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Tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients

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Abstract

In this paper, we deal with the real stochastic difference equation $Y_{n+1} = a_n Y_n + b_n,\ n \in \mathbb{Z}$, where the sequence (a_n) is a finite state space Markov chain. By means of the renewal theory, we give a precise description of the situation where the tail of its stationary solution exhibits power law behavior.
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1. Introduction

We study the following stochastic difference equation:

$$Y_{n+1} = a_n Y_n + b_n, \quad n \in \mathbb{Z}, \tag{1}$$

where (a_n) is a real, finite state space Markov chain, and (b_n) is a sequence of real i.i.d. random variables. Equations of type (1) have many applications in stochastic modeling and statistics. Most of previously studied cases deal with i.i.d.

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multiplicative coefficients (a_n): see [13,14,16,9]. For more recent work, see also [15]. Here, we study the Markovian case. In statistical literature, Eq. (1) defines a so-called Markov-switching auto-regression. See [11] for interesting applications in econometrics. Such stochastic recursions are also a basic tool in queuing theory: see [3].

We assume throughout this paper that the following conditions are satisfied:

$$\mathbb{E}\log|a_0|<0,$$

$$\mathbb{E}\log^+|b_0|<\infty. \tag{2}$$

If in addition (a_n, b_n) is stationary and ergodic, Brandt [5] proved that Eq. (1) has a unique stationary solution (Y_n), where

$$Y_n = \sum_{k=0}^{\infty} a_{n-1}a_{n-2}\cdots a_{n-k}b_{n-1-k}, \quad n\in\mathbb{Z}.$$

In the following, (Y_n) will always denote the stationary solution of Eq. (1). We deal with the tail of Y_1: we investigate the asymptotic behavior of $\mathbb{P}(xY_1>t)$, when t tends to infinity, and where $x\in\{-1,1\}$. Our approach is based on renewal-theoretic methods as developed in [16,9].

Our main results are the following two theorems, depending on the a_n being positive or not. Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}_+^*$ the set of positive real numbers.

Theorem 1. *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E=\{e_1,\dots,e_p\}\subset\mathbb{R}_+^*$, transition matrix $P=(p_{ij})$ and stationary law v. Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n). If the following conditions are satisfied:*

(1) *there is a $\lambda>0$ so that the matrix $P_\lambda=\operatorname{diag}(e_i^\lambda)P'$ has spectral radius 1 (P' denotes the transpose of P),*
(2) *the numbers $\log e_i$ are not integral multiples of the same number,*
(3) *there is a $\delta>0$ such that $\mathbb{E}|b_0|^{\lambda+\delta}<\infty$,*

then we have for $x\in\{-1,1\}$

$$t^\lambda\mathbb{P}(xY_1>t)\underset{t\to\infty}{\longrightarrow}L(x),$$

where $L(1)+L(-1)$ is positive. If $b_0\geqslant 0$ a.s., then $L(-1)=0$, and $L(1)>0$. If $b_0\leqslant 0$ a.s., then $L(1)=0$, and $L(-1)>0$.

Theorem 2. *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E=\{e_1,\dots,e_p\}\subset\mathbb{R}$ such that $e_1,\dots,e_\ell$ are positive and $e_{\ell+1},\dots,e_p$ are negative for a $0\leqslant\ell\leqslant p-1$ ($\ell=0$ means that all the e_i are negative). Let $P=(p_{ij})$ be its transition matrix and v its stationary law. Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n). If the following conditions* *are satisfied*:

(1) *there is a* $\lambda > 0$ *so that* $P_\lambda = \operatorname{diag}(|e_i|^\lambda)P'$ *has spectral radius* 1,
(2) *the numbers* $\log|e_i|$ *are not integral multiples of the same number*,
(3) *there is a* $\delta > 0$ *such that* $\mathbb{E}|b_0|^{\lambda+\delta} < \infty$,

then we have, *for* $x \in \{-1, 1\}$,

$$t^\lambda \mathbb{P}(xY_1 > t) \underset{t\to\infty}{\longrightarrow} L(x),$$

where $L(1) + L(-1)$ *is positive. If in addition* P' is *ℓ-irreducible* (*see Definition* 3) *then* $L(1) = L(-1) > 0$.

The last two hypotheses in these theorems are the same as in the i.i.d. case. In particular, Hypothesis (2) ascertains that the distribution of Y_1 is non-lattice, and it is equivalent to requiring that the subgroup generated by the $\log e_i$ be dense in $\mathbb{R}$. On the contrary, Assumption (1) comes from the Markovian dependence considered here. Indeed, we will prove in Section 4.1 that the spectral radius $\rho(P_\lambda)$ of the matrix P_λ can be computed from the formula $\rho(P_\lambda) = \lim(\mathbb{E}|a_0, \ldots, a_{1-n}|^\lambda)^{1/n}$. Therefore, this assumption is a suitable substitute for the classical relation $\mathbb{E}|a_0|^\lambda = 1$ assumed in the i.i.d. case.

Note that the assumption of independence between the two sequences (a_n) and (b_n) can be avoided. Let $\mathscr{F}_{-n}$ be the σ-field generated by $a_0, \ldots, a_{-n}$ and $b_0, \ldots, b_{-n}$. Then (b_n) is only required to be a sequence of random variables such that (a_n, b_n) is a stationary process, and $b_{-(n+1)}$ is independent of $\mathscr{F}_{-n}$. We also need one more assumption (also assumed in the i.i.d. case): for all i, $\mathbb{P}(b_0 + a_0 x = x \mid a_0 = e_i) < 1$. The proofs run exactly the same, except that of Lemma 3, where $\min_{1\leqslant i\leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)$ must be replaced by $\min_{1\leqslant i\leqslant p} \mathbb{P}(|b_0 + (a_0 - 1)m_0| > \varepsilon \mid a_0 = e_i)$. And thanks to the new assumption, we can again choose a positive ε such that the latter minimum is positive.

As the mapping $\lambda \longmapsto \log\rho(P_\lambda)$ is convex (see Section 4.1), that its right-hand derivative at 0 is negative and $\rho(P_0) = \rho(P) = 1$, only two cases may occur. Either for all $\lambda > 0$, $\rho(P_\lambda) < 1$, in which case we can prove that $\mathbb{E}|Y_1|^\lambda < \infty$ for all λ, provided $\mathbb{E}|b_0|^\lambda < \infty$ (see Proposition 3), and therefore $\mathbb{P}(|Y_1| > t) = o(t^{-\lambda})$ for all λ; or there is a unique $\lambda > 0$ so that $\rho(P_\lambda) = 1$, this is the case we study here.

Similar results have already been proved in the i.i.d. multidimensional case: a_n are matrices and Y_n and b_n vectors. Renewal theory is used by Kesten [13] when the a_n either have a density or are non-negative. These results were extended by Le Page [16] to all i.i.d. random matrices satisfying similar assumptions as in our theorems. Finally Goldie [9] proved a new specific implicit renewal theorem and derived the same results as Kesten in the i.i.d. one-dimensional case. He also studies the tails of the stationary solutions of several other one-dimensional random equations with i.i.d. coefficients.

The paper is organized as follows. In Section 2, we introduce some notation and state a new renewal theorem. In Section 3 we derive the renewal equations corresponding to our problem. In Sections 4 and 5, we prove Theorem 1, Section 5 being dedicated to the proof that the sum of the limits is non-zero. And finally in Section 6 we prove Theorem 2.

2. A renewal theorem

Our approach is based on a new renewal theorem for systems of renewal equations. First, we give some notation and conventions that will apply throughout.

Let $F = (F_{ij})_{1 \leqslant i,j \leqslant p}$ be a matrix of distributions: non-decreasing, right-continuous functions from $\mathbb{R}$ to $\mathbb{R}_+$ with limit 0 at $-\infty$.

Definition 1. For all $r \geqslant 1$ and all $p \times r$ vector or matrix H of Borel measurable, real valued functions H_{ij} on $\mathbb{R}$ that are bounded on compact intervals, we define the convolution product $F * H$ by

$$(F * H)_{ij}(t) = \sum_{k=1}^{p} \int_{-\infty}^{\infty} H_{kj}(t-u) F_{ik}(\mathrm{d}u),$$

where it exists.

We study the renewal equation $Z = F * Z + G$, where $G = (G_1, \ldots, G_p)'$ is a vector of Borel measurable, real valued functions, bounded on compact intervals, and $Z = (Z_1, \ldots, Z_p)'$ is a vector of functions. The renewal theorem will give the limit of Z at $+\infty$.

For all real t, set:

- $B = (b_{ij})_{1 \leqslant i,j \leqslant p}$ where $b_{ij} = \int u F_{ij}(\mathrm{d}u)$ if it exists, the expectation of F,
- $F^{(0)}(t) = (\delta_{ij}(t))_{1 \leqslant i,j \leqslant p}$ where $\delta_{ij}(t) = \mathbf{1}_{t \geqslant 0}$ if $i = j$ and 0 otherwise, so that $F^{(0)} * H = H$ for all H as in the definition above,
- $F^{(n)}(t) = F * F^{(n-1)}(t)$, the n-fold convolution of F,
- $U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$, the *renewal function* associated with F.

Assume that all the measures F_{ij} are finite:

$$F_{ij}(\infty) = \lim_{t \to \infty} F_{ij}(t) < \infty,$$

and that $F(\infty)$ is an irreducible matrix (see e.g. [12] for a definition and Perron–Frobenius theory). By Perron–Frobenius theorem, the spectral radius $\rho(F(\infty))$ of $F(\infty)$ is a simple eigenvalue with right and left positive eigenvectors. Assume that $\rho(F(\infty)) = 1$, and let m and u be two positive eigenvectors such that:

$$F(\infty)m = m, \quad u'F(\infty) = u', \quad \sum_{i=1}^{p} m_i = 1, \quad \sum_{i=1}^{p} u_i m_i = 1.$$

Assume also that the sequence $(\|F(\infty)^n\|)$ is bounded (for instance if $F(\infty)$ is aperiodic, this is true). We recall the following definition:

Definition 2. The matrix of distributions F is lattice if the following conditions are satisfied:

- For all $i \neq j$, F_{ij} is concentrated on a set of the form $b_{ij} + \lambda_{ij}\mathbb{Z}$.
- For all i, F_{ii} is concentrated on a set of the form $\lambda_{ii}\mathbb{Z}$.

- Each λ_{ii} is an integral multiple of the same number.
  We take λ to be the largest such number.
- For all $a_{ij},\ a_{jk},\ a_{ik}$ points of increase of $F_{ij},\ F_{jk},\ F_{ik}$, respectively, $a_{ij}+a_{jk}-a_{ik}$ is an integral multiple of λ.

Our basic tool is the following renewal theorem from [17]. It extends a previous result of Crump [7] and Athreya and Rama Murthy [4] which deals with the case where each distribution F_{ij} has support on $\mathbb{R}_+$.

Renewal Theorem A. *Assume that F is a matrix of distributions satisfying the assumptions above, that it is non-lattice, and that*

(1) *its expectation B exists,*
(2) *for all $t \in \mathbb{R}$, $U(t)$ is finite.*

If in addition G is directly Riemann integrable (see [8]), and $Z = U * G$ exists, then for all i, we have:

$$\lim_{t\to\infty} Z_i(t) = cm_i \sum_{j=1}^{p} \left[u_j \int_{-\infty}^{\infty} G_j(y)\,\mathrm{d}y\right],$$

where m and u are the eigenvectors defined above and $c=(u'Bm)^{-1}$ (under these assumptions, $u'Bm \neq 0$).

We also recall Theorem 2.3 of [4] that will be used in Section 5.

Renewal Theorem B. *Let F be a non-lattice matrix of distributions with support on the positive half-line, such that*

(1) $\rho(F(0))<1$,
(2) *$F(\infty)$ is finite, irreducible and aperiodic.*

Assume also that there is a $\alpha>0$ such that $\rho(F_\alpha)=1$, where $(F_\alpha)_{ij}=\int_0^\infty \mathrm{e}^{-\alpha u}F_{ij}(\mathrm{d}u)$. Then for all $h>0$, and all i,j, we have

$$\lim_{t\to\infty} \int_t^{t+h} \mathrm{e}^{-\alpha y} U_{ij}(\mathrm{d}y) = cm_i u_j h,$$

where m and u are right and left eigenvectors of F_α, with the same normalization as above, $c=(u'Bm)^{-1}$, and $B=(b_{ij})$ with $b_{ij}=\int_0^\infty u\mathrm{e}^{-\alpha u}F_{ij}(\mathrm{d}u)$, c being interpreted as zero if some b_{ij} is equal to infinity.

Note that this theorem can now be seen as a corollary of Theorem A. Indeed, the first assumption ascertains that $U(t)$ is finite for all t. In the positive case, the expectation B and the convolution product $U*G$ are always defined (possibly infinite). Applying Theorem A with $F=F_\alpha$ and $G=\mathbf{1}_{[t,t+h]}$ (which is obviously directly Riemann integrable) yields Theorem B.

3. The renewal equations

Let

$$z(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xY_1 > u)\,\mathrm{d}u.$$

For all $(x,t) \in \{-1,1\} \times \mathbb{R}$, we have: $z(x,t) = \sum_{i=1}^p Z_i(x,t)$, where

$$Z_i(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xY_1 > u, a_0 = e_i)\,\mathrm{d}u.$$

Besides, $Y_1 = a_0 Y_0 + b_0$, thus for all $(x,u) \in \{-1,1\} \times \mathbb{R}$, and for all i we have

$$\mathbb{P}(xY_1 > u, a_0 = e_i) = \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i) + \psi_i(x,u),$$

where

$$\psi_i(x,t) = \mathbb{P}(t - xb_0 < xa_0 Y_0 \leqslant t, a_0 = e_i) - \mathbb{P}(t < xa_0 Y_0 \leqslant t - xb_0, a_0 = e_i).$$

Let $G_i(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \psi_i(x,u)\,\mathrm{d}u$. We get

$$z(x,t) = \sum_{i=1}^p \left[\mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i)\,\mathrm{d}u + G_i(x,t) \right].$$

Now we need to distinguish two cases. Indeed, we make a change of variable that involves the sign of a_0. We start with the easier special case when all the states of our Markov chain are positive, therefore the sign of a_0 is non-random.

3.1. Positive case

Suppose all the states of our Markov chain are positive. Then for all (x,t) in $\{-1,1\} \times \mathbb{R}$, and all i, we have, thanks to a simple change of variable,

$$\mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i)\,\mathrm{d}u = \mathrm{e}^{-(t - \log e_i)} e_i^\lambda \int_0^{\mathrm{e}^{t - \log e_i}} u^\lambda \mathbb{P}(xY_0 > u, a_0 = e_i)\,\mathrm{d}u. \tag{3}$$

The Markov property and the stationarity of (Y_n) yield

$$\begin{aligned}
\mathbb{P}(xY_0 > u, a_0 = e_i) &= \sum_{j=1}^p \mathbb{P}(xY_0 > u, a_0 = e_i, a_{-1} = e_j) \\
&= \sum_{j=1}^p \mathbb{P}(xY_0 > u | a_{-1} = e_j) \nu(e_j) p_{ji} \\
&= \sum_{j=1}^p \mathbb{P}(xY_1 > u | a_0 = e_j) \nu(e_j) p_{ji}.
\end{aligned}$$

Thus, we get the following formula for Z_i:

$$
\begin{aligned}
Z_i(x,t) &= \sum_{j=1}^{p}\left[\mathrm{e}^{-(t-\log e_i)}e_i^{\lambda}\int_0^{\mathrm{e}^{t-\log e_i}} u^{\lambda}\mathbb{P}(xY_1>u, a_0=e_j)p_{ji}\,\mathrm{d}u\right] + G_i(x,t)\\
&= e_i^{\lambda}\sum_{j=1}^{p}[p_{ji}Z_j(x,t-\log e_i)] + G_i(x,t).
\end{aligned}
$$

We can rewrite this system of equations as follows:

$$
\forall 1\leqslant i\leqslant p,\quad Z_i(x,t) = \sum_{j=1}^{p} F_{ij} * Z_j(x,t) + G_i(x,t),
$$

where $F_{ij}(t) = e_i^{\lambda}p_{ji}\mathbf{1}_{t\geqslant \log e_i}$ are distribution functions. Let $Z=(Z_1,\dots,Z_p)'$, $G=(G_1,\dots,G_p)'$ and F be the matrix $F=(F_{ij})$. With the notations of Section 2 we have the following system of renewal equations for fixed x:

$$
Z(x,t) = F * Z(x,t) + G(x,t). \tag{4}
$$

3.2. General case

Now we study the general case. In order to determine the sign of a_0, we classify our states according to their sign: assume there is a $0\leqslant \ell\leqslant p-1$ so that $e_1,\dots,e_\ell>0$ and $e_{\ell+1},\dots,e_p<0$. Then Eq. (3) becomes

$$
\begin{aligned}
&\mathrm{e}^{-t}\int_0^{\mathrm{e}^t} u^{\lambda}\mathbb{P}(xa_0Y_0>u, a_0=e_i)\,\mathrm{d}u\\
&\quad = \mathrm{e}^{-(t-\log|e_i|)}|e_i|^{\lambda}\int_0^{\mathrm{e}^{t-\log|e_i|}} u^{\lambda}\mathbb{P}(x\cdot e_iY_0>u, a_0=e_i)\,\mathrm{d}u,
\end{aligned}
$$

where $x\cdot e_i$ denotes the sign of xe_i. To get similar equations as in the positive case, we introduce $2p$ new functions:

$$
\begin{aligned}
&\forall 1\leqslant i\leqslant p,\quad Z_i^{+}(t) = Z_i(1,t) = \mathrm{e}^{-t}\int_0^{\mathrm{e}^t} u^{\lambda}\mathbb{P}(Y_1>u, a_0=e_i)\,\mathrm{d}u,\\
&\forall 1\leqslant i\leqslant p,\quad Z_i^{-}(t) = Z_i(-1,t) = \mathrm{e}^{-t}\int_0^{\mathrm{e}^t} u^{\lambda}\mathbb{P}(-Y_1>u, a_0=e_i)\,\mathrm{d}u.
\end{aligned}
$$

Following the same steps as in the positive case, we get

$$
\begin{aligned}
&\forall 1\leqslant i\leqslant l,\quad Z_i^{+}(t) = |e_i|^{\lambda}\sum_{j=1}^{p} p_{ji}Z_j^{+}(t-\log|e_i|) + G_i(1,t),\\
&\forall l+1\leqslant i\leqslant p,\quad Z_i^{+}(t) = |e_i|^{\lambda}\sum_{j=1}^{p} p_{ji}Z_j^{-}(t-\log|e_i|) + G_i(1,t),
\end{aligned}
$$

$$\forall 1 \leqslant i \leqslant l, \quad Z_i^-(t) = |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^-(t - \log|e_i|) + G_i(-1,t),$$

$$\forall l+1 \leqslant i \leqslant p, \quad Z_i^-(t) = |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^+(t - \log|e_i|) + G_i(-1,t), \tag{5}$$

that we can also rewrite as a system of renewal equations: set

$$\widetilde{Z} = (Z_1^+, \ldots, Z_p^+, Z_1^-, \ldots, Z_p^-)' \quad \text{and} \quad \widetilde{G} = (G_1^+, \ldots, G_p^+, G_1^-, \ldots, G_p^-)',$$

where $G_i^+(t) = G_i(1,t)$ and $G_i^-(t) = G_i(-1,t)$. Define the $2p \times 2p$ matrix $\widetilde{F} = (\widetilde{F}_{ij})$ by:

$$\begin{aligned} \widetilde{F}_{ij}(t) &= |e_{\bar{i}}|^\lambda p_{\bar{j}\bar{i}} \mathbf{1}_{t \geqslant \log|e_{\bar{i}}|} && \text{if } 1 \leqslant i \leqslant l \text{ and } 1 \leqslant j \leqslant p, \\ & && \text{or } p+l+1 \leqslant i \leqslant 2p \text{ and } 1 \leqslant j \leqslant p, \\ & && \text{or } l+1 \leqslant i \leqslant p+l \text{ and } p+1 \leqslant j \leqslant 2p, \\ \widetilde{F}_{ij}(t) &= 0 && \text{otherwise,} \end{aligned}$$

where $\bar{i} = i \bmod p$ (see Eq. (19) for an explicit matrix form of $\widetilde{F}$). Now Eq. (5) becomes

$$\widetilde{Z}(t) = \widetilde{F} * \widetilde{Z}(t) + \widetilde{G}(t).$$

4. Part I of the proof of Theorem 1

Throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. In order to apply Renewal Theorem A, we have to check that F and G satisfy its hypotheses. Note first that $F_{ij}(\infty) = e_i^\lambda p_{ji} < \infty$ and that B the expectation of F is well defined. Indeed, $b_{ij} = e_i^\lambda p_{ji} \log e_i < \infty$. The assumption that the $\log e_i$ are not integral multiples of the same number implies that F is non-lattice. The other points are proved in the following sections.

4.1. *Finiteness of U*

Remember that $U = \sum_{k=0}^\infty F^{(k)}$. We have to check that $U(t) < \infty$ for all real t. First, we study the spectral radius of the matrices $P_\alpha = \mathrm{diag}(e_i^\alpha)P'$, i.e. $(P_\alpha)_{ij} = e_i^\alpha p_{ji}$, for $\alpha > 0$.

Proposition 1. *For all* $\alpha > 0$, *we have*

$$\rho(P_\alpha) = \lim_k (\mathbb{E}|a_0, \ldots, a_{-k}|^\alpha)^{1/k}.$$

Proof. We have

$$\mathbb{E}|a_0 a_{-1}, \ldots, a_{-k}|^\alpha = \sum_{i_1, \ldots, i_{k+1}} \mathbb{P}(a_0 = e_{i_1}, \ldots, a_{-k} = e_{i_{k+1}}) |e_{i_1}, \ldots, e_{i_{k+1}}|^\alpha$$

$$= \sum_{i_1,\dots,i_{k+1}} p_{i_2 i_1},\dots,p_{i_{k+1} i_k} v(e_{i_{k+1}})|e_{i_1},\dots,e_{i_{k+1}}|^\alpha$$
$$= \sum_{i,j} (P_\alpha^k)_{ij} v(e_j) e_j^\alpha,$$

where P_α^k is the kth power of the matrix P_α. Rewrite this equation as

$$\mathbb{E}|a_0, a_{-1},\dots,a_{-k}|^\alpha = \mathbf{1} P_\alpha^k D_\alpha, \tag{6}$$

where $\mathbf{1}$ denotes the constant row vector with all coordinates equal to 1, and D_α is the column vector with coordinates $v(e_j)e_j^\alpha$. As P, and thus P_α, is aperiodic, Theorem 8.5.1 of [12] yields

$$\frac{P_\alpha^k}{\rho^k(P_\alpha)} \underset{k\to\infty}{\longrightarrow} A_\alpha \tag{7}$$

where A_α is a constant positive matrix. Thus $(\mathbf{1} P_\alpha^k D_\alpha)^{1/k} \underset{k\to\infty}{\longrightarrow} \rho(P_\alpha)$. $\square$

The following corollary is obvious.

Corollary 1. *The mapping* $\alpha \longmapsto \log(\rho(P_\alpha))$ *is convex on* $\mathbb{R}_+$.

Proposition 2. *The right-hand derivative of* $\alpha \longmapsto \log(\rho(P_\alpha))$ *at zero is negative.*

To prove this proposition, we need another expression for $\rho(P_\alpha)$. We set $\mathbb{E}_e[\cdot] = \mathbb{E}[\cdot \mid a_0 = e]$ for all $e \in E$.

Lemma 1. *Set* $h_n(\alpha) = \max_{e\in E} \mathbb{E}_e[(a_{-1},\dots,a_{-n})^\alpha]$. *Then we have* $\rho(P_\alpha) = \inf_n (h_n(\alpha))^{1/n}$.

Proof. We first prove that the sequence (h_n) is sub-multiplicative. Indeed, set $e \in E$. We have

$$\begin{aligned}\mathbb{E}_e[(a_{-1},\dots,a_{-n}a_{-n-1},\dots,a_{-n-m})^\alpha] &= \mathbb{E}_e[(a_{-1},\dots,a_{-n})^\alpha \mathbb{E}_{a_{-n}}[(a_{-1},\dots,a_{-m})^\alpha]]\\ &\leqslant h_m(\alpha)\mathbb{E}_e[(a_{-1},\dots,a_{-n})^\alpha]\\ &\leqslant h_m(\alpha)h_n(\alpha),\end{aligned}$$

as $\mathbb{E}_{a_{-n}}[(a_{-1},\dots,a_{-m})^\alpha] \leqslant h_m(\alpha)$. Thus $\lim_n (h_n(\alpha))^{1/n} = \inf_n (h_n(\alpha))^{1/n}$. Besides, we have

$$\begin{aligned}\mathbb{E}|a_0 a_{-1},\dots,a_{-n}|^\alpha &= \sum_{e\in E} \mathbb{E}_e |a_{-1},\dots,a_{-n}|^\alpha e^\alpha v(e)\\ &\leqslant h_n(\alpha) \sum_{e\in E} e^\alpha v(e).\end{aligned}$$

As $\sum_{e\in E} e^\alpha v(e) > 0$, Proposition 1 yields

$$\rho(P_\alpha) \leqslant \lim_n (h_n(\alpha))^{1/n}.$$

On the other hand, set e_n such that $h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1},\dots,a_{-n})^\alpha]$. The equation above then yields

$$\begin{aligned}\mathbb{E}|a_0 a_{-1},\dots,a_{-n}|^\alpha &\geqslant h_n(\alpha) e_n^\alpha v(e_n)\\ &\geqslant C h_n(\alpha),\end{aligned}$$

where $C = \min_{e \in E} e^{\alpha} v(e) > 0$. Hence we also have

$$\rho(P_\alpha) \geqslant \lim_n (h_n(\alpha))^{1/n}.$$

As $\lim_n (h_n(\alpha))^{1/n} = \inf_n (h_n(\alpha))^{1/n}$, the lemma is proved. $\square$

Proof of Proposition 2. For any fixed n, set $e_n \in E$ such that $h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1}, \ldots, a_{-n})^\alpha]$. As the product $a_{-1}, \ldots, a_{-n}$ is bounded for a fixed n, we have

$$\frac{\partial}{\partial \alpha} h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1}, \ldots, a_{-n})^\alpha \log(a_{-1}, \ldots, a_{-n})],$$

hence

$$\left.\frac{\partial}{\partial \alpha}\right|_{\alpha=0} \frac{1}{n} \log h_n(\alpha) = \frac{1}{n} \mathbb{E}_{e_n}[\log(a_{-1}, \ldots, a_{-n})].$$

For all $e \in E$, the Ergodic Theorem for stationary Markov chains yields

$$\frac{1}{n} \mathbb{E}_e[\log(a_{-1}, \ldots, a_{-n})] \underset{n \to \infty}{\longrightarrow} \mathbb{E} \log a_0 = \gamma < 0. \tag{8}$$

As the state space E is finite, this convergence is also uniform on E. Thus, for any sequence (e_n) in E we have

$$\frac{1}{n} \mathbb{E}_{e_n}[\log(a_{-1}, \ldots, a_{-n})] \underset{n \to \infty}{\longrightarrow} \gamma < 0.$$

Hence, there is an integer N such that

$$\left.\frac{\partial}{\partial \alpha}\right|_{\alpha=0} \frac{1}{N} \log h_N(\alpha) \leqslant \frac{\gamma}{2} < 0.$$

In particular, the mapping $\alpha \longmapsto \frac{1}{N} \log h_N(\alpha)$ is negative on an interval of the form $]0, \varepsilon[$, with $\varepsilon > 0$. The preceding lemma then yields

$$\begin{aligned} \log \rho(P_\alpha) &= \inf_n \frac{1}{n} \log h_n(\alpha) \\ &\leqslant \frac{1}{N} \log h_N(\alpha), \end{aligned}$$

which is negative for all $\alpha \in]0, \varepsilon[$. But the mapping $\alpha \longmapsto \log \rho(P_\alpha)$ is convex and continuous on $\mathbb{R}_+$, and takes the value 0 at 0. The result above implies that its right-hand derivative at 0 is negative (possibly $-\infty$). $\square$

We have $\rho(P_0) = 1$, and in addition, in the case we study here, $\rho(P_\lambda) = 1$, thus Proposition 2 and Corollary 1 easily yield the following corollary:

Corollary 2. *For all* $0 < \alpha < \lambda$, *we have* $\rho(P_\alpha) < 1$.

Now we can study U. By definition, $F(\infty) = P_\lambda$ is irreducible as P is and all e_i are non-zero. We have chosen λ so that $\rho(P_\lambda) = \rho(F(\infty)) = 1$. For all $\alpha \in]0, \lambda[$, we have $P_{\lambda-\alpha} = (e_i^{\lambda-\alpha} p_{ji}) = (\int \mathrm{e}^{-\alpha u} F_{ij}(\mathrm{d}u))$. Corollary 2 yields $\rho(P_{\lambda-\alpha}) < 1$, so that the series $\sum_{n=0}^{\infty} (P_{\lambda-\alpha}^n)_{ij}$ is convergent for all i,j. As for all n, $(P_{\lambda-\alpha}^n)_{ij} = \int \mathrm{e}^{-\alpha u} F_{ij}^{(n)}(\mathrm{d}u)$ holds, then we have

$$(P^n_{\lambda-\alpha})_{ij} \geqslant \int_{-\infty}^{t} \mathrm{e}^{-\alpha u} F^{(n)}_{ij}(\mathrm{d}u) \geqslant \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} F^{(n)}_{ij}(\mathrm{d}u) = \mathrm{e}^{-\alpha t} F^{(n)}_{ij}(t).$$

Thus, for all i,j and t, we have $U_{ij}(t) = \sum F^{(n)}_{ij}(t) \leqslant e^{\alpha t} \sum (P^n_{\lambda-\alpha})_{ij} < \infty$.

4.2. *Proof of* $Z = U * G$

Iterating Eq. (4) yields:

$$Z = \sum_{k=0}^{n-1} [F^{(k)} * G] + F^{(n)} * Z.$$

It is thus sufficient to prove that $F^{(n)} * Z \to 0$. As seen in Section 3 we have

$$\begin{aligned}(F * Z)_i(x,t) &= \sum_{j=1}^{p} \left[\mathrm{e}^{-(t-\log e_i)} \int_0^{\mathrm{e}^{t-\log e_i}} e_i^{\lambda} p_{ji} u^{\lambda} \mathbb{P}(xY_1 > u, a_0 = e_j)\,\mathrm{d}u \right] \\ &= \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i)\,\mathrm{d}u.\end{aligned}$$

Similarly, we get for all n

$$(F^{(n)} * Z)_i(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(xa_0, \ldots, a_{1-n} Y_{1-n} > u, a_0 = e_i)\,\mathrm{d}u.$$

And thus we have

$$\sum_{i=1}^{p} (F^{(n)} * Z)_i(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(xa_0, \ldots, a_{1-n} Y_{1-n} > u)\,\mathrm{d}u.$$

But $a_0, \ldots, a_{1-n} = \exp(\sum_{k=1}^{n} \log a_{1-k})$, thus Eq. (8) and Assumption (2) yield $a_0, \ldots, a_{1-n} \to 0$. Hence for all $u > 0$, the bounded convergence theorem yields:

$$\mathbb{P}(xa_{1-n}, \ldots, a_0 Y_{1-n} > u) \underset{n\to\infty}{\longrightarrow} 0,$$

because $Y < \infty$ a.s. and is stationary. Thus $\sum_{i=1}^{p} (F^{(n)} * Z)_i(x,t) \to 0$ holds a.s. As all the terms in the sum are non-negative, each one tends to zero and we have $Z = U * G$ as required.

4.3. *G is directly Riemann integrable*

We first consider the moments of Y_1.

Proposition 3. *For all* $0 \leqslant s < \lambda$, $\mathbb{E}|Y_1|^s < \infty$.

Proof. If $s < \min\{1, \lambda\}$, then convexity and independence yield:

$$\mathbb{E}|Y_1|^s \leqslant \sum_{k=0}^{\infty} \mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s \mathbb{E}|b_{-k}|^s,$$

and if $1 \leqslant s < \lambda$, Hölder inequality yields:

$$(\mathbb{E}|Y_1|^s)^{1/s} \leqslant \sum_{k=0}^{\infty} (\mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s)^{1/s} (\mathbb{E}|b_{-k}|^s)^{1/s}.$$

But we have $\mathbb{E}|b_{-k}|^s \leqslant (\mathbb{E}|b_0|^{\lambda+\delta})^{s/(\lambda+\delta)} < \infty$, with δ given by Theorem 1. Besides, the series $\sum_k (\mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s)^{1/s}$ converges thanks to Proposition 1 and Corollary 2. Hence $\mathbb{E}|Y_1|^s < \infty$. $\square$

Proposition 4. *For all i and x, the mappings $t \longmapsto G_i(x,t)$ are directly Riemann integrable on $\mathbb{R}$.*

Proof. As G_i are clearly continuous in t, it is sufficient to prove that

$$\sum_{l=-\infty}^{\infty} \sup_{l \leqslant t < l+1} |G_i(x,t)| < \infty,$$

(see [8]). For all i, x, t, we have $G_i(x,t) = G_i^1(x,t) - G_i^2(x,t)$, where

$$G_i^1(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(u - xb_0 < xY_0 a_0 \leqslant u, a_0 = e_i)\,\mathrm{d}u \geqslant 0,$$

$$G_i^2(x,t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(u < xY_0 a_0 \leqslant u - xb_0, a_0 = e_i)\,\mathrm{d}u \geqslant 0.$$

For all real t, we have $G_i(x,t) \leqslant G_i^1(x,t) \leqslant \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda\,\mathrm{d}u = \mathrm{e}^{t\lambda}(\lambda+1)^{-1}$. In particular, G_i is directly Riemann integrable on $\mathbb{R}_-$. We still have to study G_i^1 and G_i^2 on $\mathbb{R}_+$. These two functions being of the same kind, we only study G_i^1 here.

The rest of the proof is adapted from [16]. Set $\varepsilon \in]0,1[$ so that $-1 < \lambda - (\lambda+\delta)\varepsilon < 0$, with $\delta > 0$ given by Theorem 1. We have

$$0 \leqslant \mathrm{e}^t G_i^1(x,t) \leqslant \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i)\,\mathrm{d}u + \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(u - u^\varepsilon < xY_0 a_0 \leqslant u, a_0 = e_i)\,\mathrm{d}u. \tag{9}$$

We are going to give an upper bound for each one of these two terms.

- *First term*:
  As $\mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i) \leqslant \mathbb{P}(xb_0 > u^\varepsilon)$ we have, as in [16]

$$\int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i)\,\mathrm{d}u \leqslant \mathbb{E}|b_0|^{\lambda+\delta} \frac{\mathrm{e}^{t(1+\lambda-\varepsilon(\lambda+\delta))}}{1+\lambda-\varepsilon(\lambda+\delta)}. \tag{10}$$

- *Second term*:
  For all $u>0$ we have $\mathbb{P}(xY_0a_0>u, a_0=e_i)\leqslant \frac{\mathbb{E}|Y_0e_i|^s}{u^s}$ which is finite by Proposition 3. With this slight change in [16], we get

$$\int_0^{\mathrm{e}^t} u^{\lambda}\mathbb{P}(u-u^{\varepsilon}<xa_0Y_0\leqslant u, a_0=e_i)\,\mathrm{d}u\leqslant C\mathrm{e}^{t(\lambda+\varepsilon-s)}, \tag{11}$$

where C is a positive constant, and $s\in\,]0,\lambda[$ is chosen such that $-1<\lambda+\varepsilon-1-s<0$.

Now let $\alpha=\max\{\lambda+\varepsilon-s;\ 1+\lambda-(\lambda+\delta)\varepsilon\}\ \in\,]0,1[$. Eqs. (9)–(11) yield $\mathrm{e}^tG_i^1(x,t)\leqslant C\mathrm{e}^{t\alpha}$ for all positive t, C being another positive constant. Thus $G_i^1(x,t)\leqslant C\mathrm{e}^{t(\alpha-1)}$ is directly Riemann integrable on $\mathbb{R}_+$. $\square$

4.4. Tail of the stationary distribution

We have proved that F and G satisfy the conditions of Theorem A. Hence for all i,x,t, we have, with the notation of this theorem,

$$Z_i(x,t)\underset{t\to\infty}{\longrightarrow} cm_i\sum_{j=1}^{p}\left[u_j\int_{-\infty}^{\infty}G_j(x,y)\,\mathrm{d}y\right]. \tag{12}$$

Summing up these terms, we get

$$z(x,t)\underset{t\to\infty}{\longrightarrow} c\sum_{j=1}^{p}\left[u_j\int_{-\infty}^{\infty}G_j(x,y)\,\mathrm{d}y\right], \tag{13}$$

as $\sum m_i=1$. This limit is also the limit of $t^{\lambda}\mathbb{P}(xY_1>t)$ by Lemma 9.3 of [9] which is valid under our assumptions (see also Lemma 3.7 of [16] for a similar result). Now it remains to prove that the sum of the two limits for $x\in\{-1,1\}$ is non-zero.

5. Part II of the proof of Theorem 1

5.1. Special case: b_0 has a constant sign

In Section 3, we have defined the functions

$$\psi_i(x,t)=\mathbb{P}(t-xb_0<xa_0Y_0\leqslant t, a_0=e_i)-\mathbb{P}(t<xa_0Y_0\leqslant t-xb_0, a_0=e_i).$$

If $b_0\geqslant 0$ a.s. and $x=1$, or $b_0\leqslant 0$ a.s. and $x=-1$, we have $xb_0\geqslant 0$ a.s. and for all i and t,

$$\begin{aligned}G_i(x,t)&=\mathrm{e}^{-t}\int_0^{\mathrm{e}^t}u^{\lambda}\psi_i(x,u)\,\mathrm{d}u\\&=\mathrm{e}^{-t}\int_0^{\mathrm{e}^t}u^{\lambda}\mathbb{P}(u-xb_0<xa_0Y_0\leqslant u, a_0=e_i)\,\mathrm{d}u\geqslant 0.\end{aligned}$$

Similarly, if $xb_0 \leqslant 0$ a.s. we have for all i and t:

$$G_i(x,t) = -\mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^\lambda \mathbb{P}(u < xa_0 Y_0 \leqslant u - xb_0, a_0 = e_i)\, \mathrm{d}u \leqslant 0.$$

Thus, if b_0 has constant sign, for fixed x all $G_i(x,\cdot)$ have constant sign, and have the same sign. Now assume that $\lim z(x,t) = 0$. Then Eq. (13) yields

$$c \sum_{j=1}^{p} \left[u_j \int_{-\infty}^{\infty} G_j(x,y)\, \mathrm{d}y \right] = 0.$$

As c and all u_j are positive, this yields $G_j(x,t) = 0$ for all j and $t \in \mathbb{R}$. Thus, $Z(x,t) = U * G(x,t) = 0$ for all t, and $z(x,t) = 0$. Hence $\mathbb{P}(xY_1 > t) = 0$ a.s.

- If $b_0 \geqslant 0$, we have $Y_1 \geqslant 0$, which contradicts the statement above if $x = 1$. Thus $\lim z(1,t) > 0$. And obviously $\lim z(-1,t) = 0$.
- If $b_0 \leqslant 0$, we have $Y_1 \leqslant 0$, which contradicts the statement above if $x = -1$. Thus $\lim z(-1,t) > 0$. And obviously $\lim z(1,t) = 0$.

5.2. *Lower bound for* $\mathbb{P}(|Y_1| > t)$

Now we study the general case where b_0 is allowed to change sign. We want to prove that there is a positive constant C such that $t^\lambda \mathbb{P}(|Y_1| > t) \geqslant C > 0$ when t tends to infinity. In the author's opinion, this lower bound is far from obvious. Here we adapt a method proposed by Goldie [9].

Proposition 5. *There is a positive* ε *and a corresponding positive constant* C *such that for all large enough* t, *we have*

$$\mathbb{P}(|Y_1| > t) \geqslant C\mathbb{P}\left(\sup_n |a_0, \ldots, a_{1-n}| > \frac{2t}{\varepsilon} \right).$$

As explained by Goldie [9] for the i.i.d. case, the key for such a lower bound is an inequality established by Grincevičius [10] corresponding to an extension of Lévy's symmetrization inequality: see [6]. We first extend Grincevičius' inequality to the Markovian case.

Recall that $Y_1 = \sum_{k=0}^{\infty} a_0, \ldots, a_{1-k} b_{-k}$ and set for $n \geqslant 1$,

$$Y_1^n = \sum_{k=0}^{n-1} a_0, \ldots, a_{1-k} b_{-k} \quad \text{and} \quad \Pi_n = a_0, \ldots, a_{1-n}.$$

Let $\mathscr{F}_j$ be the σ-field generated by $(a_{-j}, a_{-j-1}, \ldots)$, and X a $\mathscr{F}_j$-measurable random variable. Let $\mathrm{med}_i(X)$ be a median of X conditionally to $a_{-j} = e_i$, so that $\mathbb{P}(\mathrm{med}_i(X) \leqslant X \mid a_{-j} = e_i) \geqslant \frac{1}{2}$, and $\mathbb{P}(\mathrm{med}_i(X) \geqslant X \mid a_{-j} = e_i) \geqslant \frac{1}{2}$. Set also $\mathrm{med}_-(X) = \min_{1 \leqslant i \leqslant p} \{\mathrm{med}_i(X)\}$.

Lemma 2. *For all* $t>0$ *and* $n\geqslant 1$, *we have*

$$\mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{Y_1^j+\Pi_j\,\mathrm{med}_-\left(\frac{Y_1^n-Y_1^j}{\Pi_j}\right)\right\}>t\right)\leqslant 2\mathbb{P}(Y_1^n>t).$$

Proof. Set $T=\inf\{j\leqslant n \text{ s.t. } Y_1^j+\Pi_j\,\mathrm{med}_-((Y_1^n-Y_1^j)\Pi_j^{-1})>t\}$ if this set is not empty, $n+1$ otherwise, and $B_j=\{\mathrm{med}_-((Y_1^n-Y_1^j)\Pi_j^{-1})\leqslant(Y_1^n-Y_1^j)\Pi_j^{-1}\}$. The event $(T=j)$ is in the σ-field generated by $a_0,\dots,a_{1-j},b_0,\dots,b_{1-j}$, and B_j is in the σ-field generated by $a_{-j},\dots,a_{1-n},b_{-j},\dots,b_{1-n}$. Therefore they are independent conditionally to a_{-j}. Moreover, for all i and j we have

$$\mathbb{P}(B_j\,|\,a_{-j}=e_i)\geqslant\mathbb{P}\left(\mathrm{med}_i\left(\frac{Y_1^n-Y_1^j}{\Pi_j}\right)\leqslant\frac{Y_1^n-Y_1^j}{\Pi_j}\right)\geqslant\frac{1}{2}.$$

Thus, as Π_j is positive, we have

$$\begin{aligned}\mathbb{P}(Y_1^n>t)&\geqslant\mathbb{P}\left(\bigcup_{j=1}^n[B_j\cap(T=j)]\right)\\&=\sum_{j=1}^n\sum_{i=1}^p\mathbb{P}(B_j\,|\,a_{-j}=e_i)\mathbb{P}(T=j\,|\,a_{-j}=e_i)v(e_i)\\&\geqslant\frac{1}{2}\,\mathbb{P}(T\leqslant n)\\&=\frac{1}{2}\,\mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{Y_1^j+\Pi_j\,\mathrm{med}_-\left(\frac{Y_1^n-Y_1^j}{\Pi_j}\right)\right\}>t\right).\qquad\square\end{aligned}$$

Under our assumptions, when n tends to infinity, Y_1^n tends to Y_1, and for fixed j, $\Pi_j^{-1}(Y_1^n-Y_1^j)$ tends to a random variable $\widehat{Y}$ that has the same distribution as Y_1. Set $m_0=\mathrm{med}_-(Y_1)=\mathrm{med}_-(\widehat{Y})$, and letting n tend to infinity in Lemma 2, yields, for all $t>0$,

$$\mathbb{P}(\sup_j\{Y_1^j+\Pi_j m_0\}>t)\leqslant 2\mathbb{P}(Y_1>t).$$

Replacing Y_1 by $-Y_1$ yields a similar formula for all $t<0$, hence for all $t>0$, we have

$$\mathbb{P}(\sup_j|Y_1^j+\Pi_j m_0|>t)\leqslant 2\mathbb{P}(|Y_1|>t). \tag{14}$$

Furthermore, as proved in Goldie [9, p. 157], we have for all $t>|m_0|$,

$$\mathbb{P}(\sup_n\{Y_1^n+\Pi_n m_0\}>t)\geqslant\mathbb{P}(\exists n \text{ s.t. } |(Y_1^{n+1}+\Pi_{n+1}m_0)-(Y_1^n+\Pi_n m_0)|>2t),$$

where $Y_1^0=0$ and $\Pi_0=1$ by convention. Now notice that:

$$\begin{aligned}(Y_1^{n+1}+\Pi_{n+1}m_0)-(Y_1^n+\Pi_n m_0)&=a_0,\dots,a_{1-n}b_{-n}+(\Pi_{n+1}-\Pi_n)m_0\\&=\Pi_n(b_{-n}+(a_{-n}-1)m_0).\end{aligned}$$

Thus Eq. (14) yields, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(|Y_1| > t) &\geqslant \frac{1}{2}\,\mathbb{P}(\exists n \text{ s.t. } |\Pi_n(b_{-n} + (a_{-n} - 1)m_0)| > 2t) \\ &\geqslant \frac{1}{2}\,\mathbb{P}\left(\exists n \text{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\right). \end{aligned} \tag{15}$$

Now we extend Feller–Chung inequality (see [6]).

Lemma 3. *We have, for all $t > |m_0|$ and $\varepsilon > 0$*

$$\begin{aligned}&\mathbb{P}\left(\exists n \textit{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon} \textit{ and } |b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\right) \\ &\quad \geqslant \min_{1 \leqslant i \leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)\mathbb{P}\left(\exists n \textit{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon}\right).\end{aligned}$$

Proof. Set $A_0 = \emptyset$, $A_n = \{|\Pi_n| > \frac{2t}{\varepsilon}\}$ and $B_n = \{|b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\}$. Conditionally to a_{-n}, B_n is independent from $A_0, \ldots, A_n$. Therefore, we have

$$\begin{aligned}\mathbb{P}\left(\bigcup_{n=1}^{\infty} [A_n \cap B_n]\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(B_n \cap A_n \bigcap_{j=0}^{n-1} [B_j \cap A_j]^{\mathrm{c}}\right) \\ &\geqslant \sum_{n=1}^{\infty} \mathbb{P}\left(B_n \cap A_n \bigcap_{j=0}^{n-1} A_j^{\mathrm{c}}\right) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{p} \mathbb{P}(B_n \,|\, a_{-n} = e_i)\mathbb{P}\left(A_n \bigcap_{j=0}^{n-1} A_j^{\mathrm{c}} \,|\, a_{-n} = e_i\right)\nu(e_i).\end{aligned}$$

where A^{c} denotes the complementary set of A. But the stationarity of (a_n, b_n), and the independence of these two sequences yield $\mathbb{P}(B_n \,|\, a_{-n} = e_i) = \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)$. Thus, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} [A_n \cap B_n]\right) \geqslant \min_{1 \leqslant i \leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right). \qquad \square$$

Proof of Proposition 5. Eq. (15) and Lemma 3 yield, for all $t > |m_0|$ and for all $\varepsilon > 0$,

$$\mathbb{P}(|Y_1| > t) \geqslant \frac{1}{2} \min_{1 \leqslant i \leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)\mathbb{P}\left(\exists n \text{ s.t. } |\Pi_{n-1}| > \frac{2t}{\varepsilon}\right).$$

If b_0 is not constant (otherwise we get a special case studied in Section 5.1), we can find a $\varepsilon > 0$ such that $\min_{1 \leqslant i \leqslant p}\{\mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)\} > 0$. Thus, as expected, there is a positive constant C such that for all $t > |m_0|$, we have

$$\mathbb{P}(|Y_1| > t) \geqslant C\mathbb{P}\left(\sup_n |\Pi_n| > \frac{2t}{\varepsilon}\right). \qquad \square$$

5.3. Study of the product $a_0, \ldots, a_{1-n}$

To estimate the probability $\mathbb{P}(\sup_n |\Pi_n| > t)$, we use the method of Arjas and Speed [1], and Renewal Theorem B. First, we introduce some notation. Let $S_0 = 0$ and for all positive n,

$$S_n = \sum_{k=1}^{n} \log(a_{1-k}) = \log(a_0, \ldots, a_{1-n}) = \log \Pi_n.$$

The process (a_{1-n}, S_n) is called a Markov-modulated random walk: see [3,2], or a Markov renewal process: see [1], with semi-Markov matrix $Q = (q_{ij})$, where:

$$q_{ij}(t) = \mathbb{P}(a_{-n} = e_j, \log a_{-n} \leqslant t \mid a_{1-n} = e_i) = \mathbf{1}_{t \geqslant \log e_j} \frac{v(e_j)}{v(e_i)} p_{ji}.$$

The *first ladder epoch* of the random walk (S_n) is $\tau = \tau_1 = \inf\{n \geqslant 1 \text{ s.t. } S_n > 0\}$, and the *first ladder height* is S_τ. Let $H(t)$ be the semi-Markov matrix of this ladder process:

$$H_{ij}(t) = \mathbb{P}(\tau < \infty, S_\tau \leqslant t, a_{1-\tau} = e_j \mid a_1 = e_i).$$

As $S_\tau > 0$, H is distributed on the positive half-line.

We have $S_{\tau-1} \leqslant 0$ and $S_\tau > 0$, which implies that $\log(a_{1-\tau}) > 0$, i.e. $a_{1-\tau} > 1$. Let us rearrange the e_i such that $e_1, \ldots, e_q > 1$ and $e_{q+1}, \ldots, e_p \leqslant 1$ (they cannot be all smaller than or equal to one, for otherwise P'_α would be a sub-stochastic matrix for all α which is impossible as $\rho(P'_\alpha) > 1$ for all $\alpha > \lambda$ thanks to the convexity property). Thus, for all $j > q$, we have $H_{ij}(t) = 0$ for all t. Let $\overline{H}$ be the sub-matrix $(H_{ij})_{1 \leqslant i,j \leqslant q}$. Besides, S_τ cannot be greater than $\max_i \log(e_i)$, thus H (and $\overline{H}$) have finite support.

We define also the nth ladder epoch by $\tau_n = \inf\{k > \tau_{n-1} \text{ s.t. } S_k > S_{\tau_{n-1}}\}$, and S_{τ_n} is the corresponding ladder height. We check that

$$H_{ij}^{(n)}(t) = \mathbb{P}(\tau_n < \infty, S_{\tau_n} \leqslant t, a_{1-\tau_n} = e_j \mid a_1 = e_i),$$

where $H^{(n)}$ is the n-fold convolution of H. We also have $\overline{H^{(n)}} = \overline{H}^{(n)}$, with obvious notation. Let $\Psi = \sum_{n=0}^{\infty} H^{(n)}$ be the renewal function associated with H and $\overline{\Psi}$ the one associated with $\overline{H}$. Finally, let $M = \sup_n S_n = \sup_n S_{\tau_n}$ be the maximum of our random walk. We have, for all $1 \leqslant i \leqslant p$:

$$\mathbb{P}(M \leqslant t \mid a_1 = e_i) = \sum_{j=1}^{p} \left[\Psi_{ij}(t) \left(1 - \sum_{k=1}^{p} H_{jk}(\infty) \right) \right],$$

and if $i \leqslant q$ it reduces to

$$\mathbb{P}(M \leqslant t \mid a_1 = e_i) = \sum_{j=1}^{q} \left[\overline{\Psi}_{ij}(t) \left(1 - \sum_{k=1}^{q} \overline{H}_{jk}(\infty) \right) \right]. \tag{16}$$

Now, we are going to apply renewal Theorem B, with $F = \overline{H}$ and $\alpha = \lambda$ (here it is easier to apply Theorem B than to check the four assumptions of Arjas and Speed [1]). As $H(0) = (0)$, we have $\rho(\overline{H}(0)) < 1$, and as all H_{ij} are probabilities, $\overline{H}(\infty)$ is finite. In addition, $\overline{B}$, the expectation of $\overline{H}_\lambda(\infty) = \int_0^\infty \mathrm{e}^{-\lambda u} \overline{H}(\mathrm{d}u)$ is finite as $\overline{H}$ has finite support. The assumption that the $\log e_i$ are not integral multiples of the same number also implies that $\overline{H}$ is non-lattice.

We have

$$\begin{aligned} \overline{H}_{ij}(\infty) &= \mathbb{P}(\tau < \infty, a_{1-\tau} = e_j \mid a_1 = e_i) \\ &\geqslant \mathbb{P}(\tau = 1, a_{1-\tau} = e_j \mid a_1 = e_i) \\ &= \mathbb{P}(a_0 = e_j \mid a_1 = e_i) \ = \ p_{ji} \frac{v(e_j)}{v(e_i)}. \end{aligned}$$

As all $v(e_i)$ are positive, and P is irreducible and aperiodic, this implies that $\overline{H}(\infty)$ also is irreducible and aperiodic.

Note that $H(\infty)$ and $\overline{H}(\infty)$ have the same spectral radius. Indeed, $H(\infty)$ is a block-triangular matrix with first diagonal block $\overline{H}(\infty)$ and second diagonal block (0). Therefore $\rho(H(\infty)) = \rho(\overline{H}(\infty))$.

To compute the spectral radius of $\overline{H}_\lambda(\infty)$, we introduce $\widehat{Q}(s) = (\hat{q}_{ij}(s))$, the moment generating function of Q, as in [1]:

$$\hat{q}_{ij}(s) = \int \mathrm{e}^{st} q_{ij}(\mathrm{d}t) = \mathrm{e}_j^s \frac{v(e_j)}{v(e_i)} p_{ji} = \Delta^{-1} P_\mathrm{s} \Delta,$$

where $\Delta = \mathrm{diag}(e_i^s v(e_i))$. Thus P_s and $\widehat{Q}(s)$ have the same spectral radius, and in particular $\rho(\widehat{Q}(\lambda)) = 1$. In addition, $\widehat{Q}(\lambda)$ is a non-negative irreducible matrix, as P_λ is, therefore, by Perron–Frobenius Theorem it possesses a right eigenvector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)'$ with positive coordinates. Set $E = \mathrm{diag}(\varepsilon_i)$. Then

$$Q_\lambda(t) = E^{-1} \left(\int_{-\infty}^{t} \mathrm{e}^{\lambda u} Q(\mathrm{d}u) \right) E$$

is a semi-Markov matrix, and let $({}^\lambda a_{1-n}, {}^\lambda S_n)$ be its associated Markov renewal process. As proved in [1], $EH_\lambda(\infty)E^{-1}$ is the semi-Markov matrix of the ascending ladder process of $({}^\lambda S_n)$, and the mean of $\log {}^\lambda a_{1-n}$ is the derivative of $s \longmapsto \log \rho(P_s)$ at λ. But we have $\log \rho(P_0) = \log \rho(P_\lambda) = 0$, its right-hand derivative at zero is negative (Proposition 2) and this function is convex (Corollary 1). Thus its derivative at λ is positive, and ${}^\lambda S_n$ drifts to $+\infty$. Proposition 4.2 of [2] then implies that $\rho(EH_\lambda(\infty)E^{-1}) = 1 = \rho(\overline{H}_\lambda(\infty))$.

We have proved that all assumptions of Theorem B are valid. Thus Eq. (16) yields, when t tends to infinity

$$\begin{aligned}1-\mathbb{P}(M\leqslant t) &= \sum_{j=1}^{q}\left(1-\sum_{k=1}^{q}\overline{H}_{jk}(\infty)\right)\int_t^{\infty}\mathrm{e}^{-\lambda u}(\mathrm{e}^{\lambda u}\overline{\Psi}_{ij})(\mathrm{d}u)\\ &\sim\sum_{j=1}^{q}\left(1-\sum_{k=1}^{q}\overline{H}_{jk}(\infty)\right)\int_t^{\infty}\mathrm{e}^{-\lambda u}\overline{cm}_i\overline{u}_j\,\mathrm{d}u\\ &= \sum_{j=1}^{q}\left(1-\sum_{k=1}^{q}\overline{H}_{jk}(\infty)\right)\frac{\overline{cm}_i\overline{u}_j}{\lambda}\,\mathrm{e}^{-\lambda t},\end{aligned} \tag{17}$$

where $\overline{m}$ and $\overline{u}$ are right and left positive eigenvectors of $\overline{H}_\lambda(\infty)$, with the same normalization as in Section 2, and $\overline{c}=(\overline{u}'\overline{B}\overline{m})^{-1}>0$. Proposition 4.2 of [2] implies that $\rho(\overline{H}(\infty))=\rho(H(\infty))<1$ as $\mathbb{E}\log|a_0|<0$ (Assumption (2)). Therefore $\overline{H}(\infty)$ is strictly sub-stochastic and there is a $j\leqslant q$ such that $1-\sum_{k=1}^{q}\overline{H}_{jk}(\infty)>0$. Hence the right-hand side term of Eq. (17) is positive, thus we have, when t tends to infinity,

$$\mathrm{e}^{\lambda t}\mathbb{P}(M>t)\geqslant\sum_{i=1}^{q}\mathrm{e}^{\lambda t}\mathbb{P}(M>t\,|\,a_1=e_i)v(e_i)\geqslant C>0. \tag{18}$$

Now Eq. (18) and Proposition 5 yield, for large enough t:

$$t^{\lambda}\mathbb{P}(|Y_1|>t)\geqslant C>0$$

and thus with the notation of Theorem 1 we have $L(-1)+L(1)>0$.

6. Proof of Theorem 2

Assume that the hypotheses of Theorem 2 are satisfied. Our aim is to apply Theorem A to the distribution matrix $\widetilde{F}$ and the vector $\widetilde{G}$ defined in Section 3.2. As in the positive case, notice that $\widetilde{F}_{ij}(\infty)<\infty$ and that the expectation $\widetilde{B}$ of $\widetilde{F}$ is well defined. The assumption that the $\log|e_i|$ are not integral multiples of the same number implies again that $\widetilde{F}$ is non-lattice.

For the other points, we use the previous results obtained in the positive case. For all real t, set $F(t)=(|e_i|^{\lambda}p_{ji}\mathbf{1}_{t\geqslant\log|e_i|})_{1\leqslant i,j\leqslant p}$. It is non-negative, and

$$\widetilde{F}=\begin{pmatrix}(F)_{1\leqslant i\leqslant\ell,\ 1\leqslant j\leqslant p} & (0)\\ (0) & (F)_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p}\\ (0) & (F)_{1\leqslant i\leqslant\ell,\ 1\leqslant j\leqslant p}\\ (F)_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p} & (0)\end{pmatrix}. \tag{19}$$

6.1. Irreducibility

We have seen in the positive case that $F(\infty)$ is irreducible. Unfortunately, this does not always imply that $\widetilde{F}(\infty)$ is also irreducible.

Definition 3. Let $A=(a_{ij})_{1\leqslant i,j\leqslant p}$ be a positive matrix, and $0\leqslant \ell\leqslant p-1$ an integer. A is ℓ-reducible if there is (I,J) a (possibly trivial) partition of $\{1,\dots,p\}$ such that
For all $1\leqslant i\leqslant \ell$

$$\text{if } i\in I, \text{ then } a_{ij}=0\ \forall j\in J,$$
$$\text{if } i\in J, \text{ then } a_{ij}=0\ \forall j\in I.$$

For all $\ell+1\leqslant i\leqslant p$

$$\text{if } i\in I, \text{then } a_{ij}=0\ \forall j\in I,$$
$$\text{if } i\in J, \text{then } a_{ij}=0\ \forall j\in J.$$

If A is not ℓ-reducible, we say that A is ℓ-irreducible.

We gave this definition in order to have the following proposition.

Proposition 6. *Let $A=(a_{ij})_{1\leqslant i,j\leqslant p}$ be a positive irreducible matrix, and $0\leqslant \ell\leqslant p-1$ an integer. Then, the matrix B defined as follows:*

$$B=\begin{pmatrix} (a_{ij})_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant p} & (0)\\ (0) & (a_{ij})_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p}\\ (0) & (a_{ij})_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant p}\\ (a_{ij})_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p} & (0)\end{pmatrix}$$

is irreducible if and only if A is ℓ-irreducible.

Proof. Suppose A is ℓ-reducible for a partition (I,J). Set $\bar I=I\cup(J+p)$ and $\bar J=J\cup(I+p)$, so that $(\bar I,\bar J)$ is a non-trivial partition of $\{1,\dots,2p\}$. Then for all $(i,j)\in\bar I\times\bar J$ we can prove that $b_{ij}=0$ and $b_{ji}=0$. Thus B is reducible.

Suppose that B is reducible for the non trivial partition (I,J). Set:

$$I_1=I\cap\{1,\dots,p\},\quad I_2 = I\cap\{p+1,\dots,2p\},$$
$$J_1=J\cap\{1,\dots,p\},\quad J_2 = J\cap\{p+1,\dots,2p\}.$$

We can prove that $I_1=J_2-p$ and $I_2=J_1+p$, and we check that A is ℓ-reducible for the partition (I_1,J_1). $\square$

Now we distinguish two cases according to whether P' is ℓ-reducible or not.

6.2. First case: P' is ℓ-irreducible

In this case $F(\infty)$ is also ℓ-irreducible for λ given by Theorem 2 and $\widetilde{F}(\infty)$ is irreducible. In addition, we have $\|\widetilde{F}(\infty)^n\|\leqslant\|F(\infty)^n\|$ for all n. As $F(\infty)$ is aperiodic, this sequence is bounded. We know that $F(\infty)$ has spectral radius 1. The same also holds for $\widetilde{F}(\infty)$ thanks to the following lemma:

Lemma 4. *If the matrix* $A = (a_{ij})_{i \leqslant i,j \leqslant p}$ *is non-negative, then the matrix* B *of Proposition* 6 *has the same spectral radius as* A.

Proof. Let us compute $\mathscr{P}(X) = \det(B - XI_{2p})$ the characteristic polynomial of B. Adding the last p columns of $B - XI_{2p}$ to the first p columns, then subtracting the first p rows to the last p rows, we get $\det(B - XI_{2p}) = \det(A - XI_p)\det(A_1 - XI_p)$, where A_1 is the following matrix:

$$A_1 = \begin{pmatrix} (a_{ij})_{1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant p} \\ (-a_{ij})_{l+1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p} \end{pmatrix}.$$

Thus the spectral radius of B is the maximum of that of A and that of A_1. But A is non-negative, and component-wise $|A_1| = A$, so Theorem 8.1.18 of [12] yields $\rho(A_1) \leqslant \rho(A)$. Thus $\rho(B) = \rho(A)$. $\square$

Note that if λ is an eigenvalue of A with eigenvector X, then we have $B(X', X')' = ((AX)', (AX)')' = \lambda(X', X')'$, thus $(X', X')'$ is an eigenvector of B for the same eigenvalue. Let m and u be positive right and left eigenvectors of F for the eigenvalue 1, so that $\sum m_i = \sum m_i u_i = 1$. Then $\widetilde{m} = 2^{-1}(m', m')'$ is a right eigenvector of $\widetilde{F}$ for the eigenvalue 1, and satisfies $\sum \widetilde{m_i} = 1$. And $\widetilde{u} = (u', u')'$ is a left eigenvector so that $\sum_{i=1}^{2p} \widetilde{u}_i \widetilde{m}_i = \sum_{i=1}^{p} u_i m_i = 1$.

6.2.1. Properties of $\widetilde{F}$ and $\widetilde{G}$

Let $\widetilde{U} = \sum_{k=0}^{\infty} \widetilde{F}^{(k)}$. As $\widetilde{F}_{ij} \leqslant F_{\overline{ij}}$, the same holds for their k-fold convolution. Set $U = \sum_{k=0}^{\infty} F^{(k)}$, then $U(t) < \infty$ as in the positive case, and thus $\widetilde{U}(t) < \infty$.

To prove that $\widetilde{Z} = \widetilde{U} * \widetilde{G}$, it is sufficient to prove that $\widetilde{F}^{(n)} * \widetilde{Z} \underset{n \to \infty}{\longrightarrow} 0$. But we have seen in Section 3 that

$$(\widetilde{F} * \widetilde{Z})_i(t) = \sum_{j=1}^{p} \mathrm{e}^{-(t - \log|e_i|)} \int_0^{\mathrm{e}^{t - \log|e_i|}} |e_i|^{\lambda} p_{ji} u^{\lambda} \mathbb{P}(\pm Y_1 > u, a_0 = e_j)\,\mathrm{d}u$$
$$= \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(\pm a_0 Y_0 > u, a_0 = e_i)\,\mathrm{d}u.$$

Similarly, we get

$$(\widetilde{F}^{(n)} * \widetilde{Z})_i(t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(\pm a_{1-n}, \ldots, a_0 Y_{1-n} > u, a_0 = e_i)\,\mathrm{d}u.$$

And thus, as in the positive case, we have

$$\sum_{i=1}^{p} (\widetilde{F}^{(n)} * \widetilde{Z})_i(t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} u^{\lambda} \mathbb{P}(\pm a_{1-n}, \ldots, a_0 Y_{1-n} > u)\,\mathrm{d}u.$$

But Eq. (8) implies $a_{1-n}, \ldots, a_0 \to 0$. Thus for all $u > 0$, the bounded convergence theorem yields $\mathbb{P}(\pm a_{1-n}, \ldots, a_0 Y_{1-n} > u) \to 0$, because $Y < \infty$ a.s. and is stationary.

Thus $\sum_{i=1}^{p}(\widetilde{F}^{(n)} * \widetilde{Z})_i(t) \to 0$, and as all the terms in the sum are non-negative, each one tends to 0 and we have, as expected $\widetilde{Z} = \widetilde{U} * \widetilde{G}$.

We have $\widetilde{G}_i(t) = G_{\bar{i}}(\pm 1, t)$ which is directly Riemann integrable under the assumptions of Theorem 2 as seen for the positive case.

6.2.2. *Tail of the distribution*

We have proved that $\widetilde{F}$ and $\widetilde{G}$ satisfy the assumptions of Theorem A. Hence for all i, t, we have, with obvious notations,

$$\widetilde{Z}_i(t) \underset{t\to\infty}{\longrightarrow} \widetilde{c}\widetilde{m}_i \sum_{j=1}^{2p} \left[\widetilde{u}_j \int_{-\infty}^{\infty} \widetilde{G}_j(y)\,\mathrm{d}y \right]. \tag{20}$$

Notice that $\widetilde{c} = c$. Indeed, we have

$$\widetilde{u}'\widetilde{B}\widetilde{m} = \frac{1}{2}(u', u') \begin{pmatrix} (b_{ij})_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant p} & 0 \\ 0 & (b_{ij})_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p} \\ 0 & (b_{ij})_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant p} \\ (b_{ij})_{\ell+1\leqslant i\leqslant p,\ 1\leqslant j\leqslant p} & 0 \end{pmatrix} \begin{pmatrix} m \\ m \end{pmatrix}.$$

Hence $\widetilde{c}^{-1} = \frac{1}{2}(u'Bm + u'Bm) = c^{-1}$, where B is the expectation of F. Thus summing up the term in Eq. (20), we get

$$z(x,t) \underset{t\to\infty}{\longrightarrow} c \sum_{j=1}^{p} \left[u_j \int_{-\infty}^{\infty} (G_j(-1,y) + G_j(1,y))\,\mathrm{d}y \right].$$

And we use again Lemma 9.3 of [9] to conclude that $t^{\lambda}\mathbb{P}(xY_1 > t)$ has the same limit. Note that here this limit does not depend on x, therefore both $t^{\lambda}\mathbb{P}(Y_1 > t)$ and $t^{\lambda}\mathbb{P}(Y_1 < -t)$ have the same limit.

6.3. *Second case: P' is l-reducible*

As seen in the proof of Proposition 6, there is (I, J) a non-trivial partition of $\{1, \ldots, 2p\}$ such that for all (i,j) in $I \times J$ we have $\widetilde{F}_{ij}(\infty) = \widetilde{F}_{ji}(\infty) = 0$. Suppose that 1 belongs to I. Then System (5) splits into two independent systems of size p, one with the components $(\widetilde{Z}_i)_{i\in I}$ and the other with $(\widetilde{Z}_i)_{i\in J}$. Each of these systems has associated matrix F that satisfies the hypothesis of Renewal Theorem A, as seen in the positive case. For all i, $\widetilde{G}_i$ is also directly Riemann integrable as seen in the preceding section. Thus Theorem A yields

$$\forall i \in I, \quad \widetilde{Z}_i(t) \underset{t\to\infty}{\longrightarrow} cm_{\bar{i}} \sum_{\substack{1\leqslant j\leqslant 2p \\ j\in I}} u_{\bar{j}} \int_{-\infty}^{\infty} \widetilde{G}_j(y)\,\mathrm{d}y,$$

$$\forall i \in J, \quad \widetilde{Z}_i(t) \underset{t\to\infty}{\longrightarrow} cm_{\bar{i}} \sum_{\substack{1\leqslant j\leqslant 2p \\ j\in J}} u_{\bar{j}} \int_{-\infty}^{\infty} \widetilde{G}_j(y)\,\mathrm{d}y,$$

where $\bar{i}$ denotes i if $i \leqslant p$ and $i-p$ if $i<p$. Summing up these equalities, we get

$$z(1,t) \underset{t\to\infty}{\longrightarrow} c\sum_{j=1}^{p} u_j \int_{-\infty}^{\infty} (\mathbf{1}_I(j)G_j(1,y) + \mathbf{1}_J(j)G_j(-1,y))\,\mathrm{d}y$$

and

$$z(-1,t) \underset{t\to\infty}{\longrightarrow} c\sum_{j=1}^{p} u_j \int_{-\infty}^{\infty} (\mathbf{1}_J(j)G_j(1,y) + \mathbf{1}_I(j)G_j(-1,y))\,\mathrm{d}y.$$

Again, $t^{\lambda}\mathbb{P}(xY_1 > t)$ has the same limit as $z(x,t)$ for $x \in \{-1,1\}$. Note that here these two limits are possibly different.

6.4. The sum of the limits is non-zero

The proof is the same for both cases. The results of Section 5 can be extended to the present case. The result of Section 5.1 about the special case when b_0 has constant sign is valid here. Thus if both limits are zero then $Y_0 = 0$ almost surely which is impossible as 0 is not a solution of Eq. (1).

If X is a random variable, set $\mathrm{med}_+(X) = \max_{1\leqslant i\leqslant p}\{\mathrm{med}_i(X)\}$. The analogous of Lemma 2 is as follows:

Lemma 5. *For all $t>0$ and $n \geqslant 1$, we have*

$$2\mathbb{P}(Y_1 > t) \geqslant \mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{\mathbf{1}_{\Pi_j>0}\left[Y_1^j + \Pi_j\,\mathrm{med}_-\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right)\right]\right\} > t\right)$$
$$+\,\mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{\mathbf{1}_{\Pi_j<0}\left[Y_1^j + \Pi_j\,\mathrm{med}_+\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right)\right]\right\} > t\right)$$

Proof. As Π_j is not always positive, we introduce new events, depending on the sign of Π_j: set $T_+ = \inf\{j \leqslant n \text{ s.t. } \Pi_j > 0 \text{ and } Y_1^j + \Pi_j\,\mathrm{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) > t\}$ if this set is not empty, $n+1$ otherwise, $T_- = \inf\{j \leqslant n \text{ s.t. } \Pi_j < 0 \text{ and } Y_1^j + \Pi_j\,\mathrm{med}_+((Y_1^n - Y_1^j)\Pi_j^{-1}) > t\}$ if it is not empty, $n+1$ otherwise, $B_j^+ = \{\mathrm{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) \leqslant (Y_1^n - Y_1^j)\Pi_j^{-1}\}$, and $B_j^- = \{\mathrm{med}_+((Y_1^n - Y_1^j)\Pi_j^{-1}) \geqslant (Y_1^n - Y_1^j)\Pi_j^{-1}\}$. The events $(T_+ = j)$ and $(T_- = j)$ on the one hand andB_j^+ and B_j^-on the other hand are independent conditionally to a_{-j}. Moreover, for all i,j we have,

$$\mathbb{P}(B_j^+ \mid a_{-j} = e_i) \geqslant \mathbb{P}\left(\mathrm{med}_i\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \leqslant \frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geqslant \frac{1}{2}$$

and

$$\mathbb{P}(B_j^- \mid a_{-j} = e_i) \geqslant \mathbb{P}\left(\mathrm{med}_i\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geqslant \frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geqslant \frac{1}{2}.$$

Thus we get, as in the proof of Lemma 2:

$$\begin{aligned}\mathbb{P}(Y_1^n > t) &\geqslant \mathbb{P}\left(\bigcup_{j=1}^{n}[[(T_+ = j)\cap B_j^+]\cup[(T_- = j)\cap B_j^-]]\right)\\ &\geqslant \frac{1}{2}(\mathbb{P}(T_+ \leqslant n) + \mathbb{P}(T_- \leqslant n))\\ &= \frac{1}{2}\left[\mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{\mathbf{1}_{\Pi_j>0}\left[Y_1^j + \Pi_j \operatorname{med}_-\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right)\right]\right\} > t\right)\right.\\ &\quad\left. + \mathbb{P}\left(\max_{1\leqslant j\leqslant n}\left\{\mathbf{1}_{\Pi_j<0}\left[\left[Y_1^j + \Pi_j \operatorname{med}_+\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right)\right]\right]\right\} > t\right)\right]. \qquad \square\end{aligned}$$

The rest of the proof runs the same as in the positive case for each of these two terms. Set $m_- = \operatorname{med}_-(Y_1)$ and $m_+ = \operatorname{med}_+(Y_1)$. For all $\varepsilon > 0$ and $t > \max\{|m_+|, |m_-|\}$, we get

$$\begin{aligned}&\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n > \frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_-| > \varepsilon\right)\\ &\quad\geqslant \min_{1\leqslant i\leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_-| > \varepsilon)\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n > \frac{2t}{\varepsilon}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n < -\frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_+| > \varepsilon\right)\\ &\quad\geqslant \min_{1\leqslant i\leqslant p} \mathbb{P}(|b_0 + (e_i - 1)m_+| > \varepsilon)\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n < -\frac{2t}{\varepsilon}\right).\end{aligned}$$

If b_0 is not constant, we can again find $\varepsilon > 0$ such that $\min_{1\leqslant i\leqslant p}\{\mathbb{P}(|b_0 + (e_i - 1)m_-| > \varepsilon)\} > 0$ and $\min_{1\leqslant i\leqslant p}\{\mathbb{P}(|b_0 + (e_i - 1)m_+| > \varepsilon)\} > 0$. Thus, we get the analogous of Proposition 5: there is a constant $C > 0$ and $\varepsilon > 0$ such that for all large enough t:

$$\mathbb{P}(|Y_1| > t) \geqslant C\mathbb{P}\left(\sup_n |\Pi_n| > \frac{2t}{\varepsilon}\right).$$

Define the new random walk $S_n = \log|a_0, \ldots, a_{1-n}|$. With this slight change in Section 5.3, the proof is the same.

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