_t = \big(-\lambda_1 F_t + \lambda_2(1-F_t)\big)dt + \frac{\mu_1 - \mu_2}{\sigma} F_t(1-F_t)d\bar{B}_t,$$

Criterion: expected utility of the terminal wealth
Utility: $U(x) = x^\alpha$, $\alpha \in]0,1[$

Outline

Continuity

Value Function

$$
\begin{aligned}
V^0(t,x,f) &= \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 0, W_{t-}^\pi = x, F_t = f] \\
V^1(t,x,f) &= \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 1, W_{t-}^\pi = x, F_t = f]
\end{aligned}
$$

Continuity

For all $i \in \{0;1\}$, $0 \leq t \leq \hat{t} \leq T$, $x, \hat{x} > 0$, $0 \leq f, \hat{f} \leq 1$:

$$
\begin{aligned}
&\left|V^i(\hat{t},\hat{x},\hat{f}) - V^i(t,x,f)\right| \\
&\quad \leq \; C(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})(|\hat{x}-x| + x(|\hat{f}-f| + |\hat{t}-t|^{1/2}))
\end{aligned}
$$

Continuity

Value Function

$$V^0(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 0, W_{t-}^\pi = x, F_t = f]$$

$$V^1(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 1, W_{t-}^\pi = x, F_t = f]$$

Continuity

For all $i \in \{0;1\}$, $0 \le t \le \hat{t} \le T$, $x, \hat{x} > 0$, $0 \le f, \hat{f} \le 1$:

$$\left|V^i(\hat{t},\hat{x},\hat{f}) - V^i(t,x,f)\right| \le C(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})\left(|\hat{x} - x| + x(|\hat{f} - f| + |\hat{t} - t|^{1/2})\right)$$

Dynamic Programming Principle

Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and x, f, i:

$$V^i(s,x,f) = \sup_\pi \mathbb{E}[V^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f})]$$

Proof:

$$\begin{aligned} J^i(s,x,f,\pi) &= \mathbb{E}[U(W_T^{s,x,f,\pi})] = \mathbb{E}[\mathbb{E}[U(W_T^{s,x,f,\pi}) \mid \mathcal{F}_{s,t}]] \\ &= \mathbb{E}[J^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f}, \pi)] \\ &\leq \mathbb{E}[V^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f})] \end{aligned}$$

Dynamic Programming Principle

Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and x, f, i:

$$V^i(s,x,f) = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

Proof:

$$\begin{aligned} J^i(s,x,f,\pi) &= \mathbb{E}[U(W_T^{s,x,f,\pi})] = \mathbb{E}\big[\mathbb{E}[U(W_T^{s,x,f,\pi}) \mid \mathcal{F}_{s,t}]\big] \\ &= \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi)] \\ &\leq \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \end{aligned}$$

Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and x, f, i:

$$V^i(s,x,f) = \sup_\pi \mathbb{E}[V^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f})]$$

Proof:

$$\begin{aligned} J^i(s,x,f,\pi) &= \mathbb{E}[U(W_T^{s,x,f,\pi})] = \mathbb{E}\big[\mathbb{E}[U(W_T^{s,x,f,\pi}) \mid \mathcal{F}_{s,t}]\big] \\ &= \mathbb{E}[J^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f}, \pi)] \\ &\leq \mathbb{E}[V^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_t^{s,f})] \end{aligned}$$

Dynamic Programming Principle

Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and x, f, i:

$$V^i(s,x,f) = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

Proof:

$$\begin{aligned} J^i(s,x,f,\pi) &= \mathbb{E}[U(W_T^{s,x,f,\pi})] = \mathbb{E}\left[\mathbb{E}[U(W_T^{s,x,f,\pi}) \mid \mathcal{F}_{s,t}]\right] \\ &= \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi)] \\ &\leq \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \end{aligned}$$

Dynamic Programming Principle
Proof I

Fix π such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

$(\mathcal{B}_p)_{p\in\mathbb{N}}$ partition of $]0;+\infty[\times[0;1]$ such that for all i, (x,f) and $(\hat{x},\hat{f})$ in $\mathcal{B}_p$ and for all π:

$$|V^i(t,x,f) - V^i(t,\hat{x},\hat{f})| \leq \varepsilon, \; |J^i(t,x,f,\pi) - J^i(t,\hat{x},\hat{f},\pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

fix (x_p, f_p) in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Dynamic Programming Principle
Proof I

Fix π such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

$(\mathcal{B}_p)_{p \in \mathbb{N}}$ partition of $]0; +\infty[\times [0; 1]$ such that
for all i, (x, f) and $(\hat{x}, \hat{f})$ in $\mathcal{B}_p$ and for all π:

$$|V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \varepsilon, \; |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

fix (x_p, f_p) in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Dynamic Programming Principle

Proof I

Fix π such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

$(\mathcal{B}_p)_{p\in\mathbb{N}}$ partition of $]0;+\infty[\times[0;1]$ such that for all i, (x,f) and $(\hat{x},\hat{f})$ in $\mathcal{B}_p$ and for all π:

$$|V^i(t,x,f) - V^i(t,\hat{x},\hat{f})| \leq \varepsilon, \; |J^i(t,x,f,\pi) - J^i(t,\hat{x},\hat{f},\pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

fix (x_p, f_p) in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Dynamic Programming Principle

Proof I

Fix π such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

$(\mathcal{B}_p)_{p\in\mathbb{N}}$ partition of $]0; +\infty[\times[0;1]$ such that for all i, (x, f) and $(\hat{x}, \hat{f})$ in $\mathcal{B}_p$ and for all π:

$$|V^i(t,x,f) - V^i(t,\hat{x},\hat{f})| \leq \varepsilon, \ |J^i(t,x,f,\pi) - J^i(t,\hat{x},\hat{f},\pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f})\in\mathcal{B}_p}]$$

fix (x_p, f_p) in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f})\in\mathcal{B}_p}]$$

Dynamic Programming Principle
Proof I

Fix π such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f})]$$

$(\mathcal{B}_p)_{p\in\mathbb{N}}$ partition of $]0; +\infty[\times [0; 1]$ such that for all i, (x, f) and $(\hat{x}, \hat{f})$ in $\mathcal{B}_p$ and for all π:

$$|V^i(t,x,f) - V^i(t,\hat{x},\hat{f})| \leq \varepsilon, \ |J^i(t,x,f,\pi) - J^i(t,\hat{x},\hat{f},\pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

fix (x_p, f_p) in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Dynamic Programming Principle
Proof II

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p)\mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

For fixed p, i, let $\pi^{p,i}$ be a strategy on $[t, T]$ such that

$$V^i(t, x_p, f_p) \leq \varepsilon + J^i(t, x_p, f_p, \pi^{p,i})$$

$$\begin{aligned} \mathcal{V} &\leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, x_p, f_p, \pi^{p,\pi_{t-}})\mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \\ &\leq \varepsilon + 2\varepsilon + \varepsilon \\ &\quad + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}})\mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \end{aligned}$$

Dynamic Programming Principle

Proof II

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

For fixed p, i, let $\pi^{p,i}$ be a strategy on $[t, T]$ such that

$$V^i(t, x_p, f_p) \leq \varepsilon + J^i(t, x_p, f_p, \pi^{p,i})$$

$$\begin{aligned}\mathcal{V} &\leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, x_p, f_p, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \\ &\leq \varepsilon + 2\varepsilon + \varepsilon \\ &\quad + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]\end{aligned}$$

Dynamic Programming Principle

Proof II

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

For fixed p, i, let $\pi^{p,i}$ be a strategy on $[t, T]$ such that

$$V^i(t, x_p, f_p) \leq \varepsilon + J^i(t, x_p, f_p, \pi^{p,i})$$

$$\begin{aligned}\mathcal{V} &\leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, x_p, f_p, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \\ &\leq \varepsilon + 2\varepsilon + \varepsilon \\ &\quad + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]\end{aligned}$$

Dynamic Programming Principle
Proof II

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_{t-}}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

For fixed p, i, let $\pi^{p,i}$ be a strategy on $[t, T]$ such that

$$V^i(t, x_p, f_p) \leq \varepsilon + J^i(t, x_p, f_p, \pi^{p,i})$$

$$\begin{aligned} \mathcal{V} &\leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, x_p, f_p, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \\ &\leq \varepsilon + 2\varepsilon + \varepsilon \\ &\quad + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}] \end{aligned}$$

Dynamic Programming Principle
Proof III

$$\mathcal{V} \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}})\mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Combining step

$$\hat{\pi}_u = \begin{cases} \pi_u & \text{if } s \leq u < t \\ \pi_u^{p,\hat{\pi}_{t-}} & \text{if } u \geq t, \ \text{ and } (W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}) \in \mathcal{B}_p \end{cases}$$

$$\begin{aligned} \mathcal{V} &\leq 4\varepsilon + \mathbb{E}[J^{\hat{\pi}_{t-}}(t, W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}, \hat{\pi})] \\ &= 4\varepsilon + \mathbb{E}\left[\mathbb{E}[U(W_T^{s,x,f,\hat{\pi}}) \mid \mathcal{F}_{s,t}]\right] \\ &= 4\varepsilon + \mathbb{E}[U(W_T^{s,x,f,\hat{\pi}})] \\ &\leq 4\varepsilon + \mathbb{E}[V^i(s, x, f)] \end{aligned}$$

Dynamic Programming Principle

Proof III

$$\mathcal{V} \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}}) \mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f}) \in \mathcal{B}_p}]$$

Combining step

$$\hat{\pi}_u = \begin{cases} \pi_u & \text{if } s \leq u < t \\ \pi_u^{p,\hat{\pi}_{t-}} & \text{if } u \geq t, \quad \text{and } (W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}) \in \mathcal{B}_p \end{cases}$$

$$\begin{aligned} \mathcal{V} &\leq 4\varepsilon + \mathbb{E}[J^{\hat{\pi}_{t-}}(t, W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}, \hat{\pi})] \\ &= 4\varepsilon + \mathbb{E}\left[\mathbb{E}[U(W_T^{s,x,f,\hat{\pi}}) \mid \mathcal{F}_{s,t}]\right] \\ &= 4\varepsilon + \mathbb{E}[U(W_T^{s,x,f,\hat{\pi}})] \\ &\leq 4\varepsilon + \mathbb{E}[V^i(s,x,f)] \end{aligned}$$

Dynamic Programming Principle
Proof III

$$\mathcal{V} \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_t^{s,f}, \pi^{p,\pi_{t-}})\mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_t^{s,f})\in\mathcal{B}_p}]$$

Combining step

$$\hat{\pi}_u = \begin{cases} \pi_u & \text{if } s \leq u < t \\ \pi_u^{p,\hat{\pi}_{t-}} & \text{if } u \geq t, \quad \text{and } (W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}) \in \mathcal{B}_p \end{cases}$$

$$\begin{aligned} \mathcal{V} &\leq 4\varepsilon + \mathbb{E}[J^{\hat{\pi}_{t-}}(t, W_{t-}^{s,x,f,\hat{\pi}}, F_t^{s,f}, \hat{\pi})] \\ &= 4\varepsilon + \mathbb{E}\left[\mathbb{E}[U(W_T^{s,x,f,\hat{\pi}}) \mid \mathcal{F}_{s,t}]\right] \\ &= 4\varepsilon + \mathbb{E}[U(W_T^{s,x,f,\hat{\pi}})] \\ &\leq 4\varepsilon + \mathbb{E}[V^i(s,x,f)] \end{aligned}$$

Hamilton Jacobi Bellman Equations

$$\begin{cases} \min\left\{-\dfrac{\partial\varphi^0}{\partial t}-\mathcal{L}^0\varphi^0;\ \varphi^0(t,x,f)-\varphi^1(t,x(1-g_{01}),f)\right\}=0\\ \min\left\{-\dfrac{\partial\varphi^1}{\partial t}-\mathcal{L}^1\varphi^1;\ \varphi^1(t,x,f)-\varphi^0(t,x(1-g_{10}),f)\right\}=0\end{cases}$$

$$\begin{aligned}\mathcal{L}^0\varphi(t,x,f) &= xr\frac{\partial\varphi}{\partial x}(t,x,f)+\big(-\lambda_1 f+\lambda_2(1-f)\big)\frac{\partial\varphi}{\partial f}(t,x,f)+\\ &\quad \frac{1}{2}\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 f^2(1-f)^2\frac{\partial^2\varphi}{\partial f^2}(t,x,f)\end{aligned}$$

$$\begin{aligned}\mathcal{L}^1\varphi(t,x,f) &= x(\mu_1 f+\mu_2(1-f))\frac{\partial\varphi}{\partial x}(t,x,f)+\frac{1}{2}x^2\sigma^2\frac{\partial^2\varphi}{\partial x^2}(t,x,f)\\ &\quad +\big(-\lambda_1 f+\lambda_2(1-f)\big)\frac{\partial\varphi}{\partial f}(t,x,f)+x(\mu_1-\mu_2)f(1-f)\frac{\partial^2\varphi}{\partial x\partial f}(t,x,f)\\ &\quad +\frac{1}{2}\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 f^2(1-f)^2\frac{\partial^2\varphi}{\partial f^2}(t,x,f)\end{aligned}$$

Viscosity Solutions

$$\text{(P)} \qquad F(t,x,v(t,x),D_t v(t,x),Dv(t,x),D^2 v(t,x)) = 0$$

Definition

- v is a viscosity sub-solution of (P) if

$$F(\bar t,\bar x,v(\bar t,\bar x),D_t\varphi(\bar t,\bar x),D\varphi(\bar t,\bar x),D^2\varphi(\bar t,\bar x)) \leq 0$$

  for all $(\bar t,\bar x)$ and all functions φ $C^{1,2}$ such that $(\bar t,\bar x)$ is a local maximum of $v-\varphi$

- v is a viscosity super-solution of (P) if

$$F(\bar t,\bar x,v(\bar t,\bar x),D_t\varphi(\bar t,\bar x),D\varphi(\bar t,\bar x),D^2\varphi(\bar t,\bar x)) \geq 0$$

  for all $(\bar t,\bar x)$ and all functions φ $C^{1,2}$ such that $(\bar t,\bar x)$ is a local minimum of $v-\varphi$

Identification of the Value Function

$\mathcal{V}_\alpha$: set of continuous functions φ on $[0;T]\times[0;+\infty[\times[0;1]$ satisfying $\varphi(t,0,f)=0$ and

$$\sup_{[0;T]\times]0;+\infty[^2\times[0;1]^2}\frac{|\varphi(t,x,f)-\varphi(t,\hat{x},\hat{f})|}{(1+x^{\alpha-1}+\hat{x}^{\alpha-1})(|x-\hat{x}|+x|f-\hat{f}|)}<\infty.$$

Theorem

(V^0,V^1) is the unique viscosity solution of HJB equation in $\mathcal{V}_\alpha\times\mathcal{V}_\alpha$ satisfying

$$V^0(T,x,f)=V^1(T,x,f)=U(x)=x^\alpha$$

Outline

Discretization Scheme

Dependence on x:

$$V^i(t,x,f) = \sup_{\pi} \mathbb{E}[U(W_T^{t,x,f,\pi})] = x^{\alpha} V^i(t,1,f)$$

Numerical Scheme

- $\hat{V}^0(T,f) = \hat{V}^1(T,f) = 1$
- With the PDE part in HJB, compute $\overline{V^0}(t,\cdot)$ et $\overline{V^1}(t,\cdot)$ from $\hat{V}^0(t+dt,\cdot)$ and $\hat{V}^1(t+dt,\cdot)$
- Comparison
  - if $\overline{V^0}(t,f) \geq (1-g_{01})^{\alpha}\overline{V^1}(t,f)$, set $\hat{V}^0(t,f) = \overline{V^0}(t,f)$ otherwise $\hat{V}^0(t,f) = (1-g_{01})^{\alpha}\overline{V^1}(t,f)$
  - if $\overline{V^1}(t,f) \geq (1-g_{10})^{\alpha}\overline{V^0}(t,f)$, set $\hat{V}^1(t,f) = \overline{V^1}(t,f)$ otherwise $\hat{V}^1(t,f) = (1-g_{10})^{\alpha}\overline{V^0}(t,f)$

Discretization Scheme

Dependence on x:

$$V^i(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^{t,x,f,\pi})] \;=\; x^\alpha V^i(t,1,f)$$

Numerical Scheme

- $\hat{V}^0(T,f) = \hat{V}^1(T,f) = 1$
- With the PDE part in HJB, compute $\overline{V^0}(t,\cdot)$ et $\overline{V^1}(t,\cdot)$ from $\hat{V}^0(t+dt,\cdot)$ and $\hat{V}^1(t+dt,\cdot)$
- Comparison
  - if $\overline{V^0}(t,f) \geq (1-g_{01})^\alpha \overline{V^1}(t,f)$, set $\hat{V}^0(t,f) = \overline{V^0}(t,f)$ otherwise $\hat{V}^0(t,f) = (1-g_{01})^\alpha \overline{V^1}(t,f)$
  - if $\overline{V^1}(t,f) \geq (1-g_{10})^\alpha \overline{V^0}(t,f)$, set $\hat{V}^1(t,f) = \overline{V^1}(t,f)$ otherwise $\hat{V}^1(t,f) = (1-g_{10})^\alpha \overline{V^0}(t,f)$

Discretization Scheme

Dependence on x:

$$V^i(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^{t,x,f,\pi})] \; = \; x^\alpha V^i(t,1,f)$$

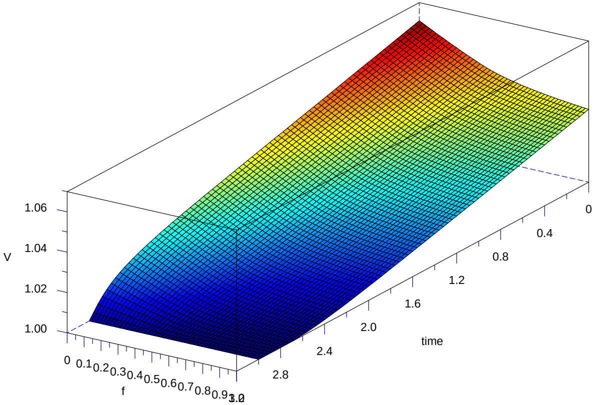
Numerical Scheme

- $\hat{V}^0(T,f) = \hat{V}^1(T,f) = 1$
- With the PDE part in HJB, compute $\overline{V^0}(t,\cdot)$ et $\overline{V^1}(t,\cdot)$ from $\hat{V}^0(t+dt,\cdot)$ and $\hat{V}^1(t+dt,\cdot)$
- Comparison
  - if $\overline{V^0}(t,f) \geq (1-g_{01})^\alpha \overline{V^1}(t,f)$, set $\hat{V}^0(t,f) = \overline{V^0}(t,f)$ otherwise $\hat{V}^0(t,f) = (1-g_{01})^\alpha \overline{V^1}(t,f)$
  - if $\overline{V^1}(t,f) \geq (1-g_{10})^\alpha \overline{V^0}(t,f)$, set $\hat{V}^1(t,f) = \overline{V^1}(t,f)$ otherwise $\hat{V}^1(t,f) = (1-g_{10})^\alpha \overline{V^0}(t,f)$

Value Function V^0

Shape

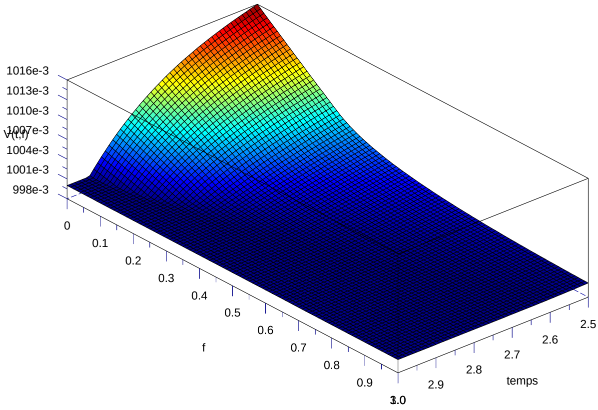
Parameters: $T = 3$, $\mu_2 = -\mu_1 = 0.2$, $\lambda_1 = \lambda_2 = 2$, $\sigma = 0.15$, $g_{01} = g_{10} = 0.001$



Value Function V^0

Regularity I

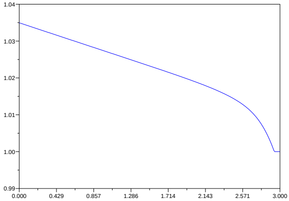
Transaction costs $g_{01} = g_{10} = 0.01$
Zoom between $t = 2.5$ and $t = 3 = T$



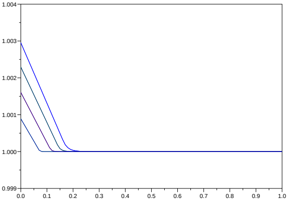
Value Function V^0

Regularity II

Section at $f = 0.05$

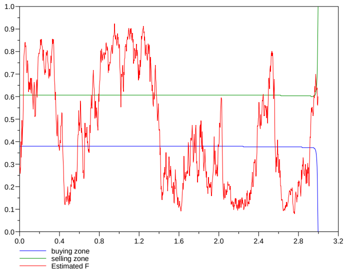


Sections at $t = 2.90$, $t = 2.91$, $t = 2.92$, $t = 2.93$,



Efficient Strategy

- Compute $\hat{V}^0$, $\hat{V}^1$
- Estimate $\hat{F}_t$ from the stock
- Compare $\hat{V}^0(t, \hat{F}_t)$ et $\hat{V}^1(t, \hat{F}_t)$:
  - buy if $\hat{V}^0(t, \hat{F}_t) = (1 - g_{01})^\alpha \hat{V}^1(t, \hat{F}_t)$
  - sell if $\hat{V}^1(t, \hat{F}_t) = (1 - g_{10})^\alpha \hat{V}^0(t, \hat{F}_t)$



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda_1 = 2$, $\lambda_2 = 2$, $T = 3$

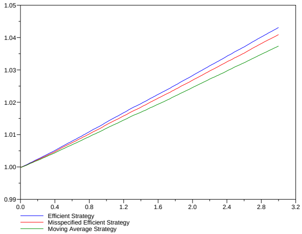
Efficient Strategy vs Value Function

- Computation of the value function:
  - time discretization step 10^{-6}
  - space discretization step 10^{-3}
- 10^5 Monte Carlo simulations of the Efficient Strategy

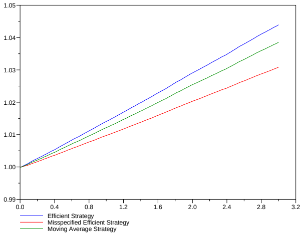
F_0	0	0.1	0.2	0.3	0.4	0.5
$\hat{V}^0$	1.061	1.057	1.053	1.049	1.045	1.043
Strategy	1.061	1.056	1.052	1.049	1.045	1.043

F_0	0.6	0.7	0.8	0.9	1
$\hat{V}^0$	1.041	1.039	1.038	1.037	1.036
Strategy	1.040	1.039	1.038	1.037	1.036

Miscalibrated Efficient Strategy vs Moving Average



miscalibrated parameters:
$\mu_1 = -1.8$, $\mu_2 = 1.8$, $\sigma = 0.15$,
$\lambda_1 = 4$, $\lambda_2 = 4$



miscalibrated parameters:
$\mu_1 = -1.8$, $\mu_2 = 1.8$, $\sigma = 0.25$,
$\lambda_1 = 4$, $\lambda_2 = 4$

Real parameters: $\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda_1 = 2$, $\lambda_2 = 2$, $T = 3$, $\delta = 0.8$
100000 Monte Carlo Simulations