Tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients

Queue de la solution stationnaire de l'équation $Y_{n+1} = a_n Y_n + b_n$ à coefficients markoviens

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Abstract

In this paper, we deal with the real stochastic difference equation $Y_{n+1} = a_n Y_n + b_n,\ n \in \mathbb{Z}$, where the sequence (a_n) is a finite state space Markov chain. By means of the renewal theory, we give a precise description of the situation where the tail of its stationary solution exhibits power law behavior. *To cite this article: B. de Saporta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

On étudie la queue de la solution stationnaire de léquation $Y_{n+1} = a_n Y_n + b_n,\ n \in \mathbb{Z}$, où (a_n) est une chaîne de Markov à espace d'états fini. Par des méthodes de renouvellement, on donne une caractérisation détaillée du cas où la queue est polynômiale. *Pour citer cet article : B. de Saporta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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1. Introduction

We study the following stochastic difference equation:

$$Y_{n+1} = a_n Y_n + b_n, \qquad n \in \mathbb{Z}, \tag{1}$$

where (a_n) is a real, finite state space Markov chain, and (b_n) is a sequence of real i.i.d. random variables. Random Equations of this type have many applications in stochastic modeling and statistics. Most of previously studied cases deal with i.i.d. coefficients (a_n) : see [6], [7], [9] and [3]. For more recent work, see also [8]. Here we study the Markovian case. In statistical literature, it is called a Markov-switching

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auto-regression, see [5] for interesting applications in econometrics. Such stochastic recursions are also a basic tool in queuing theory, see [1].

2. Main theorems

Assume that (a_n, b_n) is stationary and ergodic, and that we have:

$$\mathbb{E} \log |a_0| < 0, \quad \mathbb{E} \log^+ |b_0| < \infty. \tag{2}$$

Then it is proved in [2] that Eq. (1) has a unique stationary solution (Y_n), where

$$Y_n = \sum_{k=0}^{\infty} a_{n-1} a_{n-2} \cdots a_{n-k} b_{n-1-k}, \quad n \in \mathbb{Z}.$$

To deal with the tail of Y_1, we investigate the asymptotic behavior of $\mathbb{P}(xY_1 > t)$, when t tends to infinity, and where $x \in \{-1, 1\}$. We give two theorems, depending on the a_n being positive or not.

Theorem 2.1 *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E = \{e_1, \dots, e_p\} \subset \mathbb{R}_+^*$, transition matrix $P = (p_{ij})$ and stationary law ν. Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n). If the following conditions are satisfied:*
- *there is a $\lambda > 0$ so that the matrix $P_\lambda = diag(e_i^\lambda)P'$ has spectral radius 1 (where P' denotes the transpose of P),*
- *the $\log e_i$ are not integral multiples of a same number,*
- *there is a $\delta > 0$ such that $\mathbb{E}|b_0|^{\lambda+\delta} < \infty$,*

then we have for $x \in \{-1, 1\}$

$$t^\lambda \mathbb{P}(xY_1 > t) \xrightarrow[t\to\infty]{} L(x),$$

where $L(1) + L(-1)$ is positive. If $b_0 \geq 0$, then $L(-1) = 0$, and $L(1) > 0$. If $b_0 \leq 0$, then $L(1) = 0$, and $L(-1) > 0$.

Theorem 2.2 *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E = \{e_1, \dots, e_p\} \subset \mathbb{R}^*$ such that $\{e_1, \dots, e_l\} \subset \mathbb{R}_+$ and $\{e_{l+1}, \dots, e_p\} \subset \mathbb{R}_-$ for a $0 \leq l \leq p-1$, transition matrix $P = (p_{ij})$ and stationary law ν. Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n). If the following conditions are satisfied:*
- *there is a $\lambda > 0$ so that the matrix $P_\lambda = diag(|e_i|^\lambda)P'$ has spectral radius 1,*
- *the $\log |e_i|$ are not integral multiples of a same number,*
- *there is a $\delta > 0$ such that $\mathbb{E}|b_0|^{\lambda+\delta} < \infty$,*

then we have, for $x \in \{-1, 1\}$,

$$t^\lambda \mathbb{P}(xY_1 > t) \xrightarrow[t\to\infty]{} L(x),$$

where $L(1)+L(-1)$ is positive. If in addition P' is l-irreducible (see definition below) then $L(1) = L(-1) > 0$.

The last two hypotheses of these theorems are the same as in the i.i.d. case. In particular, the second one ascertains that the distribution of Y_1 is non-lattice, and it is equivalent to requiring that the subgroup generated by the $\log e_i$ be dense in $\mathbb{R}$. On the contrary, the first assumption comes from the Markovian dependence considered here. Indeed, we can prove that the spectral radius $\rho(P_\lambda)$ can be computed from the formula $\rho(P_\lambda) = \lim(\mathbb{E}|a_0 \cdots a_{1-n}|^\lambda)^{1/n}$. Therefore this assumption is a suitable substitute for the classical relation $\mathbb{E}|a_0|^\lambda = 1$ assumed in the i.i.d. case.

Note that the assumption of independence between the two sequences (a_n) and (b_n) can be avoided. Let $\mathcal{F}_n$ be the σ-field generated by $a_0, \dots a_{-n}$ and $b_0, \dots, b_{-n}$. Then (b_n) is only required to be a sequence of random variables such that (a_n, b_n) be a stationary process, and b_{-n} be independent of $\mathcal{F}_{n-1}$. We also need one more assumption, also assumed in the i.i.d. case: for all $1 \le i \le p$, $\mathbb{P}(b_0 + a_0 x = x \mid a_0 = e_i) < 1$.

The mapping $\lambda \longmapsto \log \rho(P_\lambda)$ being convex, its right-hand derivative in 0 being negative and as we have $\rho(P_0) = \rho(P) = 1$, only two cases may occur.

– Either for all $\lambda > 0$, $\rho(P_\lambda) < 1$, in which case we can prove that $\mathbb{E}|Y_1|^\lambda < \infty$ for all λ, provided $\mathbb{E}|b_0|^\lambda < \infty$, and therefore $\mathbb{P}(|Y_1| > t) = o(t^{-\lambda})$ for all λ.
– Or there is a unique $\lambda > 0$ so that $\rho(P_\lambda) = 1$, this is the case we study here.

3. Sketch of the proof of Theorem 2.1

Similar theorems have already been proved in the i.i.d. multidimensional case: a_n are matrices and Y_n and b_n vectors. Renewal theory is used in [6] to prove a similar theorem when the a_n either have a density or are non-negative. Kesten's results were extended in [9] to all i.i.d. random matrices satisfying similar assumptions as in our theorems. Finally in [3] a new specific implicit renewal theorem is proved and the same results as Kesten in the i.i.d. one-dimensional case are derived.

Here we follow the same steps as [9] and [3]. Our problem leads to a system of renewal equations of size p, instead of a single renewal equation. We use a new renewal theorem given in [10] to get an asymptotic equivalent of $\mathbb{P}(xY_1 > t)$, of the form $L(x)t^{-\lambda}$. However the constants $L(x)$ thus obtained are only non-negative.

The next step is to prove that $L(1) + L(-1) > 0$. To do so, we extend the method given in [3] and [4]. First we prove the following lower bound:

$$\mathbb{P}(|Y_1| > t) \ge C \ \mathbb{P}(\sup_n \ |a_0 \cdots a_{1-n}| > \frac{2t}{\varepsilon}),$$

for a positive ε and a corresponding positive constant C. And then we use a ladder height method, and again renewal theory to derive an accurate estimate of the right-hand side probability.

4. Sketch of the proof of Theorem 2.2

Now the sign of the products $a_0 \cdots a_{-n}$ is random. To be able to use the results of the positive case, we include this sign as a new dimension, and we derive a system of renewal equations of size $2p$. Unfortunately, it is not necessarily irreducible, this is why we introduce a new definition.

Definition 4.1 *Let $A = (a_{ij})_{i \le i,j \le p}$ be a positive matrix, and $1 \le l \le p-1$ an integer. A is l-*reducible *if there is (I, J) a non trivial partition of $\{1, \dots, p\}$ such that:*
- *For all $1 \le i \le l$, if $i \in I$ then $a_{ij} = 0$ $\forall j \in J$, if $i \in J$ then $a_{ij} = 0$ $\forall j \in I$.*
- *For all $l+1 \le i \le p$, if $i \in I$ then $a_{ij} = 0$ $\forall j \in I$, if $i \in J$ then $a_{ij} = 0$ $\forall j \in J$.*

*If A is not l-reducible, we say that A is l-*irreducible.

If the matrix of our system is l-irreducible, then the proof runs the same as in the positive case, and in addition we know that both limits $L(1)$ and $L(-1)$ are equal, therefore they are both positive. If the matrix is l-reducible, the system splits into two independent systems of size p, and for each of them the proof is the same as in the positive case. This time $L(1)$ and $L(-1)$ may be different.

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