Tail of a linear diffusion with Markov switching

Queue d'une diffusion linéaire à régime markovien

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Abstract

Let Y be a Ornstein-Uhlenbeck diffusion governed by a stationary and ergodic Markov jump process X, i.e. $dY_t = a(X_t)Y_t dt + \sigma(X_t)dW_t$, $Y_0 = y_0$. Ergodicity conditions for Y have been obtained. Here we investigate the tail property of the stationary distribution of this model. A characterization of the only two possible cases is established: light tail or polynomial tail. Our method is based on discretizations and renewal theory. *To cite this article: B. de Saporta, J.F. Yao, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Soit Y une diffusion de Ornstein-Uhlenbeck dirigée par un processus Markovien de saut X stationnaire et ergodique : $dY_t = a(X_t)Y_t dt + \sigma(X_t)dW_t$, $Y_0 = y_0$. On connaît des conditions d'ergodicité pour Y. Ici on s'intéresse à la queue de la loi stationnaire de ce modèle. Par des méthodes de discrétisation et de renouvellement, on donne une caractérisation complète des deux seuls cas possibles : queue polynômiale ou existence de moment à tout ordre. *Pour citer cet article : B. de Saporta, J.F. Yao, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

1 Introduction

The discrete time models $Y = (Y_n, n \in \mathbf{N})$ governed by a switching process $X = (X_n, n \in \mathbf{N})$ fit well to the situations where an autonomous process X is responsible for the dynamic (or *regime*) of Y. These models are parsimonious with regard to the number of parameters, and extend significantly the case of a single regime. Among them, the so-called Markov switching ARMA models are popular in several application fields, e.g. in econometric modeling (see [4]). More recently continuous-time version of Markov-switching models have been proposed in [1] and [3], among others where ergodicity conditions are established. Here we investigate the tail property of the stationary distribution of this continuous-time process. One of the main results states that this model can provide heavy tails which is one of the major features required in nonlinear time series modeling.

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2 Linear diffusion with Markov switching and main Theorems

The *diffusion with Markov switching* Y is constructed in two steps:

First, the *switching process* $X = (X_t)_{t\geq 0}$ is a Markov jump process defined on a probability space $(\Omega, \mathcal{A}, Q)$, with a finite state space $E = \{1, \ldots, N\}$, $N > 1$. We assume that the intensity function λ of X is positive and the jump kernel $q(i, j)$ on E is irreducible and satisfies $q(i, i) = 0$, for each $i \in E$. The process X is ergodic and will be taken stationary with an invariant probability measure denoted by μ.

Secondly, let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion defined on a probability space $(\Theta, \mathcal{B}, Q')$, and $\mathcal{F} = (\mathcal{F}_t)$ the filtration of the motion. We will consider the product space $(\Omega \times \Theta, \mathcal{A} \times \mathcal{B}, (Q_x \otimes Q'))$, $\mathbb{P} = Q \otimes Q'$ and $\mathbb{E}$ the associated expectation. Conditionally to X, $Y = (Y_t)_{t\geq 0}$ is a real-valued diffusion process, defined, for each $\omega \in \Omega$ by:

(1) Y_0 is a random variable defined on $(\Theta, \mathcal{B}, Q')$, $\mathcal{F}_0$-measurable;

(2) Y is solution of the linear SDE

$$dY_t = a(X_t)Y_t dt + \sigma(X_t)dW_t, \qquad t \geq 0. \tag{1}$$

Thus (Y_t) is a linear diffusion driven by an "exogenous" jump process (X_t).

We say a continuous or discrete time process $S = (S_t)_{t\geq 0}$ is *ergodic* if there exists a probability measure m such that when $t \to \infty$, the law of S_t converges weakly to m independently of the initial condition S_0. The distribution m is then the *limit law* of S. When S is a Markov process, m is its unique invariant law.

In [3], it is proved that the Markov-switching diffusion Y is ergodic under the condition

$$\alpha = \sum_{i \in E} a(i)\mu(i) < 0. \tag{2}$$

Note that Condition (2) will be assumed to be satisfied throughout the paper and we denote by ν the stationary (or limit) distribution of Y.

Theorem 2.1 (light tail case) *If for all i, $a(i) \leq 0$, then the stationary distribution ν of the process Y has moments of all order, i.e. for all $s > 0$ we have:*

$$\int_{\mathbb{R}} |x|^s \nu(dx) < \infty.$$

Theorem 2.2 (heavy tail case) *If there is a i such that $a(i) > 0$, one can find an exponent $s_0 > 0$ and a constant $L > 0$ such that the stationary distribution ν of the process Y satisfies*

$$\begin{aligned} t^{s_0}\nu(]t, +\infty[) &\xrightarrow[t \to +\infty]{} L, \\ t^{s_0}\nu(] - \infty, -t[) &\xrightarrow[t \to +\infty]{} L. \end{aligned}$$

Note that the two situations from Theorems 2.1 and 2.2 form a dichotomy. Moreover the characteristic exponent s_0 in the heavy tail case is completely determined as follows. Let

$$s_1 = \min\left\{\frac{\lambda(i)}{a(i)} \mid a(i) > 0\right\},$$
$$M_s = \left(q(i,j)\frac{\lambda(i)}{\lambda(i) - s a(i)}\right)_{i,j\in E} \quad \text{for } 0 \le s < s_1.$$

Then s_0 is the unique $s \in]0, s_1[$ such that the spectral radius of M_s equals to 1.

3 Discretization of the process

Our study of Y is based on the investigations of its discretization $Y^{(\delta)}$ as in [3]. First we give an explicit formula for the diffusion process. For $0 \le s \le t$, let

$$\Phi(s,t) = \Phi_{s,t}(\omega) = \exp \int_s^t a(X_u)du.$$

The process Y has the representation:

$$Y_t = Y_t(\omega) = \Phi(0,t)\left[Y_0 + \int_0^t \Phi(0,u)^{-1}\sigma(X_u)dW_u\right],$$

and for $0 \le s \le t$, Y satisfies the recursion equation:

$$\begin{aligned} Y_t &= \Phi(s,t)\left[Y_s + \int_s^t \Phi(s,u)^{-1}\sigma(X_u)dW_u\right] \\ &= \Phi(s,t)Y_s + \int_s^t \left[\exp\int_u^t a(X_v)dv\right]\sigma(X_u)dW_u. \end{aligned}$$

It is useful to rewrite this recursion as:

$$Y_t(\omega) = \Phi_{s,t}(\omega)Y_s(\omega) + V_{s,t}^{1/2}(\omega)\xi_{s,t}, \tag{3}$$

where $\xi_{s,t}$ is a standard Gaussian variable, function of $(W_u,\ s \le u \le t)$, and

$$V_{s,t}(\omega) = \int_s^t \exp\left[2\int_u^t a(X_v)dv\right]\sigma^2(X_u)du.$$

For $\delta > 0$, we will call *discretization at step size* δ of Y the discrete time process $Y^{(\delta)} = (Y_{n\delta})_n$, where $n \in \mathbb{N}$. For a fixed $\delta > 0$, the discretization $Y^{(\delta)}$ follows an $AR(1)$ equation with random coefficients:

$$Y_{(n+1)\delta}(\omega) = \Phi_{n+1}(\omega) Y_{n\delta}(\omega) + V_{n+1}^{1/2}(\omega)\xi_{n+1}, \tag{4}$$

with

$$\Phi_{n+1}(\omega) = \Phi_{n+1}(\delta)(\omega) = \exp\left[\int_{n\delta}^{(n+1)\delta} a(X_u(\omega))du\right],$$

$$V_{n+1}(\omega) = \int_{n\delta}^{(n+1)\delta} \exp\left[2\int_u^{(n+1)\delta} a(X_v(\omega))dv\right]\sigma^2(X_u(\omega))du,$$

where (ξ_n) is a standard Gaussian i.i.d. sequence defined on $(\Theta, \mathcal{B}, Q')$. Note that under Condition (2), all these discretizations are ergodic with the same limit distribution ν (see [3]).

4 Sketch of the proof

The limit distribution ν is also the law of the stationary solution of Eq. (4). To investigate the behaviour of its tail, we use the same renewal-theoretic methods as [5], [7] and [2]. In these works, the coefficients (Φ_n) form an i.i.d. sequence. Here the sequence (Φ_n) is neither i.i.d nor a Markov chain. Indeed we know only the conditional independence between Φ_n and Φ_{n+1} given $X_{n\delta}$. We thus need to adapt the mentioned methods to this special situation. Our problem leads to a system of renewal equations, and we use a new renewal theorem for systems of equations reported in [8].

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Liste des modifications

Conformément à la demande du rapporteur, les deux abstracts en français et en anglais ont été précisés: il y a deux cas, queue polynomiale ou moments à tout ordre. Nous n'avons pas de preuve que dans ce dernier cas la queue est exponentielle.

Les diverse fautes de frappe et de syntaxes signalées ont également été corrigées.