Tail of a linear diffusion with Markov switching *

Benoîte de Saporta, Jian-Feng Yao
IRMAR, Université de Rennes 1

Abbreviated title: Tail of Markov-switching diffusions

Abstract: Let Y be a Ornstein-Uhlenbeck diffusion governed by a stationary and ergodic Markov jump process X: $dY_t = a(X_t)Y_t dt + \sigma(X_t)dW_t$, $Y_0 = y_0$. Ergodicity conditions for Y have been obtained. Here we investigate the tail propriety of the stationary distribution of this model. A characterization of both heavy or light tail case is established. The method is based on a renewal theorem for systems of equations with distributions on $\mathbb{R}$.

1 Introduction

The discrete time models $Y = (Y_n, n \in \mathbf{N})$ governed by a switching process $X = (X_n, n \in \mathbf{N})$ fit well to the situations where an autonomous process X is responsible for the dynamic (or *regime*) of Y. These models are parsimonious with regard to the number of parameters, and extend significantly the case of a single regime. Among them, the so-called Markov switching ARMA models are popular in several application fields, e.g. in econometric modeling (see [Hamilton, 1989, Hamilton, 1990]). More recently continuous-time version of Markov-switching models have been proposed in [Basak et al., 1996] and [Guyon et al., 2004] where ergodicity conditions are established. In this paper we investigate the tail property of the stationary distribution of this continuous-time process. One of the main results (Theorem 2) states that this model can provide heavy tails which is one of major features required in nonlinear time series modeling. Note that heavy tails may also be obtained by using a Lévy-driven O.U. process (without Markov switching): see [Barndorff-Nielsen and Shephard, 2001] and [Brockwell, 2001].

The considered process Y, called as *diffusion with Markov switching* is constructed in two steps:

First, the *switching process* $X = (X_t)_{t\geq 0}$ is a Markov jump process (see [Feller, 1966]), defined on a probability space $(\Omega, \mathcal{A}, Q)$, with a finite state space $E = \{1, \ldots, N\}$, $N > 1$. We assume that the intensity function λ of X is positive and the jump kernel $q(i, j)$ on E is irreducible and satisfies $q(i, i) = 0$, for each $i \in E$. The process X is ergodic and will be taken stationary with an invariant probability measure denoted by μ.

Secondly, let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion defined on a

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probability space $(\Theta, \mathcal{B}, Q')$, and $\mathcal{F} = (\mathcal{F}_t)$ the filtration of the motion. We will consider the product space $(\Omega \times \Theta, \mathcal{A} \times \mathcal{B}, (Q_x \otimes Q'))$, $\mathbb{P} = Q \otimes Q'$ and $\mathbb{E}$ the associated expectation. Conditionally to X, $Y = (Y_t)_{t\geq 0}$ is a real-valued diffusion process, defined, for each $\omega \in \Omega$ by:

1. Y_0 is a random variable defined on $(\Theta, \mathcal{B}, Q')$, $\mathcal{F}_0$-measurable;

2. Y is solution of the linear SDE

$$dY_t = a(X_t)Y_t dt + \sigma(X_t)dW_t, \qquad t \geq 0. \tag{1}$$

Thus (Y_t) is a linear diffusion driven by an "exogenous" jump process (X_t).

We say a continuous or discrete time process $S = (S_t)_{t\geq 0}$ is *ergodic* if there exists a probability measure m such that when $t \to \infty$, the law of S_t converges weakly to m independently of the initial condition S_0. The distribution m is then the *limit law* of S. When S is a Markov process, m is its unique invariant law.

In [Guyon et al., 2004], it is proved that the Markov-switching diffusion Y is ergodic under the condition

$$\alpha = \sum_{i\in E} a(i)\mu(i) < 0. \tag{2}$$

The main result of the present paper is the following theorems. Note that Condition 2 will be assumed satisfied throughout the paper and we denote by ν the stationary (or limit) distribution of Y.

Theorem 1 (light tail case) *If for all i, $a(i) \leq 0$, then the stationary distribution ν of the process Y has moments of all order, i.e. for all $s > 0$ we have:*

$$\int_{\mathbb{R}} |x|^s \nu(dx) < \infty.$$

Theorem 2 (heavy tail case) *If there is a i such that $a(i) > 0$, one can find an exponent $\kappa > 0$ and a constant $L > 0$ such that the stationary distribution ν of the process Y satisfies*

$$\begin{aligned} t^\kappa \nu(]t, +\infty[) &\xrightarrow[t\to+\infty]{} L, \\ t^\kappa \nu(]-\infty, -t[) &\xrightarrow[t\to+\infty]{} L. \end{aligned}$$

Note that the two situations from Theorems 1 and 2 form a dichotomy. Moreover the characteristic exponent κ in the heavy tail case is completely determined as following. Let

$$\begin{aligned} s_1 &= \min\left\{\frac{\lambda(i)}{a(i)} \mid a(i) > 0\right\}, \\ M_s &= \left(q(i,j)\frac{\lambda(i)}{\lambda(i) - sa(i)}\right)_{i,j\in E} \quad \text{for } 0 \leq s < s_1. \end{aligned}$$

Then κ is the unique $s \in]0, s_1[$ such that the spectral radius of M_s equals to 1.

The proof of Theorem 1 is a consequence of a results of [Guyon et al., 2004], and the proof of Theorem 2 is based on a recent renewal theorem for systems of equations reported in [de Saporta, 2003] and on an AR(1) recurrence equation satisfied by the discretization of Y that we will defined in Section 2. In section 3, we study an operator related to our problem and prove Theorem 1. Sections 4-7 are devoted to the proof of Theorem 2. First we state two renewal theorems for systems of equations. Then in section 5 we derive the renewal equations associated to our problem. In sections 6 and 7 we prove Theorem 2, the latter section being dedicated to the proof that the constant L is non-zero. Finally, in section 8 we give further details on the computation of the exponent κ.

2 Discretization of the process and an AR(1) equation

First we give an explicit formula for the diffusion process. For $0 \le s \le t$, let

$$\Phi(s,t) = \Phi_{s,t}(\omega) = \exp \int_s^t a(X_u)du.$$

The process Y has the representation (see [Karatzas and Shreve, 1991]):

$$Y_t = Y_t(\omega) = \Phi(0,t)\Big[Y_0 + \int_0^t \Phi(0,u)^{-1}\sigma(X_u)dW_u\Big],$$

and for $0 \le s \le t$, Y satisfies the recursion equation:

$$\begin{aligned} Y_t &= \Phi(s,t)\Big[Y_s + \int_s^t \Phi(s,u)^{-1}\sigma(X_u)dW_u\Big] \\ &= \Phi(s,t)Y_s + \int_s^t \Big[\exp\int_u^t a(X_v)dv\Big]\sigma(X_u)dW_u. \end{aligned}$$

It is useful to rewrite this recursion as:

$$Y_t(\omega) = \Phi_{s,t}(\omega)Y_s(\omega) + V_{s,t}^{1/2}(\omega)\xi_{s,t}, \tag{3}$$

where $\xi_{s,t}$ is a standard Gaussian variable, function of $(W_u,\ s \le u \le t)$, and

$$V_{s,t}(\omega) = \int_s^t \exp\Big[2\int_u^t a(X_v)dv\Big]\sigma^2(X_u)du.$$

For $\delta > 0$, we will call *discretization at step size* δ of Y the discrete time process $Y^{(\delta)} = (Y_{n\delta})_n$, where $n \in \mathbb{N}$. Our study of Y is based on the investigations of these discretization $Y^{(\delta)}$ as in [Guyon et al., 2004].

More precisely, for a fixed $\delta > 0$, the discretization $Y^{(\delta)}$ follows an $AR(1)$ equation with random coefficients:

$$Y_{(n+1)\delta}(\omega) = \Phi_{n+1}(\omega)Y_{n\delta}(\omega) + V_{n+1}^{1/2}(\omega)\xi_{n+1}, \tag{4}$$

with

$$\Phi_{n+1}(\omega) \quad = \quad \Phi_{n+1}(\delta)(\omega) \quad = \quad \exp\Big[\textstyle\int_{n\delta}^{(n+1)\delta} a(X_u(\omega))du\Big],$$

$$V_{n+1}(\omega) \quad = \quad \int_{n\delta}^{(n+1)\delta} \exp\left[2\int_u^{(n+1)\delta} a(X_v(\omega))dv\right]\sigma^2(X_u(\omega))du,$$

where (ξ_n) is a standard Gaussian i.i.d. sequence defined on $(\Theta, \mathcal{B}, Q')$. Note that under Condition 2, all these discretization are ergodic with the same limit distribution ν (see [Guyon et al., 2004]).

3 Study of a related operator

We now introduce a related operator A and investigate its properties. Fix $s \geq 0$ and $\delta > 0$. We define the operator $A_{(s,\delta)}$ by

$$A_{(s,\delta)}\varphi(i) = \mathbb{E}_i[\Phi_1^s(\delta)\varphi(X_\delta)],$$

for every function $\varphi : E \to \mathbb{R}$ and every i in E. It has the following semi-group property:

Proposition 1 *Fix $s \geq 0$. Then for all $\delta, \gamma > 0$ we have:*

$$A_{(s,\delta)}A_{(s,\gamma)} = A_{(s,\delta+\gamma)}.$$

Proof Set $\varphi : E \to \mathbb{R}$ and i in E. We have:

$$\begin{aligned} A_{(s,\delta)}A_{(s,\gamma)}\varphi(i) &= \mathbb{E}_i[\Phi_1^s(\delta)A_{(s,\gamma)}\varphi(X_\delta)] \\ &= \mathbb{E}_i\big[\Phi_1^s(\delta)\mathbb{E}_{X_\delta}[\Phi_1^s(\gamma)\varphi(X_\gamma)]\big] \\ &= \mathbb{E}_i\Big[\exp\big(s\int_0^\delta a(X_u)du\big)\ \mathbb{E}_{X_\delta}\big[\exp\big(s\int_0^\gamma a(X_u)du\big)\ \varphi(X_\gamma)\big]\Big]. \end{aligned}$$

Then the Markov property yields:

$$\begin{aligned} A_{(s,\delta)}A_{(s,\gamma)}\varphi(i) &= \mathbb{E}_i\Big[\exp\big(s\int_0^{\delta+\gamma} a(X_u)du\big)\ \varphi(X_{\delta+\gamma})\Big] \\ &= \mathbb{E}_i[\Phi_1^s(\delta+\gamma)\varphi(X_{\delta+\gamma})] \\ &= A_{(s,\delta+\gamma)}\varphi(i). \end{aligned}$$ □

Note that $A_{(s,\delta)}\varphi(i) = \sum_{j=1}^N \mathbb{E}_i[\Phi_1^s \mathbf{1}_{X_\delta=j}]\varphi(j)$, and therefore $A_{(s,\delta)}$ can be rewritten as the matrix $((A_{(s,\delta)})_{ij})_{1\leq i,j\leq N}$ with $(A_{(s,\delta)})_{ij} = \mathbb{E}_i[\Phi_1^s \mathbf{1}_{X_\delta=j}]$. Note also that it is a positive operator.

3.1 Spectral radius

Now we investigate the properties of the spectral radius of A. First, we recall a result from [Guyon et al., 2004].

Proposition 2 *Fix $s > 0$ and $\delta > 0$. Then $A_{(s,\delta)}$ is irreducible, aperiodic and satisfies:*

$$\mathbb{E}_\mu[(\Phi_1\cdots\Phi_k)^s] = \sum_{i\in E} A_{(s,\delta)}^k \mathbf{1}(i)\mu(i) = \mu A_{(s,\delta)}^k \mathbf{1}, \tag{5}$$

where $\mathbf{1}$ is the constant function equal to 1 on E.

We denote by $\rho(X)$ the spectral radius of a matrix X. Proposition 2 yields the following corollaries.

Corollary 1 *We have:*

$$\rho(A_{(s,\delta)}) = \lim_{k\to\infty} \left(\mathbb{E}_\mu[(\Phi_1 \cdots \Phi_k)^s]\right)^{1/k}.$$

Proof As $A_{(s,\delta)}$ is a (component-wise) positive, irreducible and aperiodic matrix, Theorem 8.5.1 of [Horn and Johnson, 1985] gives the existence of a matrix $B_{(s,\delta)}$ with positive coefficients such that:

$$\frac{(A_{(s,\delta)})^n}{(\rho(A_{(s,\delta)}))^n} \xrightarrow[n\to\infty]{} B_{(s,\delta)}. \tag{6}$$

This result and Eq. (5) yield the expected result. □

Corollary 2 *For all fixed $\delta > 0$, the mapping $s \longmapsto \log \rho(A_{(s,\delta)})$ is convex on $\mathbb{R}_+$.*

Note that for all fixed $\delta > 0$ and i in E, we have $A_{(0,\delta)}\mathbf{1}(i) = \mathbb{E}_i(1) = 1$. Thus, as $A_{(0,\delta)}$ is a positive operator, it is also a stochastic matrix and $\rho(A_{(0,\delta)}) = 1$.

Proposition 3 *For all fixed $\delta > 0$, the right-hand derivative of the mapping $s \longmapsto \log \rho(A_{(s,\delta)})$ at 0 is negative.*

Proof As all the functions considered are convex, we have:

$$\begin{aligned}\frac{\partial}{\partial s} \log\left(\rho(A_{(s,\delta)})\right) &= \lim_{n\to\infty} \frac{\partial}{\partial s}\frac{1}{n} \log \mathbb{E}_\mu[(\Phi_1 \cdots \Phi_n)^\kappa] \\ &= \lim_{n\to\infty} \frac{1}{n} \frac{\mathbb{E}_\mu[(\Phi_1 \cdots \Phi_n)^\kappa \cdot \sum_{i=1}^n \log \Phi_i]}{\mathbb{E}_\mu[(\Phi_1 \cdots \Phi_n)^\kappa]}.\end{aligned}$$

The sequence (Φ_n) is stationary, thus the ergodic Theorem yields:

$$\frac{1}{n}\sum_{k=1}^n \log \Phi_k \xrightarrow[n\to\infty]{} \mathbb{E}_\mu[\log \Phi_1] \quad \mathbb{P}_\mu\text{-almost surely.} \tag{7}$$

But $\mathbb{E}_\mu[\log \Phi_1] < 0$ because of Condition 2. Indeed, we have:

$$\mathbb{E}_\mu[\log \Phi_1] = \mathbb{E}_\mu\Big[\int_0^\delta a(X_u)du\Big] = \int_0^\delta \mathbb{E}_\mu[a(X_u)]du = \delta\alpha < 0.$$

Thus we get, as expected:

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0} \log\left(\rho(A_{(s,\delta)})\right) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\mu\Big[\sum_{i=1}^n \log \Phi_i\Big] \\ &= \mathbb{E}_\mu[\log \Phi_1] < 0.\end{aligned}$$
□

Corollary 3 *Fix $\delta > 0$. We have the following dichotomy:*

$i-$ either for all $s > 0$, $\rho(A_{(s,\delta)}) < 1$,

$ii-$ or there exists a unique $\kappa > 0$ such that $\rho(A_{(\kappa,\delta)}) = 1$, and in this case $\rho(A_{(s,\delta)}) > 1$ for all $s > \kappa$ and $\rho(A_{(s,\delta)}) < 1$ for all $0 < s < \kappa$.

3.2 Choice of δ

Now we are going to prove that the preceding dichotomy is in fact independent of the value of δ.

Proposition 4 *Fix $s \geq 0$. The following propositions are equivalent:*
$i-$ there exists $\delta > 0$ such that $\rho(A_{(s,\delta)}) < 1$,
$ii-$ for all $\delta > 0$ we have $\rho(A_{(s,\delta)}) < 1$.
The same equivalence is true if we replace "< 1" by "> 1" or "$= 1$".

Proof Set $\delta > 0$ such that $\rho(A_{(s,\delta)}) < 1$, and $\gamma > 0$. For all integer $n \geq 1$ we define $m_n \in \mathbb{N}^*$ and $0 \leq \beta_n < \delta$ by $n\gamma = m_n\delta + \beta_n$ (m_n the integer part of $n\gamma/\delta$ and β_n its fractional part multiplied by δ). Thus Proposition 1 yields:

$$A^n_{(s,\gamma)} = A_{(s,n\gamma)} = A^{m_n}_{(s,\delta)} A_{(s,\beta_n)}.$$

But for all n we have

$$\begin{aligned}
\|A_{(s,\beta_n)}\| &\leq \max_i \mathbb{E}_i[\Phi^s_1(\beta_n)] \\
&\leq \exp(s\beta_n \max_i(a_i)) \\
&\leq \exp(s\delta \max_i(a_i)).
\end{aligned}$$

This upper bound is independent of n. Thus we have:

$$\log \|A^n_{(s,\gamma)}\| \leq \log \|A^{m_n}_{(s,\delta)}\| + c,$$

where c is a positive constant. We get:

$$\begin{aligned}
\log \rho(A_{(s,\gamma)}) &= \lim_n \frac{1}{n} \log \|A^n_{(s,\gamma)}\| \\
&\leq \limsup_n \frac{1}{n} \log \|A^{m_n}_{(s,\delta)}\| \\
&= \frac{\gamma}{\delta} \log \rho(A_{(s,\delta)},
\end{aligned}$$

as $m_n \sim n\gamma\delta^{-1}$. Hence $\rho(A_{(s,\gamma)}) \leq \rho(A_{(s,\delta)})^{\gamma/\delta} < 1$.

For the case "$= 1$", fix δ_0 and a corresponding κ such that $\rho(A_{(\kappa,\delta_0)}) = 1$. The mapping $s \longmapsto \rho(A_{(s,\delta_0)})$ is log-convex hence continuous. Thus we have:

$$\rho(A_{(\kappa,\delta_0)}) = \sup_{s<\kappa} \rho(A_{(s,\delta_0)}).$$

Set $\delta > 0$. We want to prove that $\rho(A_{(\kappa,\delta)}) = 1$. According to Corollary 3, for all $s < \kappa$ we have $\rho(A_{(s,\delta_0)}) < 1$. Thus the preceding study yields that for all $s < \kappa$ we also have $\rho(A_{(s,\delta)}) < 1$. Hence we have:

$$\rho(A_{(\kappa,\delta)}) = \sup_{s<\kappa} \rho(A_{(s,\delta)}) \leq 1.$$

Suppose that $\rho(A_{(\kappa,\delta)}) < 1$, then the first case implies again that $\rho(A_{(\kappa,\delta_0)}) < 1$ which is impossible. Thus we have $\rho(A_{(\kappa,\delta)}) = 1$ as expected.

The case "> 1" is a consequence of these two cases and Corollary 3. $\square$

In the following we will write A_s instead of $A_{(s,\delta)}$ each time it is non-ambiguous. We have an easy criterion to know in which case we are.

Proposition 5 *The following properties are equivalent:*
i− for all i in E, $a(i) \leq 0$,
ii− for all $s > 0$, $\rho(A_s) < 1$.

Proof Suppose that for all i in E we have $a(i) \leq 0$. Fix $\delta > 0$. Then for all $s > 0$, we have $\Phi_1^s \leq 1$. Thus for all i, $A_s\mathbf{1}(i) = \mathbb{E}_i[\Phi_1^s] \leq 1$, and component-wise we have $A_s\mathbf{1} \leq \mathbf{1}$, which implies that $\rho(A_s) \leq 1$ for all $s > 0$. Corollary 3 then yields that for all s, we have actually $\rho(A_s) < 1$.

Now suppose there exists a i_0 such that $a(i_0) > 0$. Fix $s \geq 2\lambda(i_0)a(i_0)^{-1}$. It is proved in [Guyon et al., 2004] that for all function φ from E into $\mathbb{R}$ and all i in E we have for small δ,

$$A_s\varphi(i) = [1 + \delta(sa(i) - \lambda(i))]\varphi(i) + \delta\lambda(i)\sum_{j\neq i}[q(i,j)\varphi(j)] + o(\delta). \tag{8}$$

Let ψ be the function from E into $\mathbb{R}$ such that $\psi(i_0) = 1$ and $\psi(i) = 0$ for all $i \neq i_0$. Then for all $i \neq i_0$ we have $A_s\psi(i) = \mathbb{E}_i[\Phi_1^s\mathbf{1}_{X_\delta = i_0}] \geq 0$ and for $i = i_0$ we have:

$$A_s\psi(i_0) = 1 + \delta(sa(i_0) - \lambda(i)) + o(\delta) \geq 1 + \delta\frac{sa(i_0)}{2} + o(\delta)$$

as we have chosen $s \geq 2\lambda(i)a(i_0)^{-1}$. Thus component-wise, for small enough δ, we have:

$$\begin{aligned} A_s\psi &\geq \left(1 + \delta\frac{sa(i_0)}{2} + o(\delta)\right)\psi \\ &\geq \left(1 + \delta\frac{sa(i_0)}{4}\right)\psi. \end{aligned}$$

Thus $\rho(A_s) \geq 1 + \delta\frac{sa(i_0)}{4} > 1$. □

This proposition ends the proof of Theorem 1 since we have the following result from [Guyon et al., 2004] that relates the spectral radius of A_s to the moments of the stationary law ν:

Proposition 6 *Set $s > 0$. If $\rho(A_s) < 1$, then the stationary law ν of Y has a moment of order s.*

The proof of Theorem 1 is now complete.

4 Renewal theory for systems

Now we proceed to prove Theorem 2. From now on, we will assume that there is a i such that $a(i) > 0$. Our approach is based on a new renewal theorem for systems of renewal equations. First we introduce some notation and conventions that we will apply throughout.

Let $F = (F_{ij})_{1\leq i,j\leq p}$ be a matrix of distributions: non-decreasing, right-continuous functions on $\mathbb{R}$ into $\mathbb{R}_+$ with limit 0 at $-\infty$. For all $p \times r$ matrix H of

Borel measurable, real valued functions H_{ij} on $\mathbb{R}$ that are bounded on compact intervals, we define the *convolution product* $F * H$ by:

$$(F * H)_{ij}(t) = \sum_{k=1}^{p} \int_{-\infty}^{\infty} H_{kj}(t-u)F_{ik}(du)$$

where it exists.

The transpose of a vector or matrix X will always be denoted X'. We study the renewal equation $Z = F * Z + G$, where $G = (G_1, \dots, G_p)'$ is a vector of Borel measurable, real valued functions, bounded on compact intervals, and $Z = (Z_1, \dots, Z_p)'$ is a vector of functions. The renewal theorem will give the limit of Z at $+\infty$.

For all real t, we set:

- $B = (b_{ij})_{1 \le i,j \le p}$ where $b_{ij} = \int u F_{ij}(du)$ if it exists, the expectation of F,
- $F^{(0)}(t) = (\delta_{ij}(t))_{1 \le i,j \le p}$ where $\delta_{ij}(t) = \mathbf{1}_{t \ge 0}$ if $i = j$ and 0 otherwise, so that $F^{(0)} * H = H$ for all H as in the definition above,
- $F^{(n)}(t) = F * F^{(n-1)}(t)$, the n-fold convolution of F,
- $U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$, the *renewal function* associated with F.

We will also assume that all the measures F_{ij} are finite:

$$F_{ij}(\infty) = \lim_{t \to \infty} F_{ij}(t) < \infty,$$

and that $F(\infty)$ is an irreducible matrix. $F(\infty)$ being an irreducible non-negative matrix, we can use Perron Frobenius Theorem: its spectral radius $\rho(F(\infty))$ is a simple eigenvalue with right and left positive eigenvectors. We will also assume that $\rho(F(\infty)) = 1$, and we choose two positive eigenvectors m and u so that:

$$F(\infty)m = m, \quad u'F(\infty) = u', \quad \sum_{i=1}^{p} m_i = 1, \quad \sum_{i=1}^{p} u_i m_i = 1.$$

We also assume that the sequence $(\|F(\infty)^n\|)$ is bounded (for instance if $F(\infty)$ is aperiodic, this is true). We recall the following definition: F is *lattice* if the following conditions are satisfied:

- For all $i \neq j$, F_{ij} is concentrated on a set of the form $b_{ij} + \lambda_{ij}\mathbb{Z}$.
- For all i, F_{ii} is concentrated on a set of the form $\lambda_{ii}\mathbb{Z}$.
- Each λ_{ii} is an integral multiple of the same number.

We take λ to be the largest such number.

- For all a_{ij}, a_{jk}, a_{ik} points of increase of F_{ij}, F_{jk}, F_{ik} respectively, $a_{ij} + a_{jk} - a_{ik}$ is an integral multiple of λ.

We can now state the renewal theorem. It extends a previous result of [Crump, 1970] and [Athreya and Rama Murthy, 1976] which deals with the positive case: each distribution F_{ij} has support on $\mathbb{R}_+$. The proof of this theorem is given in [de Saporta, 2003].

Renewal Theorem A *Assume that F is as above and that, in addition, it is a non-lattice matrix, that its expectation B exists, and that for all $t \in \mathbb{R}$, $U(t)$ is* *finite. If G is directly Riemann integrable (see [Feller, 1966]), and $Z = U * G$ exists, then for all i, we have:*

$$\lim_{t\to\infty} Z_i(t) = cm_i \sum_{j=1}^{p} \Big[u_j \int_{-\infty}^{\infty} G_j(y)dy\Big],$$

where m and u are the eigenvectors defined above and $c = (u'Bm)^{-1}$ (under these assumptions, $u'Bm \neq 0$).

We also recall Theorem 2.3 of [Athreya and Rama Murthy, 1976] that will be used in section 7. Note that this theorem can now be seen as a corollary of Theorem A.

Renewal Theorem B *Let F be a non-lattice matrix of distributions with support on the positive half-line, such that*

- *$\rho(F(0)) < 1$,*
- *$F(\infty)$ is finite, irreducible and aperiodic,*
- *there exist i and j such that $F_{ij}(0) < F_{ij}(\infty)$.*

Assume also that there is a $\alpha > 0$ such that $\rho(F_\alpha) = 1$, where $(F_\alpha)_{ij} = \int_0^\infty e^{-\alpha u} F_{ij}(du)$. Then for all $h > 0$, and all i, j, we have

$$\lim_{t\to\infty} \int_t^{t+h} e^{-\alpha y} U_{ij}(dy) = cm_i u_j h,$$

where m and u are right and left eigenvectors of F_α, with the same normalization as above, $c = (u'Bm)^{-1}$, and $B = b_{ij}$ with $b_{ij} = \int_0^\infty ue^{-\alpha u} F_{ij}(du)$, c being interpreted as zero if some b_{ij} is equal to infinity.

5 The renewal equations

Now we are going to derive the renewal equations associated to our problem. In the following, we will suppose that the assumptions of Theorem 2 are satisfied. We set $\delta = 1$, and κ will denote the unique positive solution of $\rho(A_s) = 1$. We are going to study the discretization $Y^{(1)}$.

5.1 Notation

As X is a stationary process, we can exetnd it to negative t and define the coefficients Φ_n, V_n and ξ_n for negative values of n. Let $b_n = V_n^{1/2}\xi_n$ and

$$R_n = \sum_{k=0}^{\infty} \Phi_n \Phi_{n-1} \cdots \Phi_{n-k+1} b_{n-k},$$

(instead of $\widetilde{Y}_n$) be the unique stationary solution of Equation (4): $R_{n+1} = \Phi_{n+1}R_n + b_{n+1}$. The limit law ν of Y is also the law of R_1. Thus we are going to study the random variable R_1.

The tail of the stationary solution of such recursive equations has already been studied in various cases. In the i.i.d. multidimensional case: Φ_n are matrices and R_n and b_n vectors, renewal theory is used in [Kesten, 1973] to prove a heavy tail property when the Φ_n either have a density or are non-negative. These results were extended in [LePage, 1983] to a wider class of i.i.d. random matrices. Finally in [Goldie, 1991] a new specific implicit renewal theorem is proved and the same results are derived in the i.i.d. one-dimensional case. This theorem also applies to the study of the tail of several other random recurrences implying i.i.d. random variables. Recently Goldie's results were extended in [de Saporta, 2004] to the case where (Φ_n) is a finite state space Markov chain. Here, (Φ_n) is not a Markov chain, but conditionally to X_n, Φ_n and Φ_{n+1} are independent. Our proof is thus very similar to that of [de Saporta, 2004], but we will repeat all the details for completeness.

Note that ξ_n are standard Gaussian random variables, thus they are symmetric, and they are also independent from the sequences (Φ_n) and (V_n). Hence we have

$$\begin{aligned}\mathbb{P}_\mu\Big(\sum_{k=0}^{\infty}\Phi_1\Phi_0\cdots\Phi_{2-k}b_{1-k}>t\Big) &= \mathbb{P}_\mu\Big(\sum_{k=0}^{\infty}\Phi_1\Phi_0\cdots\Phi_{2-k}V_{1-k}^{1/2}\xi_{1-k}>t\Big)\\ &= \mathbb{P}_\mu\Big(\sum_{k=0}^{\infty}\Phi_1\Phi_0\cdots\Phi_{2-k}V_{1-k}^{1/2}(-\xi_{1-k})>t\Big)\\ &= \mathbb{P}_\mu\Big(-\sum_{k=0}^{\infty}\Phi_1\Phi_0\cdots\Phi_{2-k}b_{1-k}>t\Big).\end{aligned}$$

Thus we have $\nu(]t,+\infty[) = \nu(]-\infty,-t[)$ for all t, hence if one of the limits stated in Theorem 2 exists, the other exists too and equals to the same value. Therefore we need study only one limit.

To study the tail of R_1, we introduce a new function. For all t in $\mathbb{R}$, we set:

$$z(t) = e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}(R_1>u)du.$$

Lemma 9.3 of [Goldie, 1991] ascertains that if $z(t)$ has a limit when t tends to infinity, then $t^\kappa\mathbb{P}(R_1>t)$ also has the same limit.

For all i in E and t in $\mathbb{R}$, we also set:

$$Z_i(t) = e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}(R_1>u, X_1=i)du,$$

so that $z(t)=\sum_{i=1}^N Z_i(t)$. We are now going to prove that $Z = {}^t(Z_1,\ldots,Z_N)$ satisfies a system of renewal equations.

5.2 The renewal equations

As R_n satisfies Eq. (4), we have $R_1=\Phi_1R_0+b_1$, thus for all t in $\mathbb{R}$, we have:

$$\mathbb{P}_\mu(R_1>u, X_1=i) = \mathbb{P}_\mu(\Phi_1R_0>u, X_1=i)+\psi_i(u),$$

where

$$\psi_i(t) = \mathbb{P}_\mu(t-b_1<\Phi_1R_0\le t, X_1=i) - \mathbb{P}_\mu(t<\Phi_1R_0\le t-b_1, X_1=i).$$

We set $G_i(t) = e^{-t}\int_0^{e^t} u^\kappa \psi_i(u)du$, and $G = {}^t(G_1, \dots, G_N)$. Then we have:

$$z(t) = \sum_{i=1}^{N} \Big[e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_\mu(\Phi_1 R_0 > u, X_1 = i)du + G_i(t) \Big].$$

We have $\Phi_1 \geq 0$ and conditionally to X_0, Φ_1 and R_0 are independent. Thus, a simple change of variable and stationarity yield:

$$\begin{aligned}
& e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_\mu(\Phi_1 R_0 > u, X_1 = i)du \\
= & \sum_{j=1}^{N} e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_\mu(\Phi_1 R_0 > u, X_1 = i \mid X_0 = j)\mu(j)du \\
= & \sum_{j=1}^{N} e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_j(\Phi_1 R_0 > u, X_1 = i)\mu(j)du \\
= & \sum_{j=1}^{N} \mathbb{E}_j\Big[\Phi_1^\kappa \mathbf{1}_{X_1=i} e^{-(t-\log\Phi_1)}\int_0^{e^{t-\log\Phi_1}} u^\kappa \mathbb{P}_j(R_0 > u)du\Big]\mu(j) \\
= & \sum_{j=1}^{N} \mathbb{E}_j\Big[\Phi_1^\kappa \mathbf{1}_{X_1=i} e^{-(t-\log\Phi_1)}\int_0^{e^{t-\log\Phi_1}} u^\kappa \mathbb{P}_\mu(R_0 > u \mid X_0 = j)du\Big]\mu(j) \\
= & \sum_{j=1}^{N} \mathbb{E}_j\Big[\Phi_1^\kappa \mathbf{1}_{X_1=i} e^{-(t-\log\Phi_1)}\int_0^{e^{t-\log\Phi_1}} u^\kappa \mathbb{P}_\mu(R_1 > u, X_1 = j)du\Big].
\end{aligned}$$

Thus we get the following system of equations: for all i in E, we have:

$$\begin{aligned}
Z_i(t) &= \sum_{j=1}^{N} \Big[\mathbb{E}_j\left[\Phi_1^\kappa \mathbf{1}_{X_1=i} Z_j(t - \log\Phi_1)\right]\Big] + G_i(t) \\
&= \sum_{j=1}^{N} \Big[F_{ij} * Z_j(t)\Big] + G_i(t), \qquad (9)
\end{aligned}$$

where $F_{ij}(t) = \mathbb{E}_j[\Phi_1^\kappa \mathbf{1}_{X_1=i}\mathbf{1}_{t\geq\log\Phi_1}]$. Thus $F = (F_{ij})_{i,j\in E}$ is a matrix of distributions in the sense of section 4, and System (9) is a system of renewal equations that can be rewritten as $Z = F * Z + G$. To apply Theorem A, we now have to prove that F and G satisfy its assumptions.

6 Proof of Theorem 2, part I

As E is a finite set, Φ_1 is bounded. Therefore, for all i, j in E, the measures F_{ij} are finite and $F_{ij}(\infty) = \mathbb{E}_j[\Phi_1^\kappa \mathbf{1}_{X_1=i}]$. Note that $F(\infty) = A'_\kappa$. As A_κ is irreducible and aperiodic by Proposition 2, so is $F(\infty)$, and its spectral radius also equals to 1. Besides, we have $b_{ij} = \mathbb{E}_j[\Phi_1^\kappa \mathbf{1}_{X_1=i}\log\Phi_1]$ thus the F_{ij} have finite expectation.

We are going to prove that the other assumptions of Theorem A are valid here in the following sections.

6.1 F is non-lattice

Set $a_m = \min_{i\in E}\{a(i)\}$, $a_M = \max_{i\in E}\{a(i)\}$ and i_0, j_0 in E such that $a(i_0) = a_m$ and $a(j_0) = a_M$.

Proposition 7 *For all i, j in E, $x \in \,]a_m, a_M[$ and small enough $\varepsilon > 0$, we have*

$$\mathbb{P}_i\Big(\int_0^1 a(X_u)du \;\in\,]x-\varepsilon; x+\varepsilon[,\;\; X_1 = j\Big) > 0,$$

i.e. x is a point of increase of $\log\Phi_1$ conditionally to $X_0 = i$ and $X_1 = j$.

Proof Set $x \in \,]a_m, a_M[$ and $0 < t < 1$ such that $x = ta_m + (1-t)a_M$. Fix i and j in E. As q is an irreducible matrix, we can find integers $0 \le l \le m \le n$ and $k_1, \dots, k_n$ in E such that $q_{i,k_1}q_{k_1,k_2}\cdots q_{k_l,i_0} > 0$, $q_{i_0,k_{l+1}}q_{k_{l+1}k_{l+2}}\cdots q_{k_m,j_0} > 0$ and $q_{j_0,k_{m+1}}q_{k_{m+1}k_{m+2}}\cdots q_{k_n,j} > 0$. Set also $y = a(i)+a(k_1)+\cdots+a(k_l)-(l+1)a_m + a(k_{l+1}) + \cdots + a(k_m) - (n-l+1)a_M + a(k_{m+1}) + \cdots + a(k_n) + a(j)$, and $z = \min\left\{\varepsilon|y|^{-1}, t(l+1)^{-1}, (1-t)(n-l+1)^{-1}\right\}$. Then we have:

$$\begin{aligned}
&\mathbb{P}_i\Big(\int_0^1 a(X_u)du \;\in]x-\varepsilon; x+\varepsilon[,\;\; X_1 = j\Big)\\
&\quad\ge\;\; \mathbb{P}_i\Big(X_u = i \text{ on } [0;\eta[,\;\; X_u = k_1 \text{ on } [\eta; 2\eta[, \dots,\;\; X_u = k_l \text{ on } [l\eta; (l+1)\eta[,\\
&\qquad\quad X_u = i_0 \text{ on } [(l+1)\eta, t[,\;\; X_u = k_{l+1} \text{ on } [t; t+\eta[, X_u = k_{l+2} \text{ on}\\
&\qquad\quad [t+\eta; t+2\eta[, \dots,\;\; X_u = k_m \text{ on } [t+(m-l-1)\eta; t+(m-l)\eta[,\\
&\qquad\quad X_u = j_0 \text{ on } [t+(m-l)\eta; 1-(n-m+1)\eta[,\;\; X_u = k_{m+1} \text{ on}\\
&\qquad\quad [1-(n-m+1)\eta; 1-(n-m)\eta[, \dots,\;\; X_u = k_n \text{ on } [1-2\eta; 1-\eta[,\\
&\qquad\quad X_u = j \text{ on } [1-\eta; 1]\;;\;\; \eta \in \,]0; z[\,\Big). \qquad (10)
\end{aligned}$$

Indeed, on this event we have:

$$\begin{aligned}
\int_0^1 a(X_u)du &= \eta a(i) + \eta a(k_1) + \cdots + \eta a(k_l) + (t-(l+1)\eta)a_m + \eta a(k_{l+1})\\
&\quad + \cdots + \eta a(k_m) + ((1-t)-(n-l+1)\eta)a_M + \eta a(k_{m+1})\\
&\quad + \cdots + \eta a(k_n) + \eta a(j)\\
&= ta_m + (1-t)a_M + \eta y\\
&= x + \eta y,
\end{aligned}$$

thus if $\eta < \varepsilon|y|^{-1}$ then we have $\int_0^1 a(X_u)du \in \,]x-\varepsilon; x+\varepsilon[$. Probability (10) can be computed (see e.g. [Norris, 1998]):

$$\begin{aligned}
(10) \;\;=\;\; & \mu(i)q_{i,k_1}q_{k_1,k_2}\cdots q_{k_l,i_0}q_{i_0,k_{l+1}}\cdots q_{k_m,j_0}q_{j_0,k_{m+1}}\cdots q_{k_n,j}\times\\
& \lambda(i)\lambda(k_1)\cdots\lambda(k_n)\lambda(i_0)(l-1)\lambda(j_0)(n-l+1)\int_0^z \Big[e^{-\lambda(i)\eta}\times\\
& e^{-\lambda(k_1)\eta}\cdots e^{-\lambda(k_n)\eta}e^{-\lambda(i_0)(t-(l-1)\eta)}e^{-\lambda(j_0)(1-t-(n-l+1)\eta}e^{-\lambda(j)\eta}\Big]d\eta.
\end{aligned}$$

Thus our choice of $k_1, \dots, k_n$ and z ascertains that this probability is positive, which proves the proposition. □

Therefore none of the $F_{ij}(\cdot) = \mathbb{E}_j[\Phi_1^\kappa \mathbf{1}_{X_1=i}\mathbf{1}_{\cdot\geq\log\Phi_1}]$ can be concentrated on a lattice set, and in particular F is non-lattice.

6.2 Finiteness of U

We are going to prove that for all i, j in E and t in $\mathbb{R}$, $U_{ij}(t)$ is finite. We start with computing the n-fold convolution of F.

Lemma 1 *For all n, i, j, t we have:*

$$F_{ij}^{(n)}(t) = \mathbb{E}_j[\Phi_1^\kappa \cdots \Phi_n^\kappa \mathbf{1}_{\log\Phi_1\cdots\Phi_n\geq t}\mathbf{1}_{X_n=i}].$$

Proof For $n = 1$, it is the definition of F. Suppose the formula is true for a fixed n. Then the Markov property and stationarity yield:

$$\begin{aligned}
&F_{ij}^{(n+1)}(t)\\
&= \sum_{k=1}^N F_{ik} * F_{kj}^{(n)}(t) \;=\; \sum_{k=1}^N \int F_{kj}^{(n)}(t-u)F_{ik}(du)\\
&= \sum_{k=1}^N \int \mathbb{E}_j[\Phi_1^\kappa\cdots\Phi_n^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\geq t-u}\mathbf{1}_{X_n=k}]\mathbb{E}_k[\Phi_1^\kappa\delta_u(\log\Phi_1)\mathbf{1}_{X_1=i}]\\
&= \sum_{k=1}^N \int \mathbb{E}_\mu[\Phi_1^\kappa\cdots\Phi_n^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\geq t-u}\mathbf{1}_{X_n=k}\mathbf{1}_{X_0=j}] \times\\
&\quad \mathbb{E}_\mu[\Phi_{n+1}^\kappa\delta_u(\log\Phi_{n+1})\mathbf{1}_{X_{n+1}=i}\mathbf{1}_{X_n=k}]\frac{1}{\mu(k)\mu(j)}\\
&= \sum_{k=1}^N \mathbb{E}_\mu[\Phi_1^\kappa\cdots\Phi_n^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\geq t-\log\Phi_{n+1}}\mathbf{1}_{X_0=j}\mid\mathbf{1}_{X_n=k}] \times\\
&\quad \mathbb{E}_\mu[\Phi_{n+1}^\kappa\mathbf{1}_{X_{n+1}=i}\mid\mathbf{1}_{X_n=k}]\frac{\mu(k)}{\mu(j)}\\
&= \sum_{k=1}^N \mathbb{E}_\mu[\Phi_1^\kappa\cdots\Phi_n^\kappa\Phi_{n+1}^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\geq t-\log\Phi_{n+1}}\mathbf{1}_{X_0=j}\mathbf{1}_{X_{n+1}=i}\mid\mathbf{1}_{X_n=k}]\frac{\mu(k)}{\mu(j)}\\
&= \mathbb{E}_\mu[\Phi_1^\kappa\cdots\Phi_n^\kappa\Phi_{n+1}^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\Phi_{n+1}\geq t}\mathbf{1}_{X_0=j}\mathbf{1}_{X_{n+1}=i}]\frac{1}{\mu(j)}\\
&= \mathbb{E}_j[\Phi_1^\kappa\cdots\Phi_n^\kappa\Phi_{n+1}^\kappa\mathbf{1}_{\log\Phi_1\cdots\Phi_n\Phi_{n+1}\geq t}\mathbf{1}_{X_{n+1}=i}].
\end{aligned}$$

Thus the formula is also true for $n+1$ and the lemma is proved. □

We have seen that $F(\infty) = A_\kappa'$. Proposition 1 and the preceding lemma also imply that for all n we have $F^{(n)}(\infty) = (A_\kappa^n)' = F(\infty)^n$. We can prove a more general result.

Lemma 2 *For all n and $0 \leq r < \kappa$ we have:*

$$\int_\infty^\infty e^{-ru}F^{(n)}(du) = (A_{\kappa-r}^n)'.$$

Proof For all i, j in E, Proposition 1 and the preceding lemma yield

$$\begin{aligned}\int_{\infty}^{\infty} e^{-ru} F_{ij}^{(n)}(du) &= \int_{\infty}^{\infty} e^{-ru} \mathbb{E}_j[\Phi_1^\kappa \cdots \Phi_n^\kappa \delta_u(\log \Phi_1 \cdots \Phi_n) \mathbf{1}_{X_n=i}] \\ &= \mathbb{E}_j[\Phi_1^\kappa \cdots \Phi_n^\kappa e^{-r \log \Phi_1 \cdots \Phi_n} \mathbf{1}_{X_n=i}] \\ &= \mathbb{E}_j[\Phi_1^{\kappa-r} \cdots \Phi_n^{\kappa-r} \mathbf{1}_{X_n=i}] \\ &= (A_{\kappa-r}^n)_{ji}. \end{aligned}$$

□

Now fix $0 < r < kappa$. We have:

$$\begin{aligned} U_{ij}(t) &= \sum_{n=0}^{\infty} F_{ij}^{(n)}(t) \\ &\leq e^{rt} \int_{-\infty}^{t} e^{-ru} \sum_{n=0}^{\infty} F_{ij}^{(n)}(du) \\ &\leq e^{rt} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-ru} F_{ij}^{(n)}(du) \\ &= e^{rt} \sum_{n=0}^{\infty} (A_{\kappa-r}^n)_{ji}, \end{aligned} \tag{11}$$

according to the preceding lemma. But Corollary 3 says that $\rho(A_{\kappa-r}) < 1$. Thus the series in (11) converges. Hence $U_{ij}(t) < \infty$ for all i, j in E and t in $\mathbb{R}$.

6.3 Proof of $\mathbf{Z} = \mathbf{U} * \mathbf{G}$

Iterating the renewal equation (9) yields, for all n,

$$Z = F^{(n)} * Z + \sum_{k=0}^{n-1} F^{(k)} * G. \tag{12}$$

The same change of variable as in section 5.2 yields:

$$\sum_{i=1}^{N} (F^{(n)} * Z)_i(t) = e^{-t} \int_0^{e^t} u^\kappa \mathbb{P}_\mu(\Phi_1 \Phi_2 \cdots \Phi_n R_0 > u) du.$$

But we have seen at (7) that we have $\Phi_1 \cdots \Phi_n \to 0$ when n tends to infinity. Thus the bounded convergence Theorem yields $\sum_{i=1}^{N} (F^{(n)} * Z)_i(x, t) \to 0$ as n tends to infinity. Each term of this sum is non-negative, thus each term tends to 0. Letting n tend to infinity in Equation (12) we thus get $Z = U * G$.

6.4 G is directly Riemann integrable

As the G_i are clearly continuous in t, it is sufficient to prove that :

$$\sum_{l=-\infty}^{\infty} \sup_{l \leq t < l+1} |G_i(t)| < \infty,$$

(see [Feller, 1966]). But for all i, t, we have $G_i(t) = G_i^1(t) - G_i^2(t)$, where

$$\begin{aligned} G_i^1(t) &= e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_\mu(u - b_1 < \Phi_1 R_0 \le u, X_1 = i)du \ \ge\ 0, \\ G_i^2(t) &= e^{-t}\int_0^{e^t} u^\kappa \mathbb{P}_\mu(u < \Phi_1 R_0 \le u - b_1, X_1 = i)du \ \ge\ 0. \end{aligned}$$

For all real t, we have $G_i(t) \le G_i^1(t) \le e^{-t}\int_0^{e^t} u^\kappa du = e^{t(\kappa+1)}(\kappa+1)^{-1}$. In particular, G_i is directly Riemann integrable on $\mathbb{R}_-$. We still have to study G_i^1 and G_i^2 on $\mathbb{R}_+$. These two functions being of the same kind we only give the detailed study of the first one.

The proof is adapted from [LePage, 1983]. Set $\varepsilon \in]0;1[$ such that $-1 < \kappa - (1-\varepsilon) < 0$. Thus we have:

$$\begin{aligned} 0 \le e^t G_i^1(t) \le \int_0^{e^t} & u^\kappa \mathbb{P}_\mu(b_1 > u^\varepsilon, X_1 = i)du + \\ & \textstyle\int_0^{e^t} u^\kappa \mathbb{P}_\mu(u - u^\varepsilon < \Phi_1 R_0 \le u, X_1 = i)du. \end{aligned} \tag{13}$$

We are going to give an upper bound for each one of these two terms.

- *First term:*

Chebychev inequality yields:

$$\int_0^{e^t} u^\kappa \mathbb{P}_\mu(b_1 > u^\varepsilon, X_1 = i)du \le \mathbb{E}_\mu|b_1|^\kappa \frac{e^{t(1+\kappa(1-\varepsilon))}}{1+\kappa(1-\varepsilon)}. \tag{14}$$

Note that b_1 has moments of all order. Indeed, we have, by independence, $\mathbb{E}_\mu|b_1|^\kappa = \mathbb{E}_\mu(V_1^{\kappa/2})\mathbb{E}_\mu|\xi_1|^\kappa$, and ξ_1 is a standard Gaussian variable and V_1 is bounded.

- *Second term:*

We have:

$$\begin{aligned} &\int_0^{e^t} u^\kappa \mathbb{P}_\mu(u - u^\varepsilon < \Phi_1 R_0 \le u, X_\delta = i)du \\ &= \int_0^{e^t} u^\kappa \mathbb{P}_\mu(\Phi_1 R_0 > u - u^\varepsilon, X_\delta = i)du - \int_0^{e^t - e^{t\varepsilon}} u^\kappa \mathbb{P}_\mu(\Phi_1 R_0 > u, X_\delta = i)du \\ &\le \int_0^{e^t} u^\kappa[1 - \mathbf{1}_{u\ge 1}(u-u^\varepsilon)^\kappa(1-\varepsilon u^{\varepsilon-1})]\mathbb{P}_\mu(\Phi_1 R_0 > u - u^\varepsilon, X_\delta = i)du. \end{aligned}$$

Set $0 < r < \kappa$. As Φ_1 is bounded, there exists a positive constant c such that for all $u > 0$ we have:

$$\mathbb{P}_\mu(\Phi_1 R_0 > u, X_1 = i) \le c\frac{\mathbb{E}_\mu|R_0|^r}{u^r},$$

which is bounded by Proposition 6. Thus we get:

$$\int_0^{e^t} u^\kappa \mathbb{P}_\mu(u - u^\varepsilon < \Phi_1 R_0 \le u, X_1 = i)du \le Ce^{t(\kappa - r + \varepsilon - 1)}, \tag{15}$$

where C is a positive constant. Now set $\beta = \max\{\kappa + \varepsilon - r;\ 1 + \kappa - \kappa\varepsilon\} \in]0; 1[$. Then Eq. (13), (14) and (15) yield $e^t G_i^1(t) \leq c e^{t\beta}$ for all $t > 0$. Thus $G_i^1(t) \leq c e^{t(\beta-1)}$ is directly Riemann integrable on $\mathbb{R}_+$.

6.5 Tail of the distribution

We have now proved that F and G satisfy the assumptions of Theorem A. Thus we get, for all i, t:

$$Z_i(t) \xrightarrow[t\to\infty]{} c m_i \sum_{j=1}^{N} u_j \int_{-\infty}^{\infty} G_j(y)dy. \tag{16}$$

Summing up these terms, we get:

$$z(t) \xrightarrow[t\to\infty]{} c \sum_{j=1}^{N} u_j \int_{-\infty}^{\infty} G_j(y)dy, \tag{17}$$

as $\sum m_i = 1$. We still have to prove that this limit is non-zero.

7 Proof of Theorem 2, part II

Now we are going to prove that there exists a positive constant C such that $t^\kappa \mathbb{P}_\mu(|R_1| > t) \geq C > 0$ when t tends to infinity. First, we give a lower bound of this probability involving the products $\Phi_1 \cdots \Phi_n$, and then we study the asymptotic behavior of such products.

7.1 Lower bound for $\nu\{\mathbf{x} : |\mathbf{x}| > \mathbf{t}\}$

The following proof is adapted from [Goldie, 1991] and [de Saporta, 2004].

Proposition 8 *There exist $\varepsilon > 0$ and a corresponding positive constant C such that for large enough t we have:*

$$\mathbb{P}_\mu(|R_1| > t) \geq C\ \mathbb{P}_\mu(\sup_n\ (\Phi_1 \cdots \Phi_n) > \frac{2t}{\varepsilon}).$$

For the i.i.d. case, the key to such a lower bound is an inequality established in [Grincevičius, 1980] that extends Lévy's symmetrization inequality (see [Chow and Teicher, 1978]). Here we need first to extend this inequality.

Recall that $R_1 = \sum_{k=0}^{\infty} \Phi_1 \Phi_0 \cdots \Phi_{2-k} b_{1-k}$. For all $n \geq 1$, we set:

$$R_1^n = \sum_{k=0}^{n-1} \Phi_1 \Phi_0 \cdots \Phi_{2-k} b_{1-k} \quad \text{and} \quad \Pi_n = \Phi_1 \Phi_0 \cdots \Phi_{2-n}.$$

If x is a $\sigma(X_t, W_t, a \leq t \leq b)$-measurable random variable, let $med_i(x)$ be a median of x conditionally to $X_b = i$ and $med_-(x) = \min_i\{med_i(x)\}$.

Lemma 3 *For all $t > 0$ and $n \geq 1$, we have*

$$\mathbb{P}_\mu\Big(\max_{1\leq j\leq n} \Big\{R_1^j + \Pi_j med_-\Big(\frac{R_1^n - R_1^j}{\Pi_j}\Big)\Big\} > t\Big) \leq 2\mathbb{P}_\mu(R_1^n > t).$$

Proof Set $T = \inf\Big\{j \leq n$ t.q. $R_1^j + \Pi_j med_-\big(\Pi_j^{-1}(R_1^n - R_1^j)\big) > t\Big\}$ if this set is not empty, $n+1$ otherwise, and $B_j = \Big\{med_-\big(\Pi_j^{-1}(R_1^n - R_1^j)\big) \leq \Pi_j^{-1}(R_1^n - R_1^j)\Big\}$. The event $(T = j)$ is in the σ-field generated by $(X_t, W_t, (1-j) \leq t \leq 1)$, and B_j is in the σ-field generated by $(X_t, W_t, (1-n) \leq t \leq (1-j))$. Therefore these events are independent conditionally to $X_{(1-j)}$. Besides, for all i and j we have $\mathbb{P}_\mu(B_j \mid X_{(1-j)} = i) \geq \mathbb{P}_\mu\Big(med_i\big(\Pi_j^{-1}(R_1^n - R_1^j)\big) \leq \Pi_j^{-1}(R_1^n - R_1^j) \mid X_{(1-j)} = i\Big) \geq 1/2$. Thus, as the products Π_j are positive or zero, we have:

$$\begin{aligned}
\mathbb{P}_\mu(R_1^n > t) &\geq \mathbb{P}_\mu\Big(\bigcup_{j=1}^n [B_j \cap (T = j)]\Big) \\
&= \sum_{j=1}^n \sum_{i=1}^N \mathbb{P}_\mu(B_j \mid X_{(1-j)} = i)\mathbb{P}(T = j \mid X_{(1-j)} = i)\mu(i) \\
&\geq \frac{1}{2}\mathbb{P}_\mu(T \leq n) \\
&= \frac{1}{2}\mathbb{P}_\mu\Big(\max_{1\leq j\leq n}\Big\{R_1^j + \Pi_j med\Big(\frac{R_1^n - R_1^j}{\Pi_j}\Big)\Big\} > t\Big). \qquad \square
\end{aligned}$$

Under our assumptions, R_1^n tends to R_1 when n tends to infinity, and for fixed j, $\Pi_j^{-1}(R_1^n - R_1^j)$ converges to a random variable $\widehat{R}$ that has the same distribution as R_1. Set $m_0 = med_-(R_1) = \min_i\{med(R_1 \mid X_1 = i)\} = med_-(\widehat{R})$, and letting n tend to infinity in Lemma 3, we get, for all $t > 0$,

$$\mathbb{P}_\mu\Big(\sup_j\{R_1^j + \Pi_j m_0\} > t\Big) \leq 2\mathbb{P}_\mu(R_1 > t).$$

Replacing R_1 by $-R_1$ yields a similar formula, thus, for all $t > 0$ we get:

$$\mathbb{P}_\mu\Big(\sup_j |R_1^j + \Pi_j m_0| > t\Big) \leq 2\mathbb{P}_\mu(|R_1| > t). \tag{18}$$

Furthermore, as proved in [Goldie, 1991], p.157, for all $t > |m_0|$ we have:

$$\begin{aligned}
&\mathbb{P}_\mu\Big(\sup_n\{R_1^n + \Pi_n m_0\} > t\Big) \\
&\quad\geq \mathbb{P}_\mu\Big(\exists n \text{ s.t. } |(R_1^{n+1} + \Pi_{n+1}m_0) - (R_1^n + \Pi_n m_0)| > 2t\Big),
\end{aligned}$$

where $R_1^0 = 0$ and $\Pi_0 = 1$. But we have:

$$\begin{aligned}
(R_1^{n+1} + \Pi_{n+1}m_0) - (R_1^n + \Pi_n m_0) &= \Phi_1\Phi_0 \cdots \Phi_{2-n}b_{1-n} + (\Pi_{n+1} - \Pi_n)m_0 \\
&= \Pi_n\big(b_{1-n} + (\Phi_{1-n} - 1)m_0\big).
\end{aligned}$$

Thus Eq. (18) yields, for all $\varepsilon > 0$

$$\begin{aligned}
\mathbb{P}_\mu(|R_1| > t) &\geq \frac{1}{2}\mathbb{P}_\mu\Big(\exists n \text{ s.t. } \big|\Pi_n\big(b_{1-n} + (\Phi_{1-n} - 1)m_0\big)\big| > 2t\Big) \\
&\geq \frac{1}{2}\mathbb{P}_\mu\Big(\exists n \text{ s.t. } \big|\Pi_n\big| > \frac{2t}{\varepsilon} \text{ and} \\
&\qquad \big|b_{1-n} + (\Phi_{1-n} - 1)m_0\big| > \varepsilon\Big).
\end{aligned} \tag{19}$$

Now we give an extension of Feller-Chung's inequality adapted to the present case (see [Chow and Teicher, 1978]):

Lemma 4 *For all $t > |m_0|$ and $\varepsilon > 0$, we have:*

$$\mathbb{P}_\mu\Big(\exists n \quad s.t. \quad \big|\Pi_n\big| > \frac{2t}{\varepsilon} \; and \; \big|b_{1-n} + (\Phi_{1-n} - 1)m_0\big| > \varepsilon\Big)$$
$$\geq \min_{1\leq i\leq N} \mathbb{P}_i(|b_0 + (\Phi_0 - 1)m_0| > \varepsilon)\mathbb{P}_\mu\big(\exists n \quad s.t. \quad |\Pi_n| > \frac{2t}{\varepsilon}\big).$$

Proof Set $A_0 = \emptyset$, $A_n = \{|\Pi_n| > 2t\varepsilon^{-1}\}$ and $B_n = \{|b_{1-n}+(\Phi_{1-n}-1)m_0| > \varepsilon\}$. Conditionnally to $X_{(1-n)1}$, B_n is independent of $A_0, \ldots, A_n$. Thus we have:

$$\begin{aligned}
&\mathbb{P}_\mu\big(\bigcup_{n=1}^{\infty}[A_n \cap B_n]\big) \\
&= \sum_{n=1}^{\infty}\mathbb{P}_\mu\big(B_n \cap A_n \bigcap_{j=0}^{n-1}[B_j \cap A_j]^c\big) \\
&\geq \sum_{n=1}^{\infty}\mathbb{P}_\mu\big(B_n \cap A_n \bigcap_{j=0}^{n-1} A_j^c\big) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{N}\Big[\mathbb{P}_\mu(B_n \mid X_{(1-n)1} = i)\mathbb{P}_\mu\big(A_n \bigcap_{j=0}^{n-1} A_j^c \mid X_{(1-n)1} = i\big)\mu(i)\Big].
\end{aligned}$$

where A^c denotes the complementary set of A. But, by stationnarity we have $\mathbb{P}_\mu(B_n \mid X_{(1-n)1} = i) = \mathbb{P}_i(|b_0 + (\Phi_0 - 1)m_0| > \varepsilon)$. Thus we get:

$$\mathbb{P}_\mu\big(\bigcup_{n=1}^{\infty}[A_n \cap B_n]\big) \geq \min_{1\leq i\leq N}\mathbb{P}_i(|b_0 + (\Phi_0 - 1)m_0| > \varepsilon)\mathbb{P}_\mu\big(\bigcup_{n=1}^{\infty} A_n\big). \qquad \square$$

Now we can give the proof of Proposition 8.

Proof of Proposition 8 Eq. (19) and Lemma 4 yield, for all $t > |m_0|$ and $\varepsilon > 0$,

$$\mathbb{P}_\mu(|R_1| > t) \geq \frac{1}{2}\min_{1\leq i\leq N}\mathbb{P}_i(|b_0 + (\Phi_0 - 1)m_0| > \varepsilon)\mathbb{P}_\mu(\exists n \text{ s.t. } |\Pi_{n-1}| > \frac{2t}{\varepsilon}).$$

As we have $b_0 = V_0^{1/2}\xi_0$, V_0 and Φ_0 are bounded, but ξ is not bounded as it is a Gaussian variable equality $b_0 + (\Phi_0 - 1)m_0 = 0$ can not hold $\mathbb{P}_i$-almost surely. Thus we can find $\varepsilon > 0$ such that $\min_{1\leq i\leq N}\mathbb{P}_i(|b_0 + (\Phi_0 - 1)m_0| > \varepsilon) > 0$. Hence, as expected there is a constant $C > 0$ such that for all $t > |m_0|$, we have:

$$\mathbb{P}_\mu(|R_1| > t) \geq C\ \mathbb{P}_\mu(\sup_n\ |\Pi_n| > \frac{2t}{\varepsilon}). \qquad \square$$

7.2 Asymptotic behaviour of the products $\mathbf{\Phi_1 \cdots \Phi_n}$

To estimate the probability $\mathbb{P}_\mu(\sup_n\ |\Pi_n| > t)$, we use the ladder height method given by [Feller, 1966] for the study of the maximum of random walks.

7.2.1 Notation

First we introduce some notation. Set $S_0 = 0$ and for all positive n, we set:

$$S_n = \sum_{k=1}^{n} \log(\Phi_{2-k}) = \log \Pi_n = \int_{(1-n)}^{1} a(X_u)du.$$

The *first ladder epoch* of this random walk is $\tau = \tau_1 = \inf\{n \geq 1 \text{ s.t. } S_n > 0\}$, and the *first ladder height* is S_τ. We denote by $H(t)$ the matrix of distributions of S_τ with the following coordinates:

$$H_{ij}(t) = \mathbb{P}_\mu(\tau < \infty, S_\tau \leq t, X_{(1-\tau)} = j \mid X_1 = i).$$

As $S_\tau > 0$, H is distributed on the positive half-line. Moreover $S_\tau > 0$, $S_{1-\tau} \leq 0$ and the Φ_n are bounded, thus we have $S_\tau \leq \sup \log \Phi_n \leq \sup a(i) < \infty$, and H has bounded support.

We define also the n^{th} ladder epoch by $\tau_n = \inf\{n > \tau_{n-1} \text{ s.t. } S_n > S_{\tau_{n-1}}\}$, and S_{τ_n} is the corresponding ladder height. We check that we have:

$$H_{ij}^{(n)}(t) = \mathbb{P}_\mu(\tau_n < \infty, S_{\tau_n} \leq t, X_{(1-\tau_n)} = j \mid X_1 = i),$$

where $H^{(n)}$ is the n-fold convolution of H. Let $\Psi = \sum_{n=0}^{\infty} H^{(n)}$ be the renewal function associated with H.

7.2.2 The random walk $\mathbf{S}_{\tau_\mathbf{n}}$

To investigate the asymptotic behaviour of (S_{τ_n}) we are going to use a renewal theorem as in [Feller, 1966] for the i.i.d. case, namely Theorem B. We want to apply it for $F = H$ and $\alpha = s$, thus we have to prove that H satisfies its assumptions.

As $H(0) = 0$, we have $\rho(H(0)) < 1$ thus the first assumption is true. In addition, H_{ij} are probability measures, therefore H is finite. We have seen that H thus $\widehat{B}$, the expectation of $H_\kappa(\infty) = \int_0^\infty e^{-\kappa u} H(du)$ is well defined. Proposition 7 yields again that H is also non-lattice.

Irreducibility and aperiodicity For all i, j in E, we have:

$$\begin{aligned} H_{ij}(\infty) &= \mathbb{P}_\mu(\tau < \infty,\ X_{1-\tau} = j \mid X_1 = i) \\ &\geq \mathbb{P}_\mu(\tau = 1,\ X_0 = j \mid X_1 = i) \\ &= \mathbb{P}_j(\log \Phi_1 > 0,\ X_1 = i)\frac{\mu(j)}{\mu(i)} \\ &= \mathbb{P}_j\Big(\int_0^1 a(X_u)du > 0,\ X_1 = i\Big)\frac{\mu(j)}{\mu(i)}, \end{aligned}$$

and Proposition 7 implies that the last term is positive as $0 \in]a_m; a_M[$. Thus the second assumption af Theorem B is valid. We have also proved that for all i and j we have $0 = H_{ij}(0) < H_{ij}(\infty)$, so that the third assumption is also valid.

Spectral radius of $\mathbf{H}_\kappa(\infty)$ Now we define a new probability law $\mathbb{P}_\kappa$ on $\Omega\times\Theta$. For all bounded $\mathcal{A}\times\mathcal{B}$-measurable functions f which first coordinate depends only on $(X_t,(1-n)\le t\le 1)$, we set:

$$\mathbb{P}_\kappa(f)=\mathbb{E}_\kappa(f)=\frac{\mathbb{E}_\mu\big(f(\Phi_1,\ldots,\Phi_{2-n},\theta)(\Phi_1\cdots\Phi_{2-n})^\kappa\big)}{\mathbb{E}_\mu\big((\Phi_1\cdots\Phi_{2-n})^\kappa\big)}.$$

Set $H_\kappa(t)=\int_0^t e^{-\kappa u}H(du)$. We have

$$\begin{aligned}(H_\kappa)_{ij}(t) &= \frac{\mathbb{P}_\kappa(\tau<\infty,S_\tau\le t,X_{(1-\tau)}=j\mid X_1=i)}{\mathbb{E}_\mu\big((\Phi_1\cdots\Phi_{1-\tau})^\kappa,\tau<\infty\big)}\\ &= \frac{(H^\sharp_\kappa)_{ij}(t)}{\mathbb{E}_\mu\big((\Phi_1\cdots\Phi_{1-\tau})^\kappa,\tau<\infty\big)},\end{aligned}$$

where $(H^\sharp_\kappa)_{ij}(t)=\mathbb{P}_\kappa(\tau<\infty,S_\tau\le t,X_{(1-\tau)}=j\mid X_1=i)$ describes the behaviour of the ladder heights of our random walk under the new probability law $\mathbb{P}_\kappa$.

The computation we made in the proof of Proposition 3 yields:

$$\begin{aligned}\frac{\partial}{\partial r}\Big|_{r=\kappa}\log\big(\rho(A_r)\big) &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_\kappa\Big(\sum_{i=1}^n\log\Phi_i\Big)\\ &= \mathbb{E}_\kappa(\log\Phi_1).\end{aligned}$$

But we have $\log\rho(A_0)=\log\rho(A_\kappa)=0$, this function is convex (Corollary 2) and its right-hand derivative at 0 is negative (Proposition 3). Thus its left-hand derivative at s is positive, i.e. $\mathbb{E}_\kappa(\log\Phi_1)>0$. Under the law $\mathbb{P}_\kappa$ our random walk thus drifts to $+\infty$, hence for all n and i, we have $(\mathbb{P}_\kappa)_i(\tau_n<\infty)=1$ and $H^\sharp$ is a stochastic matrix, therefore its spectral radius equals to 1.

For all n, we have:

$$H_\kappa^{(n)}(\infty)=(H_\kappa(\infty))^n=\frac{(H^\sharp_\kappa(\infty))^n}{\mathbb{E}_\mu\big((\Phi_1\cdots\Phi_{2-\tau_n})^\kappa,\tau<\infty\big)},$$

thus $\rho(H_\kappa(\infty))=\lim\big(\mathbb{E}_\mu((\Phi_1\cdots\Phi_{2-\tau_n})^\kappa,\tau<\infty)\big)^{-1/n}$ and we now have to prove that this limit equals to 1. But for all n, we have $\tau_n\ge n$, and the event $(\tau_n=k)$ depends only on $(X_t,(1-k)\le t\le 1)$. Thus we have:

$$\begin{aligned}\mathbb{E}_\mu((\Phi_1\cdots\Phi_{1-\tau_n})^\kappa,\tau_n<\infty) &= \sum_{k=n}^\infty\mathbb{E}_\mu((\Phi_1\cdots\Phi_{1-k})^\kappa,\tau_n=k)\\ &= \sum_{k=n}^\infty\mathbb{P}_\kappa(\tau_n=k)\mathbb{E}_\mu((\Phi_1\cdots\Phi_{1-k})^\kappa). \quad (20)\end{aligned}$$

Set $\varepsilon>0$. For large enough n, our choice of s and Eq. (5) and (6) yield:

$$\mu A_\kappa^n\mathbf{1}-\varepsilon\le\mathbb{E}_\mu((\Phi_1\cdots\Phi_{1-n})^\kappa)\le\mu A_\kappa^n\mathbf{1}+\varepsilon.$$

Thus for large enough n, Eq. (20) yields:

$$\begin{aligned}&(\mu A_\kappa^n\mathbf{1}-\varepsilon)\sum_{k=n}^\infty\mathbb{P}_\kappa(\tau_n=k)\\ &\le\ \mathbb{E}_\mu((\Phi_1\cdots\Phi_{1-\tau_n})^\kappa,\tau_n<\infty)\ \le\ (\mu A_\kappa^n\mathbf{1}+\varepsilon)\sum_{k=n}^\infty\mathbb{P}_\kappa(\tau_n=k),\end{aligned}$$

and as $\mathbb{P}_\kappa(\tau_n < \infty) = 1$, we have:

$$\mu A_\kappa^n \mathbf{1} - \varepsilon \le \mathbb{E}_\mu((\Phi_1 \cdots \Phi_{1-\tau_n})^\kappa, \tau_n < \infty) \le \mu A_\kappa^n \mathbf{1} + \varepsilon.$$

Thus as $n \to \infty$ we have, with the notation of Corollary 1, $\mathbb{E}_\mu((\Phi_1 \cdots \Phi_{1-\tau_n})^\kappa \sim \mu B_\kappa \mathbf{1}$. Hence we have, as expected $\mathbb{E}_\mu((\Phi_1 \cdots \Phi_{1-\tau_n})^\kappa, \tau_n < \infty)^{1/n} \to 1$.

Thus all the assumptions of Theorem B are valid here. We are going to use it in the following part.

7.2.3 Asymptotic behaviour of the maximum

Let $M = \sup_n S_n = \sup_n S_{\tau_n}$, be the maximum of our random walk. Using the definition of H, we get, for all $1 \le i \le N$:

$$\begin{aligned}
&\mathbb{P}_\mu(M \le t \mid X_1 = i) \\
&= \sum_{n=1}^{\infty} \mathbb{P}_\mu(\tau_n < \infty, S_{\tau_n} \le t, \tau_{n+1} = \infty \mid X_1 = i) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{N} \mathbb{P}_\mu(\tau_n < \infty, S_{\tau_n} \le t, \tau_{n+1} = \infty,\ X_1 = i \mid X_{(1-\tau_n)} = j) \frac{\mu(j)}{\mu(i)} \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{N} \big[\mathbb{P}_\mu(\tau_n < \infty, S_{\tau_n} \le t, X_{(1-\tau_n)} = j \mid X_1 = i) \times \\
&\quad (1 - \mathbb{P}_\mu(\tau_{n+1} < \infty \mid X_{(1-\tau_n)} = j)\big] \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{N} \big[H_{ij}^{(n)}(t)\big(1 - \sum_{k=1}^{N} H_{jk}(\infty)\big)\big] \\
&= \sum_{j=1}^{N} \big[\Psi_{ij}(t)\big(1 - \sum_{k=1}^{N} H_{jk}(\infty)\big)\big]. \qquad (21)
\end{aligned}$$

Thorem B applied to (21) yields, when t tends to infinity:

$$\begin{aligned}
1 - \mathbb{P}_\mu(M \le t \mid X_1 = i) &= \sum_{j=1}^{N} \Big[\big(1 - \sum_{k=1}^{N} H_{jk}(\infty)\big) \int_t^\infty e^{-\kappa u}(e^{\kappa u}\Psi_{ij})(du)\Big] \\
&\overset{t\to\infty}{\sim} \sum_{j=1}^{N} \Big[\big(1 - \sum_{k=1}^{N} H_{jk}(\infty)\big) \int_t^\infty e^{-\kappa u} \widehat{c}\widehat{m}_i \widehat{u}_j du\Big] \\
&= \sum_{j=1}^{N} \Big[\big(1 - \sum_{k=1}^{N} H_{jk}(\infty)\big)\widehat{c}\widehat{m}_i \widehat{u}_j\Big] e^{-\kappa t}, \qquad (22)
\end{aligned}$$

where $\widehat{m}$ and $\widehat{u}$ are right and left eigenvectors of $H_\kappa(\infty)$ with positive coordinates with the same normalisation as in section 4, and $\widehat{c} = ({}^t\widehat{u}\widehat{B}\widehat{m})^{-1} > 0$.

7.3 Conclusion

We still have to prove that there is a $j \le N$ such that $1 - \sum_{k=1}^{N} H_{jk}(\infty) > 0$. But the mapping $r \longmapsto H_r(\infty) = \int_0^\infty e^{ru} H(du)$ is clearly increasing componentwise. As these matrix are non negative and irreducible, Corollaries 8.1.19 and

8.1.20 of [Horn and Johnson, 1985] imply that the mapping $r \longmapsto \rho(H_r(\infty))$ is also increasing. As $\rho(H_\kappa(\infty)) = 1$, we have $\rho(H_0(\infty)) = \rho(H(\infty)) < 1$. This is a sub-stochastic, non-stochastic matrix, thus there exists a j such that we have $1 - \sum_{k=1}^N H_{jk}(\infty) > 0$.

We have now proved that the right-hand side term in (22) is positive, thus there is a constant $C > 0$ such that, when t tends to infinity, we have:

$$e^{\kappa t}\mathbb{P}_\mu(M > t) \geq \sum_{i=1}^N e^{\kappa t}\mathbb{P}_\mu(M > t \mid X_1 = i)\mu(i) \geq C. \tag{23}$$

Putting together this result and Proposition 8, we get, for large enough t:

$$t^\kappa \mathbb{P}_\mu(|R_1| > t) \ \geq K \ > \ 0. \tag{24}$$

With the notation of Theorem 2, it means that $L > 0$, which ends the proof of this theorem.

8 Determination of κ

Set $s_1 = \min\{\lambda(i)a(i)^{-1} \mid a(i) > 0\}$, and let M_s be the matrix with components $\{q(i,j)\lambda(i)(\lambda(i) - sa(i))^{-1}\}$. This matrix is well defined for all $s < s_1$. We can precisely compare the spectral radius of A_s and that of M_s.

Proposition 9 *For all $0 < s < s_1$, we have $\rho(M_s) < 1$ if and only if $\rho(A_s) < 1$, and we have $\rho(M_s) > 1$ if and only if $\rho(A_s) > 1$.*

Proof Suppose that $\rho(M_s) < 1$. M_s is a positive irreducible matrix as q is, λ being positive and $s < s_1$. Thus Perron-Frobenius Theorem (see e.g. [Horn and Johnson, 1985]) gives the existence of a vector φ with positive coordinates such that $M_s\varphi = \rho(M_s)\varphi < \varphi$. Hence for all i in E, we have:

$$\varphi(i) > \sum_j q(i,j)\frac{\lambda(i)}{\lambda(i) - sa(i)}\varphi(j),$$

that we can rewrite, since $s < s_1$, as

$$(sa(i) - \lambda(i))\varphi(i) + \lambda(i)\sum_j q(i,j)\varphi(j) < 0. \tag{25}$$

Proposition 4 enables us to choose a small enough δ such that Formula (8) is valid here. Eq.(25) thus yields:

$$\begin{aligned} A_s\varphi(i) &= [1 + \delta(sa(i) - \lambda(i))]\varphi(i) + \delta\lambda(i)\sum_{j\neq i}[q(i,j)\varphi(j)] + o(\delta) \\ &= \varphi(i) + \delta[(sa(i) - \lambda(i))\varphi(i) + \lambda(i)\sum_j q(i,j)\varphi(j)] + o(\delta) \\ &< \varphi(i). \end{aligned}$$

Thus component-wise we get $A_s\varphi < \varphi$, which implies that $\rho(A_s) < 1$. The proof that $\rho(M_s) > 1$ implies $\rho(A_s) > 1$ runs the same, the inequalities being reversed.

Suppose now that $\rho(A_s) < 1$. A_s is a positive irreducible matrix, thus Perron-Frobenius Theorem gives the existence of a vector ψ with positive coordinates such that $A_s\psi = \rho(A_s)\psi < \psi$. Hence for all i in E, and small enough δ, we have:

$$\begin{aligned} \delta[(sa(i) - \lambda(i))\psi(i) + \lambda(i)\sum_j q(i,j)\psi(j)] + o(\delta) &= A_s\psi(i) - \psi_i \\ &< 0. \end{aligned}$$

Hence, for all i, we get $(sa(i) - \lambda(i))\psi(i) + \lambda(i)\sum_j q(i,j)\psi(j) < 0$, or, as $s < s_1$,

$$\psi(i) > \frac{\lambda(i)}{\lambda(i) - sa(i)} \sum_j q(i,j)\psi(j),$$

and thus $M_s\psi < \psi$. As M_s is a positive matrix, we conclude that $\rho(M_s) < 1$. Here again the proof that $\rho(A_s) > 1$ implies $\rho(M_s) > 1$ runs the same with reversed inequalities. □

Proposition 10 *The spectral radius of M_s tends to infinity when s tends to s_1.*

Proof Set $i_0 \in E$ such that $\lambda(i_0)a(i_0)^{-1} = s_1$, and e_{i_0} the row vector with zero coordinates except the i_0^{th} which is set to be 1. Set $v_{i_0} = \lambda(i_0)(\lambda(i_0) - sa(i_0))^{-1}$. We have $e_{i_0}M_s = v_{i_0}q(i_0,\cdot) \geq v_{i_0}e_{i_0}$ as q is a positive matrix. As M_s is also positive, for all $s < s_1$, we get $\rho(M_s) \geq v_{i_0} = \lambda(i_0)(\lambda(i_0) - sa(i_0))^{-1}$. Hence this spectral radius tends to infinity when s tends to s_1. □

Corollary 4 *There is a unique $s \in]0; s_1[$ such that $\rho(M_s) = 1$, and this s equals to the unique κ such that $\rho(A_\kappa) = 1$.*

Proof For all $s < \kappa$, we have $\rho(A_s) < 1$ by Corollary 3, thus Proposition 9 yields $\rho(M_s) < 1$ for all $0 < s < \min\{\kappa, s_1\}$. As $\rho(M_s) \to \infty$ as s tends to s_1, we also have $\rho(A_s) > 1$ for s close to s_1. Therefore $\kappa < s_1$, and $\rho(A_s) > 1$ for all $\kappa < s < s_1$. Hence $\rho(M_s) > 1$ for all $\kappa < s < s_1$. As M_s has continuous coordinates, its spectral radius is also continuous, thus $\rho(M_\kappa) = 1$ and κ is the only value of $s \in]0; s_1[$ satisfying this equation. □

We now give an illustration by computing the value of κ when $E = \{1, 2\}$. The jump kernel q then equals to:

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the invariant law of the process X is $\mu = (\lambda(2), \lambda(2))/(\lambda(1) + \lambda(2))$. We suppose that $a(1)$ or $a(2)$ is positive. Condition 2 becomes

$$\lambda(1)a(2) + \lambda(2)a(1) < 0. \tag{26}$$

For all i in E, set $r_i = \frac{a(i)}{\lambda(i)}$. We have $r_1 + r_2 < 0$, $r_1r_2 > 0$, and $s_1 = \max\{r_1^{-1}, r_2^{-1}\}$. For $s \in [0, s_1[$, the matrix M_s equals to:

$$M_s = \begin{pmatrix} 0 & \frac{1}{1 - sr_1} \\ \frac{1}{1 - sr_2} & 0 \end{pmatrix},$$

and its spectral radius is $\left[(1-sr_1)(1-sr_2)\right]^{-1/2}$. It equals to 1 for $\kappa = r_1^{-1} + r_2^{-1} = \lambda(2)a(2)^{-1} + \lambda(1)a(1)^{-1}$.

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Address:
Benoîte de Saporta, Jian-Feng Yao
IRMAR, Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
France
e-mail addresses:
benoite.de-saporta@math.univ-rennes1.fr
jian-feng.yao@univ-rennes1.fr