Renewal theorem for a system of renewal equations
Théorème de renouvellement pour un système d'équations de renouvellement

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March 2, 2004

Abstract: We show that the classical renewal theorems of Feller hold in the case of a system of renewal equations, when the distributions involved are supported on the whole real line. We extend Feller's methods and also use Perron-Frobenius theory and potential theory.

Résumé: On généralise les théorèmes de renouvellement de Feller au cas d'un système d'équations de renouvellement faisant intervenir des mesures qui ont pour support toute la droite réelle. Pour cela on suit la même démarche que Feller en faisant intervenir de plus la théorie de Perron-Frobenius et la théorie du potentiel.

1 Introduction

We study the asymptotic behavior, when t tends to $+\infty$, of $Z(t) = {}^t(Z_1(t), \dots, Z_p(t))$ the solution of a system of renewal equations of the following type:

$$Z_i(t) = G_i(t) + \sum_{k=1}^{p} \int_{-\infty}^{\infty} Z_k(t-u) F_{ik}(du), \qquad \forall\, t \in \mathbb{R}, \qquad \forall\, 1 \leq i \leq p, \tag{1}$$

where $G(t) = {}^t(G_1(t), \dots, G_p(t))$ is a vector of real-valued Borel-measurable functions that are bounded on compact sets, and for each $1 \leq i, j \leq p$, F_{ij} is a distribution: non-negative, non-decreasing, right-continuous and tending to 0 in $-\infty$.

Such systems, with $F_{ij}: \mathbb{R} \to \mathbb{R}_+$, arise in the study of the tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ where (a_n) is a Markov chain on a finite state space $\{e_1, \dots, e_p\}$ with transition matrix $P = (p_{ij})$. In this case, $F_{ij}(t) = |e_i|^\lambda p_{ji} \mathbf{1}_{t \geq \log |e_i|}$. This is what motivated this study.

The standard renewal equation corresponds to the case when $p = 1$ and $F_{11}(\infty) = 1$. Then Feller's renewal theorems (see [5], XI) are available for any directly Riemann integrable G_1. The multidimensional case for measures supported on the positive real line has also already been studied by Crump in [2] and Athreya et al. in [1]. They extended Feller's ideas and methods to derive a similar theorem.

For more recent works on systems of renewal equations, see [4] and [7]. In both papers, the authors study such systems in the special case when F_{ij} are supported on the positive half-line and have a density. In [4], Engibaryan proves that the renewal theorems hold for a wider class of function G, namely integrable, essentially bounded functions tending to 0 in $+\infty$. His approach is based on similar results in dimension 1, and the Gauss triangular factorization. In [7], Tsalyuk uses complex analysis. His functions are complex-valued and he uses the Laplace transform $\widehat{F}(z)$ of F. Under suitable assumptions, mainly that $I - \widehat{F}(z)$ is not invertible at a finite number of points in the closed half-plane $Re(z) \geq 0$, he gives the structure of the resolvent R of the renewal equation ($R = U - F^{(0)}$, see our notations in the following part). However both proofs can not be extended to wider classes of F_{ij}.

In this paper, we further extend Feller's methods to the case of measures supported on the whole real line. Here we only study the case when the matrix of F_{ij} is non-lattice (see definition 2).

In the following part, we state some definitions and the main results. In parts 3 and 4, we state and prove two preliminary results that we will need in the last part to prove our renewal theorems.

2 Hypotheses and main results

We start with a list of notations we are going to use throughout this paper.

2.1 Notations

Let $F = (F_{ij})_{1\leq i,j\leq p}$ be a matrix of distributions as above.

Definition 1 *For any* $p \times r$ *matrix* H *of Borel-measurable real-valued functions that are bounded on compact intervals, we define the* convolution product $F * H$ *by:*

$$(F * H)_{ij}(t) = \sum_{k=1}^{p} \int_{-\infty}^{\infty} H_{kj}(t-u)F_{ik}(du),$$

when the integrals exist.

We can then rewrite equation (1) as

$$Z = G + F * Z.$$

For any real t we define:

• the *expectation* of F (when it exists): $B = (b_{ij})_{1\leq i,j\leq p}$ with $b_{ij} = \int uF_{ij}(du)$,

• $F^{(0)}(t) = (\delta_{ij}(t))_{1\leq i,j\leq p}$ with $\delta_{ij}(t) = \mathbf{1}_{t\geq 0}$ if $i = j$ and 0 otherwise, so that $F^{(0)} * H = H$ for any H as in the definition above,

• the *n-fold convolution* of F: $F^{(n)}(t) = F * F^{(n-1)}(t)$,

• the *renewal function* associated with F: $U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$.

We also recall the definition of a lattice matrix of distributions as given in [1].

Definition 2 *F is lattice if the following assertions are true:*

• *For each $i \neq j$, F_{ij} is concentrated on a set of the form $b_{ij} + \lambda_{ij}\mathbb{Z}$.*

• *For each i, F_{ii} is concentrated on a set of the form $\lambda_{ii}\mathbb{Z}$.*

• *The λ_{ii} are integral multiples of some same number.*

We take λ to be the largest such number.

• *If a_{ij}, a_{jk}, a_{ik} are points of increase of F_{ij}, F_{jk} and F_{ik} respectively, then $a_{ij} + a_{jk} - a_{ik}$ is an integral multiple of λ.*

2.2 Hypotheses

To get a renewal theorem similar to that of Feller in dimension 1, we need make some assumptions on the matrix F, as in [1], essentially to be able to use Perron-Frobenius theory (see [6]):

• an assumption of finiteness of measures,

$$\forall\ 1 \leq i,j \leq p, \qquad F_{ij}(\infty) = \lim_{t\to\infty} F_{ij}(t) < \infty, \tag{2}$$

• an assumption of irreducibility.
Recall that a $n \times n$ matrix $A = 9a_{ij})$ is *irreducible* if for any non-trivial partition (I, J) of $\{1, \dots, n\}$, we can find i in I and j in J so that $a_{ij} \neq 0$ (see [6]).

$$F(\infty) \text{ is an irreducible matrix.} \tag{3}$$

As $F(\infty)$ is a non-negative (component-wise) irreducible matrix, we can apply Perron-Frobenius theorem: its spectral radius $\rho(F(\infty))$ is an eigenvalue of algebraic multiplicity 1, with a right-hand and a left-hand positive (component-wise) eigenvector. In the following, we will also assume that

$$\rho(F(\infty)) = 1. \tag{4}$$

This very assumption enables us to deal with the matrix F as with a "probability". Then we denote by m and u the Perron-Frobenius eigenvectors for the eigenvalue 1:

$$\begin{array}{rcl} F(\infty)m &=& m, \\ {}^t uF(\infty) &=& {}^t u, \\ \sum_{i=1}^{p} m_i &=& 1, \\ \sum_{i=1}^{p} u_i m_i &=& 1. \end{array} \tag{5}$$

• Finally we make a transience-type assumption:

$$\forall\, t \in \mathbb{R}, \qquad U(t) < \infty. \tag{6}$$

This last assumption does not appear in [1]. Indeed, it is automatically true for measures distributed on the positive half-line. However, this is no longer so in the general case, even in dimension 1 (for example it is false if F has means zero).

2.3 Main results

We can now state the main theorems we are going to prove in the following parts.

Theorem 1 *If assumptions (2), (3), (4), and (6) are true, if, in addition, F is non-lattice and Z is a bounded continuous (component-wise) solution of $Z = F * Z$, then Z is a constant vector.*

Theorem 2 *If assumptions (2), (3), (4), and (6) are true, then for any i, j, for any bounded interval $I =]a; b]$, $U_{ij}(I + t) = U_{ij}(t + b) - U_{ij}(t + a)$ is uniformly (in t) bounded.*

These first two theorems will help us to prove the following renewal theorems:

Theorem 3 (Renewal theorem, first form) *If assumptions (2), (3), (4), and (6) are true, if, in addition, F is non lattice and B exists, then ${}^t uBm \neq 0$ and, for any i, j and for any $h > 0$, we have*

$$U_{ij}(t + h) - U_{ij}(t) \xrightarrow[t \to \infty]{} c m_i u_j h,$$

where m and u are the eigenvectors defined in (5), and $c = ({}^t uBm)^{-1}$.

Theorem 4 (Renewal theorem, second form) *Under the assumptions of theorem 3, if G is directly Riemann integrable (component-wise), and $Z = U * G$ exists, then*

$$\lim_{t \to \infty} Z_i(t) = c m_i \sum_{j=1}^{p} u_j \int_{-\infty}^{\infty} G_j(u) du.$$

This last form is the useful one when we want to derive the asymptotic behavior of a function from a renewal equation it satisfies. Note that if $Z = U * G$ exists, then Z is solution of the renewal equation (1). However in the case of measures supported on the whole line, we can not prove the uniqueness of this solution. To know the limit of a function Z satisfying a renewal equation of type (1), we have to prove first that $Z = U * G$, then we can apply the renewal theorem. A general method to prove this is iterating the renewal equation and prove that $F^{(n)} * Z \xrightarrow[n \to \infty]{} 0$.

3 Equation $Z = F * Z$

As in dimension 1, the special form of the solutions of this equation will play an important part in the proof of the renewal theorems. This whole section is almost the same as in the case of measures supported on the positive half-line.

We start with a study of the points of increase of U.

3.1 Points of increase of U

Lemma 1 *Let Σ_{ij} be the set of all points of increase of the $F_{ij}^{(k)}$ for all $k \in \mathbb{N}$, ie*

$$\Sigma_{ij} = \{a \mid \exists\, k \in \mathbb{N},\ F_{ij}^{(k)}(a+\varepsilon) - F_{ij}^{(k)}(a-\varepsilon) > 0 \quad \forall\, \varepsilon > 0\}.$$

Then for any i, j, k, $\Sigma_{ik} + \Sigma_{kj} \subset \Sigma_{ij}$.

The proof is exactly the same as in the case of measures distributed on the positive half-line.

Proof

Set x in Σ_{ik} and y in Σ_{kj}. Then we can find integers n and m so that x be a point of increase of $F_{ik}^{(n)}$ and y a point of increase of $F_{kj}^{(m)}$. According to lemma V.4.1 in [5], $x+y$ is then a point of increase of $F_{ik}^{(n)} * F_{kj}^{(m)}$, hence one of $\sum_{k=1}^{p} F_{ik}^{(n)} * F_{kj}^{(m)} = F_{ij}^{(n+m)}$. Thus $\Sigma_{ik} + \Sigma_{kj} \subset \Sigma_{ij}$. □

The definition of a lattice matrix was chosen to have the following lemma work quite similarly to lemma V.4.2 in [5] in dimension 1.

Lemma 2 *If assumption (3) is true, and if F is non-lattice and the F_{ij} are not all concentrated on $\mathbb{R}_-$, then for any i, j, Σ_{ij} is asymptotically dense at infinity in the following sense:*

$$\forall\, \varepsilon > 0, \qquad \exists\, \Delta_\varepsilon > 0 \text{ so that for any } x \geq \Delta_\varepsilon, \qquad]x; x+\varepsilon[\, \cap\, \Sigma_{ij} \neq \emptyset.$$

The proof follows the same steps as that of lemma 2 in [1].

Proof

According to lemma 1, if $\Sigma_{i_0 j_0}$ is asymptotically dense at infinity, then so is $\Sigma_{i_0 j}$ for any j and $\Sigma_{i j_0}$for any i, thus either all Σ_{ij} are asymptotically dense at infinity, or none is.

Suppose none of the Σ_{ij} is asymptotically dense at infinity, especially Σ_{ii} is not asymptotically dense at infinity. It is a closed subset of $\mathbb{R}$ for addition according to lemma 1, and it is not empty according to lemma 1 and because $F(\infty)$ is not a zero-matrix thanks to assumption (3). Thus there is a δ_{ii} so that $\Sigma_{ii} \subset \delta_{ii}\mathbb{Z}$ and it contains $n\delta_{ii}$ for all large enough n (see lemma V.4.2 in [5]).

Set c in Σ_{ij}, and d in Σ_{ji}. Set a large enough n so that $n\delta_{ii} \in \Sigma_{ii}$ and $(n+1)\delta_{ii} \in \Sigma_{ii}$, then according to lemma 1, $d - n\delta_{ii} + c$ and $d - n\delta_{ii} + c + \delta_{ii}$ are in Σ_{jj}, thus $\delta_{ii} \geq \delta_{jj}$, and by symmetry they are equal. Thus all δ_{jj} are equal. We set $\delta = \delta_{jj}$ for all j.

By a similar argument, we show that if $i \neq j$, then $\Sigma_{ij} \subset b_{ij} + \delta\mathbb{Z}$ (indeed $\Sigma_{ij} + \Sigma_{jj}$ is closed under addition), and according to lemma 1, $b_{ij} + b_{jk} = b_{ik} + n\delta$. Thus F is lattice, which is impossible. □

3.2 Proof of theorem 1

We start with studying a more regular special case.

Lemma 3 *Let K be a vector of bounded uniformly continuous functions on $\mathbb{R}$ such that $K = F * K$. Under assumptions (2), (3), (4), and (6), if in addition F is non-lattice and there is i_0 so that $a_{i_0} = sup_{t\in\mathbb{R}} K_{i_0}(t) > 0$, then there exists $\delta_{i_0} > 0$ such that for any $h > 0$, there exists an interval of length h on which $K_{i_0} > \delta_{i_0}$.*

Proof

For any $1 \leq j \leq p$ we set $a_j = sup_{t\in\mathbb{R}} K_j(t)$. Set i_0 such that $a_{i_0} > 0$ and j_0 such that $\frac{a_{j_0}}{m_{j_0}} = \max_{1\leq j\leq p} \frac{a_j}{m_j} > 0$, where m is the eigenvector of $F(\infty)$ defined in (5). As $F(\infty)m = m$, for any i, n we get $\sum_{j=1}^{p} F_{ij}^{n}(\infty)m_j = m_i$, where $F_{ij}^{n}(\infty)$ are the coordinates of the matrix $F(\infty)^n$. Then

$$\begin{aligned}
\sum_{j=1}^{p} F_{j_0 j}^{n}(\infty)a_j &= \sum_{j=1}^{p} F_{j_0 j}^{n}(\infty)m_j \frac{a_j}{m_j} \\
&\leq \Big(\sum_{j=1}^{p} F_{j_0 j}^{n}(\infty)m_j\Big)\frac{a_{j_0}}{m_{j_0}} \\
&= m_{j_0}\frac{a_{j_0}}{m_{j_0}} \\
&= a_{j_0}.
\end{aligned}$$

Thus we get

$$a_{j_0} \geq \sum_{j=1}^{p} F_{j_0 j}^{n}(\infty)a_j. \tag{7}$$

We divide the rest of the proof in two cases depending on a_{j_0} being reached or not.

First case: $\exists\ t_0 \in \mathbb{R}$ such that $K_{j_0}(t_0) = a_{j_0}$.
Iterating $K = F * K$, we get

$$\begin{aligned}
a_{j_0} = K_{j_0}(t_0) &= \sum_{r=1}^{p} \int K_r(t_0 - u)F_{j_0 r}^{(n)}(du) \\
&\leq \sum_{r=1}^{p} a_r \int F_{j_0 r}^{(n)}(du) \\
&= \sum_{r=1}^{p} a_r F_{j_0 r}^{(n)}(\infty) \\
&\leq \sum_{r=1}^{p} a_r F_{j_0 r}^{n}(\infty) \quad \text{as } F_{ij}^{(n)}(\infty) \leq F_{ij}^{n}(\infty) \\
&\leq a_{j_0} \quad \text{according to (7)}.
\end{aligned}$$

All these inequalities are thus in fact equalities. Hence $\sum_{r=1}^{p} \int \big(a_r - K_r(t_0 - u)\big)F_{j_0 r}^{(n)}(du) = 0$. As the integrated function is non-negative and continuous, we conclude that for any u, point of increase of a $F_{j_0 r}^{(n)}$, ie for any $u \in \Sigma_{j_0 r}$, we have $a_r = K_r(t_0 - u)$. But according to lemma 2, $\Sigma_{j_0 r}$ is asymptotically dense at infinity. The uniform continuity of the functions K_r now implies that

$$\lim_{t\longrightarrow -\infty} K_r(t) = a_r.$$

From the bounded convergence theorem applied to $K_i(t) = \sum_{r=1}^{p} \int K_r(t-u)F_{ir}^{(n)}(du)$ when $t \longrightarrow \infty$, we derive that $a_i = \sum_{r=1}^{p} a_r F_{ir}^{(n)}(\infty)$. Thus for any t, r we get

$$\begin{aligned}
K_r(t) - a_r &= \sum_{l=1}^{p} \int (K_l(t-u) - a_l)F_{rl}^{(n)}(du), \\
|K_r(t) - a_r| &\leq \sum_{l=1}^{p} \int |K_l(t-u) - a_l| F_{rl}^{(n)}(du) \\
&= \sum_{l=1}^{p} \int_{-\infty}^{T} |K_l(t-u) - a_l| F_{rl}^{(n)}(du) + \sum_{l=1}^{p} \int_{T}^{\infty} |K_l(t-u) - a_l| F_{rl}^{(n)}(du).
\end{aligned}$$

As $F(\infty)$ has spectral radius 1, and thus that $\lim_{n\to\infty} \|F(\infty)^n\| = 1$, we get $\sup_{n,i,j} F_{ij}^{(n)}(\infty) \leq \sup_{n,i,j} F_{ij}^n(\infty) < \infty$. Set $\varepsilon > 0$, we can choose T so that for any n, we have

$$\sum_{l=1}^{p} \int_T^\infty |K_l(t-u) - a_l| F_{rl}^{(n)}(du) < \varepsilon.$$

As K is bounded and $\lim_{n\to\infty} F^{(n)}(T) = 0$ because $U(T) < \infty$, we get

$$\lim_{n\to\infty} \int_{-\infty}^{T} |K_l(t-u) - a_l| F_{rl}^{(n)}(du) \leq M \lim_{n\to\infty} F_{rl}^{(n)}(T) = 0.$$

Thus for any $1 \leq r \leq p$, K_r is the constant function a_r. Especially, $K_{i_0}(t) = a_{i_0} > 0$, from which we derive the expected result for $\delta_{i_0} = a_{i_0}/2$.

Second case: For any t, $K_{j_0}(t) \neq a_{j_0}$.

Then we can find (t_n), a sequence tending to $\pm\infty$ such that $K_{j_0}(t_n) \to a_{j_0}$. Let $\zeta_{n,i}(x) = K_i(t_n + x)$. As K is bounded and uniformly continuous, $(\zeta_{n,i})_{n,i}$ is a uniformly bounded and uniformly equi-continuous family. Ascoli theorem then gives us a sub-sequence (t_{n_j}) of (t_n) such that for any n, i, the sequence $(\zeta_{n_j,i})_j$ converges uniformly on any compact set to η_i, a bounded uniformly continuous function. Now we get

$$\begin{aligned}
\zeta_{n_j,i}(x) &= K_i(t_{n_j} + x) \\
&= \sum_{r=1}^{p} \int K_r(t_{n_j} + x - y) F_{ir}(dy) \\
&= \sum_{r=1}^{p} \int \zeta_{n_j,r}(x-y) F_{ir}(dy).
\end{aligned}$$

When j tends to ∞, the bounded convergence theorems says

$$\eta_i(x) = \sum_{r=1}^{p} \int \eta_r(x-y) F_{ir}(dy). \tag{8}$$

In addition, for any x, i, we get $\eta_i(x) = \lim_{j\to\infty} K_i(t_{n_j} + x) \leq a_i$, and by choice of t_n, $\eta_{j_0}(0) = \lim_{j\to\infty} K_{j_0}(t_{n_j}) = a_{j_0}$. Thus $\sup \eta_{j_0} = a_{j_0} > 0$, hence η_{j_0} satisfies the assumptions of this lemma in the first case. Each η_i is thus a constant function, say c_i, with $c_{j_0} = a_{j_0}$.

From (8), we derive that $c = {}^t(c_1, \ldots, c_p)$ is a right eigenvector of $F(\infty)$ for the eigenvalue 1. As the corresponding eigenvectors sub-space is one-dimensional according to Perron-Frobenius theorem, we conclude that $c = \alpha m$. As $c_{j_0} = a_{j_0} > 0$, we get $\alpha = \frac{c_{j_0}}{m_{j_0}} > 0$ and thus c has positive coordinates.

Set $h > 0$. As $K_{i_0}(t_{n_j} + x) \to c_{i_0}$ uniformly on $[0; h]$, for any large enough j we have $K_{i_0}(x) > \frac{c_{i_0}}{2}$ for any x in $]t_{n_j}; t_{n_j} + h[$. □

Proof of theorem 1

Set $\phi_\varepsilon(t) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp(-\frac{t^2}{2\varepsilon^2})$. For any i, we set

$$f_{\varepsilon,i}(t) = \phi_\varepsilon * Z_i(t) = \int_{-\infty}^{\infty} \phi_\varepsilon(t-y) Z_i(y) dy = \int_{-\infty}^{\infty} \phi_\varepsilon(y) Z_i(t-y) dy.$$

For any $\varepsilon > 0$, and any $1 \leq i \leq p$, we have

$$\begin{aligned}
f_{\varepsilon,i}(t) &= \sum_{r=1}^{p} \int_{-\infty}^{\infty} \phi_\varepsilon(y) \int_{-\infty}^{\infty} Z_r(t-y-u) F_{ir}(du) dy \\
&= \sum_{r=1}^{p} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \phi_\varepsilon(y) Z_r(t-y-u) dy \Big) F_{ir}(du) \\
&= \sum_{r=1}^{p} \int_{-\infty}^{\infty} f_{\varepsilon,r}(t-u) F_{ir}(du).
\end{aligned}$$

In addition $f_{\varepsilon,i}$ is smooth, and its derivative is bounded, because Z is bounded and uniformly continuous. Thus, $f'_{\varepsilon,i}(t) = \sum_{r=1}^{p} \int f'_{\varepsilon,r}(t-u)F_{ir}(du)$, and we can use lemma 3.

Set $a_i = \sup f'_{\varepsilon,i}$. If there is a i such that $a_i > 0$, then we can find δ such that for any $h > 0$ there is an interval $]t; t+h[$ on which $f'_{\varepsilon,i} > \delta$. Integration on $]t; t+h[$ yields $\delta h < f_{\varepsilon,i}(t+h) - f_{\varepsilon,i}(t)$. As $f_{\varepsilon,i}$ is bounded, we get $\delta h < M$ for any $h > 0$, which is impossible. Thus for any i, $a_i \leq 0$.

Replacing Z_i by $-Z_i$, we prove similarly that for any i, $a_i \geq 0$. Thus for any i, t, ε, we have $f'_{\varepsilon,i}(t) = 0$. For any i, ε, the convolution $f_{\varepsilon,i}$ is a constant function. Letting ε tend to 0, we obtain that Z_i is a constant function for any i. □

4 Potential theory

The aim of this section is to prove theorem 2, ie that U has uniformly bounded increments. It is easily proved for measures supported on the positive half-line, or in the one-dimensional case, thanks to special renewal equations. However these methods can not be extended to the present case. This is the only technical difficulty we have met to extend the renewal theorems from the case of measures supported on the positive half-line to measures supported on the whole real line. We give here an original proof of theorem 2 that involves the one-dimensional potential theory (see [3]), by extending it to the d-dimensional case.

4.1 Definitions and notations

Definition 3 *A* kernel N *on* $\mathbb{R}$ *is a mapping of* $\mathbb{R} \times \mathcal{B}(\mathbb{R})$ *onto* $[0, +\infty]$ *such that*

- $t \longmapsto N(t, A)$ *is measurable for any* $A \in \mathcal{B}(\mathbb{R})$,
- $A \longmapsto N(t, A)$ *is a measure for any* $t \in \mathbb{R}$.

For a given non-negative measurable function f on $\mathbb{R}$, we define the mapping Nf by

$$Nf(t) = \int f(y)N(t, dy).$$

We also define the composition of kernels: given two kernels M and N on $\mathbb{R}$, their product MN is defined by

$$MN(t, A) = \int N(y, A)M(t, dy).$$

Definition 4 $N = (N_{i,j})_{1 \leq i,j \leq p}$ *is a* kernel *on* $\mathbb{R}^p$ *if each of its components* N_{ij} *is a kernel on* $\mathbb{R}$ *in the sense of definition 3.*

For any measurable non-negative (component-wise) vector of functions $f = {}^t(f_1, \ldots, f_p)$, the mapping Nf is defined by $Nf = {}^t\big((Nf)_1), \ldots, (Nf)_p\big)$, with

$$(Nf)_i(t) = \sum_{j=1}^{p} N_{ij} f_j(t).$$

If M and N are two kernels on $\mathbb{R}^p$, their product is $MN = \big((MN)_{ij}\big)$, where

$$(MN)_{ij} = \sum_{k=1}^{p} M_{ik} N_{kj}.$$

We also define a special kernel I by

$$\begin{array}{rcll} I_{ij}(t, A) & = & 0 & \text{if} \quad i \neq j, \\ I_{ii}(t, A) & = & \mathbf{1}_A(t). & \end{array}$$

where

$$\mathbf{1}_A(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \in A, \\ 0 & \text{otherwise.} \end{array} \right.$$

Thus for any function $f : \mathbb{R} \longrightarrow \mathbb{R}^p$, we have $If = f$.

In the following, N will always denote a kernel on $\mathbb{R}^p$. Let N^k be its powers for the composition product defined above, with $N^0 = I$.

Definition 5 *The* potential kernel *associated with the kernel N is the following kernel*

$$G = \sum_{k=0}^{\infty} N^k.$$

On the set of measurable function from $\mathbb{R}$ onto $\mathbb{R}^p$ we define the following partial order relationship:

$$u \preceq v \quad \text{if,} \quad \forall\, 1 \leq i \leq p, \quad u_i \leq v_i.$$

This order has the following good property: if $u \preceq v$ then for any kernel M, we have $Mu \preceq Mv$.

Definition 6 *Let $u : \mathbb{R} \longrightarrow \mathbb{R}_+^p$ be a non-negative (component-wise) function. It is* excessive *for kernel N if*

$$Nu \preceq u.$$

4.2 Maximum principle

Let $A \subset \mathbb{R}$ and A^c be its complementary set. We denote J_A the kernel on $\mathbb{R}^p$ that satisfies $(J_A f)_i(t) = f_i(t)\mathbf{1}_A(t)$, ie

$$\begin{array}{rcll} (J_A)_{ij}(t,B) & = & 0 & \text{if } i \neq j, \\ (J_A)_{ii}(t,B) & = & \mathbf{1}_{A\cap B}(t). & \end{array}$$

Notice that $J_A f$ depends only on the values of f on A.

Let G_A be the potential kernel associated with NJ_A and G^A that associated with $J_A N$. We have $NG^A = G_A N$ and $J_A G_A = G^A J_A$. We also define the similar potential kernels for A^c.

Definition 7 *We set $H_A = J_A + J_{A^c} G_{A^c} N J_A = G^{A^c} J_A$.*

We now give a series of propositions as a preliminary to the maximum principle.

Proposition 1 *The measures $(H_A)_{ij}$ are supported on A, and for any t in A,*

$$\begin{array}{rcll} (H_A)_{ij}(t,B) & = & 0 & \textit{if } i \neq j, \\ (H_A)_{ii}(t,B) & = & \mathbf{1}_B(t). & \end{array}$$

Proof
It is an easy consequence of the definition of H_A and J_A. □

Proposition 2 *If u is an excessive function, then $H_A u \preceq u$.*

Proof
We prove by induction on k that

$$J_A u + \sum_{m=0}^{k} J_{A^c}(NJ_{A^c})^m NJ_A u \preceq u. \tag{9}$$

If $k = 0$, as u is excessive and $J_A u \preceq u$, we have $NJ_A u \preceq Nu \preceq u$. Then $J_{A^c} NJ_A u \preceq J_{A^c} u$ and $J_A u + J_{A^c} NJ_A u \preceq J_A u + J_{A^c} u = u$.
Suppose it is true at rank k: $J_A u + \sum_{m=0}^{k} J_{A^c}(NJ_{A^c})^m NJ_A u \preceq u$.

At rank $k+1$, we apply N then J_{A^c} to the two members of the inequality in the induction hypotheses. We get

$$\begin{aligned} J_{A^c}u \succeq J_{A^c}Nu &\succeq J_{A^c}NJ_Au + \sum_{m=0}^{k} J_{A^c}NJ_{A^c}(NJ_{A^c})^m NJ_Au \\ &= \sum_{m=0}^{k+1} J_{A^c}(NJ_{A^c})^m NJ_Au. \end{aligned}$$

Adding J_Au to both sides of the equation, we get:

$$J_{A^c}u + J_Au = u \succeq J_Au + \sum_{M=0}^{k+1} J_{A^c}(NJ_{A^c})^m NJ_Au.$$

which ends the induction.

Letting k tend to $+\infty$ in (9), we get the expected equation $H_Au \preceq u$. □

Proposition 3

$$\begin{aligned} H_A &= J_A + J_{A^c}NH_A, \qquad \textit{thus} \quad NH_A = H_A \quad \textit{on } A^c, \\ NH_A &= G_{A^c}NJ_A. \end{aligned}$$

Proof
We have $G_{A^c} = I + NJ_{A^c}G_{A^c}$, thus $NH_A = NJ_A + NJ_{A^c}G_{A^c}NJ_A = G_{A^c}NJ_A$. It yields that $J_{A^c}NH_A = J_{A^c}G_{A^c}NJ_A = H_A - J_A$. Thus $H_A = J_A + J_{A^c}NH_A$. □

Proposition 4 *Let u be an excessive function. Then H_Au is the smallest (for $\preceq$) excessive function greater than or equal to u on A.*

Proof
Set $v = H_Au$. As u is excessive, we have $v = H_Au \preceq u$ according to proposition 2, and thus $Nv \preceq Nu \preceq u$. As $u = v$ on A by proposition 1, especially we have $Nv \preceq v$ on A. Proposition 3 yields $NH_A = H_A$ on A^c, therefore on this set $Nv = v$. Thus $Nv \preceq v$ everywhere and v is excessive.
If w is excessive and greater than or equal to u on A, proposition 1 yields $H_Au \preceq H_Aw$, and $H_Aw \preceq w$ by proposition 2. Hence $H_Au \preceq H_Aw \preceq w$ everywhere. □

Proposition 5

$$G = H_AG + J_{A^c}G_{A^c} = H_AG + G^{A^c}J_{A^c}.$$

Proof
Multiplying equality $I - J_{A^c}N = I - N + J_AN$ on the left by G^{A^c} and on the right by G yields:

$$G^{A^c}(I - J_{A^c}N)G = G^{A^c}(I - N)G + G^{A^c}J_ANG.$$

But by definition we have $G^{A^c}(I - J_{A^c}N) = I = (I - N)G$. Thus

$$\begin{aligned} G &= G^{A^c} + G^{A^c}J_ANG \\ &= G^{A^c}J_{A^c} + G^{A^c}J_A(I + NG) \\ &= G^{A^c}J_{A^c} + H_AG \\ &= J_{A^c}G_{A^c} + H_AG. \end{aligned}$$

□

Proposition 6 *If f is any non-negative (component-wise) excessive function, v an excessive function, and $A = \cup_{i=1}^{p}\{f_i > 0\}$, then*

$$Gf \preceq v \quad \text{on } A \qquad \Rightarrow \qquad Gf \preceq v \quad \text{on } \mathbb{R}.$$

Proof

As $Gf \preceq v$ on A, proposition 1 yields $H_A Gf \preceq H_A v$. But v is excessive, thus proposition 2 yields $H_A v \preceq v$. Finally proposition 5 yields $Gf = H_A Gf + G^{A^c} J_{A^c} f = H_A Gf$ as by definition of A, we have $J_{A^c} f = 0$. Thus $Gf = H_A Gf \preceq H_A v \preceq v$. □

Definition 8 *Let f be a non-negative (component-wise) function, and $A \subset \mathbb{R}$. We define $\sup_{t\in A} f(t)$ by:*

$$\sup_{t\in A} f(t) = \max_{1\le i\le p}\Big(\sup_{t\in A}(f_i(t))\Big).$$

With this definition, on the set A we have $f \preceq \sup_{t\in A} f(t)\mathbf{1}$, where $\mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1})$, the function with all coordinates equal to the constant function 1.

Corollary 1 (Maximum Principle) *If $\mathbf{1}$ is excessive, then for any non-negative (component-wise) function f, if $A = \cup_{i=1}^{p}\{f_i > 0\}$, we have*

$$\sup_{t\in\mathbb{R}} Gf(t) \;=\; \sup_{t\in A} Gf(t).$$

Proof

Set $\alpha = \sup_{t\in A} Gf(t)$. If α is infinite, it is obviously true. Otherwise, we have $Gf \preceq \alpha\mathbf{1}$ on A. As $\mathbf{1}$ is excessive, proposition 6 yields $Gf \preceq \alpha\mathbf{1}$ on $\mathbb{R}$. Thus $\sup_{t\in\mathbb{R}} Gf(t) \le \alpha$, and then $\sup_{t\in\mathbb{R}} Gf(t) = \sup_{t\in A} Gf(t)$.

□

4.3 Increments of U

Now we can give the proof of theorem 2. Let m be the eigenvector defined in (5), and $N = (N_{ij})$ the following kernel:

$$N_{ij}(t, A) = \frac{m_j}{m_i}\int \mathbf{1}_A(t-x)F_{ij}(dx).$$

Then $(Nf)_i(t) = \sum_{j=1}^{p} \frac{m_j}{m_i}\int f_j(t-x)F_{ij}(dx) = \Big((\frac{m_j}{m_i}F_{ij}) * f\Big)_i(t)$.

Function $\mathbf{1}$ is excessive for N. Indeed,

$$\begin{aligned}
(N\mathbf{1})_i(t) &= \sum_{j=1}^{p}\frac{m_j}{m_i}\int F_{ij}(dx)\\
&= \frac{1}{m_i}\sum_{j=1}^{p} m_j F_{ij}(\infty)\\
&= \frac{m_i}{m_i} \qquad \text{by definition of } m,\\
&= 1.
\end{aligned}$$

The potential kernel G associated with N satisfies $G_{ij} = \frac{m_j}{m_i}U_{ij}$.

Set $h > 0$, $A = [-h; h]$, and $f_i = \mathbf{1}_A$ for $1 \le i \le p$. Then we have

$$\begin{aligned}
(Gf)_i(t) &= \sum_{j=1}^{p}\frac{m_j}{m_i}\int \mathbf{1}_{[-h;h]}(t-x)U_{ij}(dx)\\
&= \sum_{j=1}^{p}\frac{m_j}{m_i}\big(U_{ij}(t+h) - U_{ij}(t-h)\big).
\end{aligned}$$

In the sense of definition 8, Gf has finite upper bound, say α, on the bounded interval $A = [-h;h]$, because U is finite according to assumption (6). The maximum principle yields then $\sup_{t\in\mathbb{R}} Gf(t) = \sup_{t\in[-h;h]} Gf(t)$. Denote i_0 the number of a coordinate of Gf that reaches this upper bound. Set (t_n) a series of points in A such that $(Gf)_{i_0}(t_n)$ tends to this upper bound. Then majoring $\mathbf{1}_{[-h;h]}(t_n - x)$ by $\mathbf{1}_{[-2h;2h]}(x)$, we get, for any $t \in \mathbb{R}$ and any $1 \le i \le p$,

$$\sum_{j=1}^{p} \frac{m_j}{m_i}\big(U_{ij}(t+h) - U_{ij}(t-h)\big) = (Gf)_i(t) \le \sup_{t\in[-h;h]} (Gf)_{i_0}(t) \le \sum_{j=1}^{p} \frac{m_j}{m_{i_0}}\big(U_{i_0 j}(2h) - U_{i_0 j}(-2h)\big) \ < \ \infty.$$

All these terms are non-negative and $m_i > 0$ for any i, thus each $U_{ij}(t+h) - U_{ij}(t-h)$ is uniformly (in t) bounded. To get the expected result on any finite interval I, just include I in a larger symmetric interval. □

5 The renewal theorems

Now we can prove the renewal theorems 3 and 4. Thanks to the result of the preceding section, the proof is now again the same as in the case of measures supported on the positive half-line, at least for the first two steps. The renewal equation used in the third step is slightly different as it involves $F(\infty)\mathbf{1}_{t\ge 0}$ instead of $F(\infty)$, and $Z(t) = m\mathbf{1}_{t\ge 0}$ instead of $Z(t) = m$. However the method is essentially the same.

Proof of theorem 3

For any interval $I =]a;b]$, any $1 \le i,j \le p$, and $t \in \mathbb{R}$, we set $U_{ij}^{(t)}(I) = U_{ij}(t+b) - U_{ij}(t+a)$. Theorem 2 yields that the family $\big(U_{ij}^{(t)}(I)\big)_t$ is bounded. Theorem VIII.6.2 in [5] gives us a sequence (t_n) tending to $+\infty$ and measures V_{ij} such that for any $1 \le i,j \le p$ and any interval I, $U_{ij}^{(t_n)}(I) \xrightarrow[n\to\infty]{} V_{ij}(I)$.

First step: Show that V_{ij} are multiples of Lebesgue measure.

Set $k_0 \in \{1,\dots,p\}$ and $a > 0$. Let $G(t)$ be the vector defined by $G_k(t) = 0$ for any $k \ne k_0$ and G_{k_0} is a continuous non-zero function that vanishes outside $[0;a]$. Then $Z = U * G$ is well defined, and Z is solution of the renewal equation

$$\forall\ 1 \le i \le p, \qquad Z_i(t) = G_i(t) + \sum_{k=1}^{p} \int Z_k(t-u) F_{ik}(du). \tag{10}$$

For any i, we have:

$$\begin{aligned} Z_i(t_n + x) &= \int G_{k_0}(t_n + x - y) U_{ik_0}(dy) \\ &= \int G_{k_0}(x-y) U_{ik_0}^{(t_n)}(dy) \\ &\xrightarrow[n\to\infty]{} \int G_{k_0}(x-y) V_{ik_0}(dy). \end{aligned}$$

Set $\zeta_i(x) = \int G_{k_0}(x-y)V_{ik_0}(dy)$. Then ζ_i is a bounded continuous function, and $Z_i(t_n + x) \to \zeta_i(x)$. The bounded convergence theorem applied to equation (10) yields

$$\forall\ 1 \le i \le p, \qquad \zeta_i(t) = \sum_{k=1}^{p} \int \zeta_k(t-u) F_{ik}(du).$$

Now theorem 1 yields that ζ_i is a constant function for any i. Thus $\int G_{k_0}(x-y)V_{ik_0}(dy)$ does not depend on x, and this is true for any continuous function G_{k_0} that vanishes outside a compact set. Thus V_{ik_0} is finite on compact sets, and unchanged by translation, therefore it is a multiple of Lebesgue measure. Denote Lebesgue measure by l. Hence there are $a_{ij} \in \mathbb{R}$ such that:

$$\forall\ i,j, \qquad V_{ij} = a_{ij} l.$$

Second step: Show that $a_{ij} = cm_i u_j$.

Again we set k_0 and we define G by $G_k(t) = 0$ for any $k \neq k_0$ and $G_{k_0}(t) = \mathbf{1}_{[0;1]}(t)$. Then $Z = U * G$ is well defined and Z is solution of the renewal equation $Z = G + F * Z$. For any x, we have

$$\begin{aligned} Z_i(t_n - x) &= \int G_{k_0}(t_n - x - y) U_{ik_0}(dy) \\ &= U_{ik_0}(t_n - x) - U_{ik_0}(t_n - x - 1) \\ &\xrightarrow[n\to\infty]{} a_{ik_0}. \end{aligned}$$

The bounded convergence theorem applied to equation $Z(t_n) = G(t_n) + F * Z(t_n)$ yields $a_{ik_0} = \sum_{k=1}^p a_{kk_0} F_{ik}(\infty)$. Thus $(a_{1k_0}, \dots, a_{pk_0})$ is an eigenvector of $F(\infty)$ for eigenvalue 1. As the corresponding eigenvectors subspace is one-dimensional, there is a r_{k_0} such that for any i, $a_{ik_0} = r_{k_0} m_i$. Replacing F by ${}^tF$, we prove similarly that there is a s_{k_0}such that for any j, $a_{k_0 j} = s_{k_0} u_j$. Thus for any i, k_0, we have $a_{ik_0} = r_{k_0} m_i = s_i u_{k_0}$. Hence the quotient $\frac{s_i}{m_i} = \frac{r_{k_0}}{u_{k_0}} = c$ does not depend on i, and $a_{ij} = r_j m_i = cm_i u_j$.

Third step: Identification of c.

Now we set $G(t) = \big(F(\infty)\mathbf{1}_{t\geq 0} - F(t)\big)m$. Let $Z(t) = m\mathbf{1}_{t\geq 0}$. Then

$$G_i(t) + \sum_{k=1}^p \int Z_k(t-x) F_{ik}(dx) = \begin{cases} m_i - \sum_{j=1}^p F_{ij}(t) m_j + \sum_{k=1}^p m_k F_{ik}(t), & \text{if } t \geq 0, \\ -\sum_{j=1}^p F_{ij}(t) m_j + \sum_{k=1}^p m_k F_{ik}(t), & \text{if } t < 0, \end{cases}$$

and thus

$$G_i(t) + \sum_{k=1}^p \int Z_k(t-x) F_{ik}(dx) = m_i \mathbf{1}_{t\geq 0} = Z_i(t).$$

Thus $G + F * Z = Z$. Iterating this equality yields

$$\begin{aligned} Z &= G + F * Z \\ &= G + F * G + F^{(2)} * Z \\ &= \dots \\ &= \sum_{k=0}^{n-1} F^{(k)} * G + F^{(n)} * Z. \end{aligned}$$

But we have

$$\begin{aligned} (F^{(n)} * Z)_i(t) &= \int_{-\infty}^{\infty} \sum_{k=1}^p Z_k(t-x) F_{ik}^{(n)}(dx) \\ &= \sum_{k=1}^p m_k \int_{-\infty}^t F_{ik}^{(n)}(dx) \\ &= \sum_{k=1}^p m_k F_{ik}^{(n)}(t) \\ &\xrightarrow[n\to\infty]{} 0, \end{aligned}$$

as $U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$ is finite for any t. Thus $Z = U * G$. As G is non-increasing and integrable on $\mathbb{R}_+$ and on $\mathbb{R}_-$, G is directly Riemann integrable (see [5], XI). To conclude, we need the following lemma.

Lemma 4 *Let G be directly Riemann integrable, and U a matrix of distributions such that for any real x, any $h > 0$ and any $1 \leq i, j \leq p$, $U_{ij}(t_n + x + h) - U_{ij}(t_n + h) \xrightarrow[n\to\infty]{} a_{ij} h$. If $Z = U * G$ exists, then*

$$Z_i(t_n) \xrightarrow[n\to\infty]{} \sum_{k=1}^p a_{ik} \int_{-\infty}^{\infty} G_k(y) dy.$$

This lemma and the result of the first step yield

$$m_i = Z_i(t_n) \xrightarrow[n\to\infty]{} \sum_{k=1}^{p} a_{ik} \int_{-\infty}^{\infty} G_k(y)dy.$$

But

$$\begin{aligned}
\int_{-\infty}^{\infty} G_k(y)dy &= \int_{-\infty}^{\infty} \sum_{j=1}^{p} \big(F_{kj}(\infty)\mathbf{1}_{y\geq 0} - F_{kj}(y)\big)m_j dy \\
&= \sum_{j=1}^{p} m_j \int_{-\infty}^{\infty} yF_{kj}(dy) \\
&= \sum_{j=1}^{p} m_j b_{kj}.
\end{aligned}$$

As $a_{ij} = cm_i u_j$, we get $m_i = c\sum_{k=1}^{p}\sum_{j=1}^{p} m_i u_j b_{jk} m_k$. But $\sum_{k,j} u_j b_{jk} m_k \neq 0$ as $m_i > 0$ thus $c = \big(\sum_{k,j} u_j b_{jk} m_k\big)^{-1}$. This value does not depend on the choice of the sequence (t_n). As $(U_{ij}(I+t))_t$ is bounded, from any sequence (t), we can extract a convergent sub-sequence. Hence we have proved the weak convergence of $U_{ij}^{(t)}$ to $a_{ij}l$ as t tend to $+\infty$. □

Proof of lemma 4

Set $h > 0$. For any $k \in \mathbb{Z}$, we set $g_k(x) = \mathbf{1}_{[(k-1)h;kh]}$, $G_i^k = g_k$ for any i, and $Z^k = U * G^k$. Then

$$\begin{aligned}
Z_i^k(t_n) &= \sum_{j=1}^{p} \int G_j^k(t_n - y)U_{ij}(dy) \\
&= \sum_{j=1}^{p} U_{ij}(t_n - (k-1)h) - U_{ij}(t_n - kh) \\
&\xrightarrow[n\to\infty]{} \sum_{j=1}^{p} a_{ij}h.
\end{aligned}$$

This limit is independent of n and k, thus for any $n,\ k,\ i,\ Z_i^k(t_n) \leq M_h$.

Let $\underline{m}_k^i$ and $\overline{m}_k^i$ be respectively the minimum and maximum of G_i on $[(k-1)h;kh]$. As G is directly Riemann integrable, the series $\underline{\sigma}^i = h\sum \underline{m}_k^i$ and $\overline{\sigma}^i = h\sum \overline{m}_k^i$ are absolutely convergent, and their difference tends to 0 as h tends to 0. For any i, we have:

$$\sum_{j=-k}^{k} \underline{m}_j^i g_j(t_n) \leq \quad G_i(t_n) \quad \leq \sum_{j=-k}^{k} \overline{m}_j^i g_j(t_n) + \sum_{|j|>k} \overline{m}_j^i g_j(t_n),$$

$$\sum_{r=1}^{p}\sum_{j=-k}^{k} \underline{m}_j^r \int g_j(t_n - y)U_{ir}(dy) \leq \quad Z_i(t_n) \quad \leq \sum_{r=1}^{p}\sum_{j=-k}^{k} \overline{m}_j^r \int g_j(t_n - y)U_{ir}(dy) + M_h \sum_{r=1}^{p}\sum_{|j|>k} \overline{m}_j^r,$$

$$n\to\infty, \qquad \sum_{r=1}^{p}\sum_{j=-k}^{k} \underline{m}_j^r a_{ir} h \leq \quad \limsup Z_i(t_n) \quad \leq \sum_{r=1}^{p}\sum_{j=-k}^{k} \overline{m}_j^r a_{ir} h + M_h \sum_{r=1}^{p}\sum_{|j|>k} \overline{m}_j^r,$$

$$k\to\infty, \qquad \sum_{r=1}^{p} \underline{\sigma}^r a_{ir} \leq \quad \limsup Z_i(t_n) \quad \leq \sum_{r=1}^{p} \overline{\sigma}^r a_{ir}.$$

Letting h tend to 0 we get $\limsup Z_i(t_n) = \sum_{r=1}^{p} a_{ir} \int G_r(u)du$. We get the same value for the inferior limit. Thus $\lim Z_i(t_n) = \sum_{r=1}^{p} a_{ir} \int G_r(u)du$. □

Lemma 4 and theorem 3 easily yield the second form of the renewal theorem.

Acknowledgment: The author wishes to thank J.F. Yao her PhD. director, and Y. Guivarc'h for their support, hints and corrections.

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