Optimal Portfolio Allocation under Transaction Costs

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Outline

Outline

Motivation

Main approaches to investment

- Fundamental approach
  - fundamental economic principles
- Technical Analysis approach
  - past prices behaviour
- Mathematical Approach
  - mathematical models

Aim

Compare the performance of technical analysis versus miscalibrated mathematical models

Framework

Market: risk-free asset $dS_t^0 = S_t^0 r dt$
risky asset $dS_t = \mu(t) S_t dt + \sigma S_t dB_t$

- B standard Brownian motion,
- $\mu(t) \in \{\mu_1, \mu_2\}$ independent of B,
- $\pi_t \in \{0, 1\}$ proportion of the wealth invested in the risky asset
- W_t^π wealth when strategy π is applied

Aim

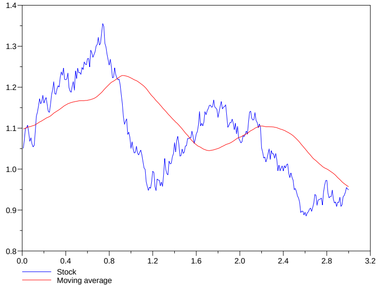
Maximise the expected utility of the terminal wealth

Technical Analyst strategy

Moving average

$$M_t^\delta = \frac{1}{\delta}\int_{t-\delta}^{t} S_u du$$

- If $S_t > M_t^\delta$ buy
- If $S_t < M_t^\delta$ sell



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\delta = 0.8$.

BLANCHET, DIOP, GIBSON, KAMINSKI, TALAY, TANRÉ (2005)

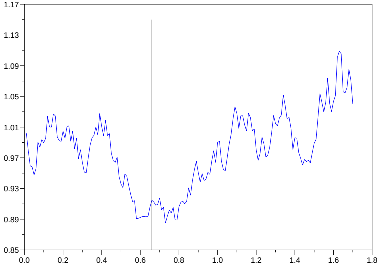
One change of drift

- $\mu(t) = \mu_1$ if $t < \tau$
- $\mu(t) = \mu_2$ if $t \geq \tau$

with $\mathbb{P}(\tau > t) = e^{-\lambda t}$

Strategy

detect τ



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda = 2$.

Previous Work
Results

- theoretical study of the value function
- theoretical study of detecting the change of drift
- numerical comparisons of strategies
  - well calibrated detection
  - miscalibrated detection
  - moving average

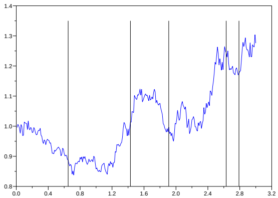
Conclusion

- Moving average strategy can overperform miscalibrated mathematical strategies
- Range of misspecifications for which this is true

Outline

- Several changes of drift
  (ξ_{2n+1}) iid $\mathrm{Exp}(\lambda_1)$
  (ξ_{2n}) iid $\mathrm{Exp}(\lambda_2)$
  $\tau_0 = 0$, $\tau_n = \xi_1 + \cdots + \xi_n$

$$\mu(t) = \begin{cases} \mu_1 & \text{if } \tau_{2n} \leq t < \tau_{2n+1} \\ \mu_2 & \text{if } \tau_{2n+1} \leq t < \tau_{2n+2} \end{cases}$$



$\mu_1 = -0.2$, $\mu_2 = 0.2$,
$\sigma = 0.15$, $\lambda_1 = \lambda_2 = 2$.

- Transaction costs
  - g_{01} buying cost
  - g_{10} selling cost

Admissible strategies

Control process: $\pi_t \in \{0, 1\}$ proportion of the wealth invested in the risky asset

$\mathcal{F}_t^S = \sigma(S_u, u \leq t)$

π_t must be $\mathcal{F}_t^S$-adapted

Problem

$$\mathcal{F}_t^S \neq \mathcal{F}_t^B = \sigma(B_u, u \leq t)$$

$\Longrightarrow$ Change of framework

Filtering Theory

Optional Projection: $F_t = \mathbb{P}(\mu(t) = \mu_1 \mid \mathcal{F}_t^S)$

$$\overline{B}_t = \frac{1}{\sigma}\Big(\log\frac{S_t}{S_0} - \int_0^t (\mu_1 F_s + \mu_2(1-F_s)) - \frac{\sigma^2}{2})ds\Big)$$

Proposition

- $\overline{B}$ is a $(\mathcal{F}^S)$ Brownian motion
- $\mathcal{F}^S = \mathcal{F}^{\overline{B}}$

$$\frac{dS_t}{S_t} = \big(\mu_1 F_t + \mu_2(1-F_t)\big)dt + \sigma d\overline{B}_t$$

Kurtz, Ocone 1988

$$dF_t \;=\; \big(-\lambda_1 F_t + \lambda_2(1-F_t)\big)dt + \frac{\mu_1-\mu_2}{\sigma}F_t(1-F_t)d\overline{B}_t$$

New Framework

Control process: π_t

State: pair (W_t, F_t)
Dynamics:

$$\frac{dW_t^\pi}{W_{t-}^\pi} = \big(\pi_t(\mu_1 F_t + \mu_2(1-F_t)) + (1-\pi_t)r\big)dt + \pi_t \sigma d\bar{B}_t - g_{01}\delta(\Delta\pi_t = 1) - g_{10}\delta(\Delta\pi_t = -1)$$

$$dF_t = \big(-\lambda_1 F_t + \lambda_2(1-F_t)\big)dt + \frac{\mu_1-\mu_2}{\sigma}F_t(1-F_t)d\bar{B}_t,$$

Criterion: expected utility of the terminal wealth
Utility: $U(x) = x^\alpha$, $\alpha \in]0,1[$

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Criterion: expected utility of the terminal wealth
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Outline

Definition

If $\pi_{t-} = 0$

$$J^0(t,x,f,\pi) = \mathbb{E}[U(W_T^\pi) \mid W_{t-}^\pi = x, F_t = f]$$

If $\pi_{t-} = 1$

$$J^1(t,x,f,\pi) = \mathbb{E}[U(W_T^\pi) \mid W_{t-}^\pi = x, F_t = f]$$

Value Function

$$V^0(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 0, W_{t-}^\pi = x, F_t = f]$$

$$V^1(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^\pi) \mid \pi_{t-} = 1, W_{t-}^\pi = x, F_t = f]$$

First results

Comparison

$$\begin{aligned} V^0(t,x,f) &\geq V^1(t,(1-g_{01})x,f) \\ V^1(t,x,f) &\geq V^0(t,(1-g_{10})x,f) \end{aligned}$$

Continuity

For all $i \in \{0;1\}$,
$0 \leq t \leq \hat{t} \leq T$,
$x, \hat{x} > 0$,
$0 \leq f, \hat{f} \leq 1$:

$$\begin{aligned} &\left|V^i(\hat{t},\hat{x},\hat{f}) - V^i(t,x,f)\right| \\ &\quad \leq \quad C(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})\left(|\hat{x} - x| + x(|\hat{f} - f| + |\hat{t} - t|^{1/2})\right) \end{aligned}$$

Hamilton Jacobi Bellman Equations

Hamilton Jacobi Bellman Equations

$$\begin{cases} \min\left\{-\frac{\partial V^0}{\partial t} - \mathcal{L}^0 V^0;\ V^0(t,x,f) - V^1(t,x(1-g_{01}),f)\right\} = 0 \\ \min\left\{-\frac{\partial V^1}{\partial t} - \mathcal{L}^1 V^1;\ V^1(t,x,f) - V^0(t,x(1-g_{10}),f)\right\} = 0 \end{cases}$$

$$\begin{aligned} \mathcal{L}^0\varphi(t,x,f) &= xr\frac{\partial\varphi}{\partial x}(t,x,f) + \big(-\lambda_1 f + \lambda_2(1-f)\big)\frac{\partial\varphi}{\partial f}(t,x,f) + \\ &\quad \frac{1}{2}\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 f^2(1-f)^2\frac{\partial^2\varphi}{\partial f^2}(t,x,f) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^1\varphi(t,x,f) &= x(\mu_1 f + \mu_2(1-f))\frac{\partial\varphi}{\partial x}(t,x,f) + \frac{1}{2}x^2\sigma^2\frac{\partial^2\varphi}{\partial x^2}(t,x,f) \\ &\quad +\big(-\lambda_1 f + \lambda_2(1-f)\big)\frac{\partial\varphi}{\partial f}(t,x,f) + x(\mu_1-\mu_2)f(1-f)\frac{\partial^2\varphi}{\partial x\partial f}(t,x,f) \\ &\quad +\frac{1}{2}\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 f^2(1-f)^2\frac{\partial^2\varphi}{\partial f^2}(t,x,f) \end{aligned}$$

Viscosity Solutions

$$\text{(P)} \qquad F(t,x,v(t,x),D_t v(t,x),Dv(t,x),D^2 v(t,x)) = 0$$

Definition

- v is a viscosity sub-solution of (P) if

$$F(\bar{t},\bar{x},v(\bar{t},\bar{x}),D_t\varphi(\bar{t},\bar{x}),D\varphi(\bar{t},\bar{x}),D^2\varphi(\bar{t},\bar{x})) \leq 0$$

  for all $(\bar{t},\bar{x})$ and all functions φ $C^{1,2}$ such that $(\bar{t},\bar{x})$ is a local maximum of $v - \varphi$

- v is a viscosity super-solution of (P) if

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Identification of the Value Function

$\mathcal{V}_\alpha$: set of continuous functions φ on $[0; T] \times [0; +\infty[\times[0; 1]$ satisfying $\varphi(t, 0, f) = 0$ and

$$\sup_{[0;T]\times]0;+\infty[^2\times[0;1]^2} \frac{|\varphi(t,x,f) - \varphi(t,\hat{x},\hat{f})|}{(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})(|x - \hat{x}| + x|f - \hat{f}|)} < \infty.$$

Theorem

(V^0, V^1) is the unique viscosity solution of HJB equation in $\mathcal{V}_\alpha \times \mathcal{V}_\alpha$ satisfying

$$V^0(T,x,f) = V^1(T,x,f) = U(x) = x^\alpha$$

Benoîte de Saporta

Outline

1. Motivation
2. Framework
3. Mathematical Results
4. Numerical Results

Discretization Scheme

Dependence on x:

$$V^i(t,x,f) = \sup_\pi \mathbb{E}[U(W_T^{t,x,f,\pi})] \; = \; x^\alpha V^i(t,1,f)$$

Numerical Scheme

- $\hat{V}^0(T,f) = \hat{V}^1(T,f) = 1$
- With the PDE part in HJB, compute $\overline{V^0}(t,\cdot)$ et $\overline{V^1}(t,\cdot)$ from $\hat{V}^0(t+dt,\cdot)$ and $\hat{V}^1(t+dt,\cdot)$
- Comparison
  - if $\overline{V^0}(t,f) \geq (1-g_{01})^\alpha \overline{V^1}(t,f)$, set $\hat{V}^0(t,f) = \overline{V^0}(t,f)$ otherwise $\hat{V}^0(t,f) = (1-g_{01})^\alpha \overline{V^1}(t,f)$
  - if $\overline{V^1}(t,f) \geq (1-g_{10})^\alpha \overline{V^0}(t,f)$, set $\hat{V}^1(t,f) = \overline{V^1}(t,f)$ otherwise $\hat{V}^1(t,f) = (1-g_{10})^\alpha \overline{V^0}(t,f)$

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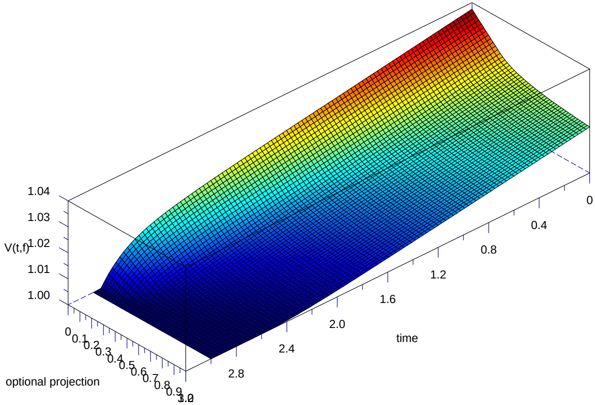
Value Function V^0

Shape

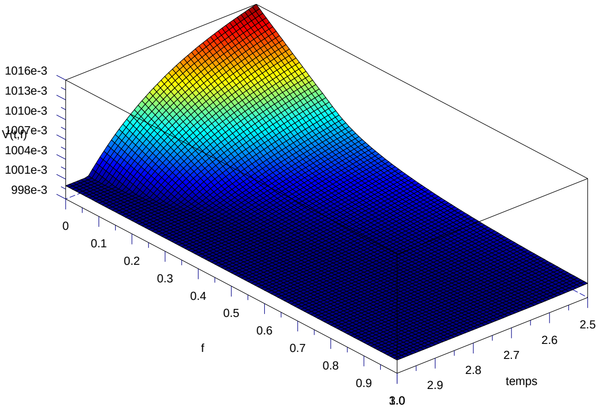
Parameters: $T = 3$, $\mu_2 = -\mu_1 = 0.2$, $\lambda_1 = \lambda_2 = 2$, $\sigma = 0.15$, $g_{01} = g_{10} = 0.01$



Value Function V^0

Regularity I

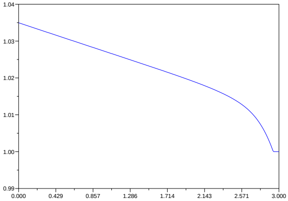
Transaction costs $g_{01} = g_{10} = 0.01$
Zoom between $t = 2.5$ and $t = 3 = T$



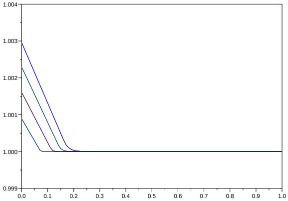
Value Function V^0

Regularity II

Section at $f = 0.05$

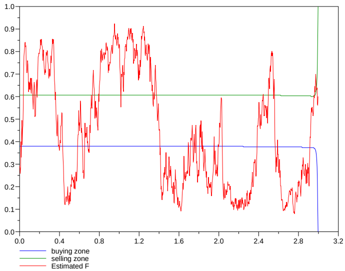


Sections at $t = 2.90$, $t = 2.91$, $t = 2.92$, $t = 2.93$,



Efficient Strategy

- Compute $\hat{V}^0$, $\hat{V}^1$
- Estimate $\hat{F}_t$ from the stock
- Compare $\hat{V}^0(t, \hat{F}_t)$ et $\hat{V}^1(t, \hat{F}_t)$:
  - buy if $\hat{V}^0(t, \hat{F}_t) = (1 - g_{01})^\alpha \hat{V}^1(t, \hat{F}_t)$
  - sell if $\hat{V}^1(t, \hat{F}_t) = (1 - g_{10})^\alpha \hat{V}^0(t, \hat{F}_t)$



$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda_1 = 2$, $\lambda_2 = 2$, $T = 3$

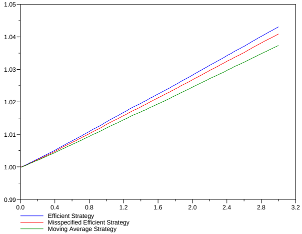
Efficient Strategy vs Value Function

- Computation of the value function:
  - time discretization step 10^{-6}
  - space discretization step 10^{-3}
- 10^5 Monte Carlo simulations of the Efficient Strategy

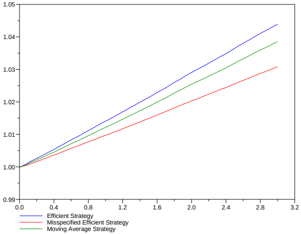
F_0	0	0.1	0.2	0.3	0.4	0.5
$\hat{V}^0$	1.061	1.057	1.053	1.049	1.045	1.043
Strategy	1.061	1.056	1.052	1.049	1.045	1.043

F_0	0.6	0.7	0.8	0.9	1
$\hat{V}^0$	1.041	1.039	1.038	1.037	1.036
Strategy	1.040	1.039	1.038	1.037	1.036

Miscalibrated Efficient Strategy vs Moving Average



miscalibrated parameters:
$\mu_1 = -0.18$, $\mu_2 = 0.18$,
$\sigma = 0.15$, $\lambda_1 = 4$, $\lambda_2 = 4$



miscalibrated parameters:
$\mu_1 = -0.18$, $\mu_2 = 0.18$,
$\sigma = 0.25$, $\lambda_1 = 4$, $\lambda_2 = 4$

Real parameters: $\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda_1 = 2$, $\lambda_2 = 2$, $T = 3$, $\delta = 0.8$

100000 Monte Carlo Simulations

transaction costs: $g_{01} = g_{10} = 0.001$