On the multidimensional stochastic equation

$$Y_{n+1} = A_n Y_n + B_n$$

Sur l'équation vectorielle stochastique

$$Y_{n+1} = A_n Y_n + B_n$$

Benoîte de Saporta [a] Yves Guivarc'h [b] Emile LePage [c]

[a] *IRMAR, Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, France*

[b] *IRMAR, Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, France*

[c] *LMAM, Université de Bretagne Sud, Centre Yves Coppens, Campus de Tohannic, BP 573, 56017 Vannes, France*

Abstract

We study the behavior at infinity of the tail of the stationary solution of a multidimensional linear auto-regressive process with random coefficients. We exhibit an extended class of multiplicative coefficients satisfying a condition of irreducibility and proximality that yield to a heavy tail behavior. *To cite this article: B. de Saporta, Y. Guivarc'h, E. LePage, C. R. Acad. Sci. Paris, Ser. I 336 (2004).*

Résumé

Preprint submitted to Elsevier Science *9 septembre 2004*

On étudie le comportement à l'infini de la queue de la solution stationnaire d'un processus auto-régressif linéaire multidimensionnel à coefficients aléatoires. On donne une vaste classe de coefficients multiplicatifs vérifiant une condition d'irréductibilité et de proximalité qui conduisent à un comportement de type queue polynomiale. *Pour citer cet article : B. de Saporta, Y. Guivarc'h, E. LePage, C. R. Acad. Sci. Paris, Ser. I 336 (2004).*

---

1 Introduction

We study the following stochastic difference equation

$$Y_{n+1} = A_n Y_n + B_n, \qquad n \in \mathbb{N}, \quad Y_n \in \mathbb{R}^d, \tag{1}$$

where (A_n, B_n) is an iid sequence of random variables, A_n is in $\mathcal{G}$ the linear group of invertible square matrices of size d, and B_n is a vector of $\mathbb{R}^d$. Here we restrict ourselves to $d \geq 2$ (see [8] and [4] for the one-dimensional case).

Under weak assumptions, the corresponding Markov process has a unique stationary solution. The behavior of its tail at infinity has been investigated by H. Kesten [8], when the coefficients are non-negative matrices and vectors. E. LePage [10] gave another result for a class of non-singular matrices. This note extends the latter result to a wide class of multiplicative coefficients, namely a class with a property of irreducibility and proximality.

Email addresses: benoite.de-saporta@univ-rennes1.fr (Benoîte de Saporta), yves.guivarch@univ-rennes1.fr (Yves Guivarc'h), lepage@univ-ubs.fr (Emile LePage).

2 Definitions and Notation

For $s \geq 0$, we denote $k(s) = \lim_n (\mathbb{E}\|A_1 \cdots A_n\|^s)^{1/n}$, and $\sigma = \sup\{s \geq 0 \; ; \; k(s) < +\infty\}$. Throughout this note, we assume that

$$\sigma > 0, \qquad \mathbb{E} \log \|A_1^{-1}\| < \infty, \quad \mathbb{E} \log \|B_1\| < \infty, \quad \alpha = \lim \frac{1}{n} \mathbb{E}[\log \|A_1 A_2 \cdots A_n\|] < 0.$$

Then, Eq. 1 has a unique stationary solution (see [1]) that has the same law as the random variable

$$R = \sum_{k=1}^{\infty} A_1 A_2 \cdots A_{k-1} B_k.$$

Let η denote the law of (A_1, B_1), S_η its support in the group $\mathcal{A} = \mathcal{G} \ltimes \mathbb{R}^d$ of affine transformations $x \mapsto Ax + B$ on $\mathbb{R}^d$, and Γ_η be the semi-group generated by S_η. Similarly, let μ be the law of A_1 (μ is the projection of η on $\mathcal{G}$), S_μ its support and Γ_μ the semi-group it generates.

Following [8], we consider the row vectors of $\mathbb{R}^d$ and the right-hand side action of $\mathcal{G}$ on the unit sphere $\mathbb{S}^{d-1}$: for all $x \in \mathbb{S}^{d-1}$ and $a \in \mathcal{G}$, the action of a on x is denoted by $x \cdot a$ that is equal to $xa\|xa\|^{-1}$.

The semi-group Γ_μ is said to be *irreducible* if it has no invariant non-trivial sub-space. It is said to be *proximal* if for all v and v' in the projective space $\mathcal{P}^{d-1} = \mathcal{P}(\mathbb{R}^d)$ (corresponding to row vectors) there is a sequence (a_n) in Γ_μ such that $\lim_n \delta(va_n, v'a_n) = 0$, where δ is a distance on $\mathcal{P}^{d-1}$. Finally, Γ_μ is said to be *expanding* (resp *contracting*) if it has at least one element with spectral radius greater than one (resp. less than one). If Γ_μ is all at once irreducible, proximal and expanding, it is said to satisfy *Condition i-p-e.*

3 The main Theorem

Theorem 3.1 *Let $d \geq 2$ and (A_n, B_n) in $\mathcal{A}$ be a sequence of iid random variable satisfying Condition* $(\mathbf{C})$*. Suppose in addition that*

(1) The semi-group Γ_μ generated by the support of the law μ of A_1 satisfies condition i-p-e.

(2) The semi-group Γ_μ has no invariant salient closed convex cone with non empty interior.

(3) The semi-group Γ_η generated by the support of the law η of (A_1, B_1) has no fixed point in $\mathbb{R}^d$.

Then Equation $k(s) = 1$ has a unique positive solution κ on $]0,\ \sigma[$.
If in addition $\mathbb{E}[\|A_1\|^\kappa \log \det |A_1|] > -\infty$ and there is a $\delta > 0$ such that $\mathbb{E}\|B_1\|^{\kappa+\delta} < \infty$, then for all $x \in \mathbb{S}^{d-1}$ we have

$$\lim_{t \to +\infty} t^\kappa \mathbb{P}(xR > t) = \ell e_\kappa(x), \tag{2}$$

where $\ell > 0$ and e_κ is a positive symmetric continuous function on $\mathbb{S}^{d-1}$.

In [8], a similar result is proved for non-negative matrices. This case is out of the scope of our theorem because of Assumption (ii). Actually, the proof of [8] can be extended to the case when the semi-group Γ_μ has an invariant cone. Therefore our result is the complement of that of [8].

In [10], the assumption made on the coefficient A_n is that the Markov chain $X_n = X_0 \cdot A_1 \cdots A_n$ on $\mathbb{S}^{d-1}$ must hit any open subset for any starting point $X_0 = x$. Our conditions (i) and (ii) are much weaker. Indeed take for instance a probability μ with two atoms a and a', a being a positive matrix and a' a negative matrix. Then the Markov chain (X_n) starting from any positive or negative vector will never hit the set of vectors that are neither negative nor positive. It is not difficult to exhibit such examples satisfying Conditions (i) and (ii). For instance, set $d = 2$, $\mu = (\delta_a + \delta_{a'})/2$ and

$$a = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \qquad a' = \begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 0 \end{pmatrix}.$$

Then the semi-group Γ_μ satisfies our hypotheses but not that of [10].

Our theorem also enables us to answer an open problem stated by H. Kesten in [8], namely: *Let* $d = 2$*, set* m_1 *and* m_2 *two positive matrices and* m_3 *a rotation. Take* $\mu = p_1\delta_{m_1} + p_2\delta_{m_2} + p_3\delta_{m_3}$ *with* $p_i > 0$ *and* $p_1 + p_2 + p_3 = 1$*. Is the limit (2) still valid?* This is out of the scope of the result of [8] as a non-trivial rotation is not a non-negative matrix. Our result enables us to answer positively this question when the ratio θ/π (where θ is the angle of the rotation m_3) is different from $0 \mod 1/2$ and is either irrational or of the form $(2k+1)/n$, where k and n are integers.

Under Condition (**C**) and if the law of B_1 is arbitrary with compact support, Conditions (ii) and (iii) can be shown to be necessary for the validity of the conclusion of our theorem. Also, Γ_μ expanding, irreducible and contracting are necessary conditions, but proximality is not. Let $V_r \subset \mathcal{A}^r$ $(r \geq 2)$ be the set of r-tuples $(g) = (g_1, \ldots, g_r)$ such that the semi-group $\Gamma_{(g)} \subset \mathcal{A}$ generated by $g_1, \ldots, g_r$ satisfies the above necessary conditions, and let U_r be the subset of V_r where $\Gamma_{(g)}$ is also proximal. If $\Gamma_{(g)}$ is Zariski dense in $\mathcal{A}$, contracting, and satisfies (ii), then it can be shown, using [5], that $(g) \in U_r$. Then, U_r contains a dense open subset of full Haar measure in V_r. In particular, distributions of the form $\eta = \sum_{i=1}^{r} p_i \delta_{g_i}$ with $r \geq 2$, $\prod p_i > 0$, $\alpha < 0$ and $(g) \in U_r$ satisfy the conclusion of our theorem. For these distributions and from a generic point of view, the conditions of our theorem are also necessary.

4 Sketch of the proof

Our proof follows the same steps as in [10] but uses the new tools given in [6]. The key point is to derive a renewal equation satisfied by $z(x,t) = e^{-t} \int_0^{e^t} \mathbb{P}(xR > u)du$ and to prove that the renewal theorem for functionals of a Markov chain given in [9] applies.

The first step is to study the operator $\mathcal{P}$ defined on the projective space $\mathcal{P}^{d-1}$ by

$$\mathcal{P}f(v) = \mathbb{E}[\|vA_1\|^{\kappa} f(vA_1)].$$

It is proved in [6] that under the assumptions of our theorem, its spectral radius is 1 and it has a unique corresponding continuous eigenfunction e_κ, which is positive. Hence we can define a Markovian operator on $\mathcal{P}^{d-1}$ by:

$$\mathcal{Q}f(v) = \frac{1}{e_\kappa(v)} \mathbb{E}[\|vA_1\|^{\kappa} e_\kappa(vA_1) f(vA_1)].$$

Under our assumption, $\mathcal{Q}$ has a spectral gap on a space of Hőlder functions.

The second step is to prove that the operator Q defined on $\mathbb{S}^{d-1}$ by:

$$Qf(x) = \frac{1}{e_\kappa(\bar{x})}\mathbb{E}[\|xA_1\|^\kappa e_\kappa(\bar{x}A_1)f(x \cdot A_1)],$$

where $\bar{x}$ is the projective image of x, has the same properties as $\mathcal{Q}$, and in particular that is has a unique invariant probability. Assumption (ii) is essential for this uniqueness.

Then we prove that the renewal theorem of [9] applies to the following operator on $\mathbb{S}^{d-1} \times \mathbb{R}$:

$$Qf(x,t) = \frac{1}{e_\kappa(\bar{x})}\mathbb{E}[\|xA_1\|^\kappa e_\kappa(\bar{x}A_1)f(x \cdot A_1, t - \log\|xA_1\|)].$$

This gives us Equation 2 with a non-negative constant ℓ. To prove that ℓ is actually positive requires a detailed study of the operator defined by Q on spaces of functions with controlled growth at infinity. Here again, we follow the original idea of [10].

References

[1] A. Brandt, The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients, Adv. Appl. Prob. 18(1986) 211–220.

[2] H. Furstenberg, Noncommuting Random Products, Trans. Amer. Math. Soc., 108(1963) 377–428.

[3] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces, in: Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI), Amer. Math. Soc., 1973, pp 193–229.

[4] C.M. Goldie, Implicit renewal theory and tails of solutions of random equations, Ann. Appl. Probab. 1(1991) 26–166.

[5] I. Ya. Goldsheid and Y. Guivarc'h, Zariski closure and the dimension of the Gaussian law of the product of random matrices, Probab. Theory Related Fields, 105(1996) 109–142.

[6] Y. Guivarc'h and E. LePage, Simplicité de spectres de Lyapunov et propriété d'isolation spectrale pour une famille d'opérateurs de transfert sur l'espace projectif, in: V. Kaimanovitch (Ed), Random walks and Geometry: Workshop Vienna 2001, De Gruyter, 2004, pp 181–259.

[7] Y. Guivarc'h and A. Raugi, Products of random matrices: convergence theorems, Random matrices and their applications, Contemp. Math. 50(1986) 31–54.

[8] H. Kesten, Random difference equations and renewal theory for products of random matrices, Acta Math 131(1973) 207–248.

[9] H. Kesten, Renewal theory for functionals of a Markov chain with general state space, Ann. Probability 2(1974) 355–386.

[10] E. Le Page, Théorèmes de renouvellement pour les produits de matrices aléatoires. Equations aux différences aléatoires, Séminaires de probabilités de Rennes, 1983.