GENERATION OF FINE TRANSITION LAYERS AND THEIR DYNAMICS FOR THE STOCHASTIC ALLEN–CAHN EQUATION

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Abstract. We study an ε-dependent stochastic Allen–Cahn equation with a mild random noise on a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Here ε is a small positive parameter that represents formally the thickness of the solution interface, while the mild noise $\xi^\varepsilon(t)$ is a smooth random function of t of order $\mathcal{O}(\varepsilon^{-\gamma})$ with $0 < \gamma < 1/3$ that converges to white noise as $\varepsilon \to 0^+$. We consider initial data that are independent of ε satisfying some non-degeneracy conditions, and prove that steep transition layers—or interfaces—develop within a very short time of order $\varepsilon^2|\ln \varepsilon|$, which we call the "generation of interface". Next we study the motion of those transition layers and derive a stochastic motion law for the sharp interface limit as $\varepsilon \to 0^+$. Furthermore, we prove that the thickness of the interface for ε small is indeed of order $\mathcal{O}(\varepsilon)$ and that the solution profile near the interface remains close to that of a (squeezed) travelling wave; this means that the presence of the noise does not destroy the solution profile near the interface as long as the noise is spatially uniform. Our results on the motion of interface improve the earlier results of Funaki (1999) and Weber (2010) by considerably weakening the requirements for the initial data and establishing the robustness of the solution profile near the interface that has not been known before.

1. Introduction

We consider a stochastic Allen–Cahn equation with a Neumann boundary condition

$$\begin{aligned}
&\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \frac{1}{\varepsilon}\xi^\varepsilon(t), \quad t > 0, \quad x \in \Omega,\\
&\frac{\partial u}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial\Omega, \qquad\qquad (1.1)\\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}$$

where Ω is a smooth open bounded domain in $\mathbb{R}^n$ ($n \geq 2$), ν is the outward unit normal vector to $\partial\Omega$, and $\varepsilon > 0$ is a small parameter. The nonlinearity f is of the *bistable* type, and the perturbation term $\xi^\varepsilon(t)$ is what we call a mild noise which is a smooth but random function of t that behaves like an irregular white noise in the limit as $\varepsilon \to 0$. As mentioned in [25], such an equation can be viewed as describing intermediate (mesoscopic) level phenomena between macroscopic and microscopic ones. In such a scale, an active noise appears as a correction term to the reaction-diffusion equation when fluctuations in the hydrodynamic limit is taken into account, see [41].

Our main goal is to make a detailed analysis of the *sharp interface limit* of the problem (1.1) as $\varepsilon \to 0$. In the deterministic case where the perturbation term ξ^ε is replaced by non-random, uniformly bounded smooth functions, the sharp interface limit of (1.1) is well understood: it is known that the solution u typically develops steep transition layers—or *interfaces*—of thickness $\mathcal{O}(\varepsilon)$ within a very short time, which we call the *generation of interface* (or one may call it the

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emergence of transition layers). Furthermore, as $\varepsilon \to 0$, those layers converge to interfaces of thickness 0 whose law of motion is given by the curvature flow with some driving force (the *propagation of interface*). See [2, 14] and the references therein for details.

In the present problem, the perturbation term $\xi^\varepsilon(t)$ is random while it is no longer uniformly bounded as $\varepsilon \to 0$, since it converges to white noise in a certain sense. This makes the analysis harder than in the classical deterministic case. Nonetheless, as we shall see, it is possible to derive a number of results that are as optimal as those established for the classical deterministic problem.

Our first result concerns the *generation of interface*. More precisely, we consider solutions of (1.1) with ε-independent initial data, and show that steep transition layers of thickness $\mathcal{O}(\varepsilon)$ emerge within a very short time. This thickness estimate of order $\mathcal{O}(\varepsilon)$ is the same optimal estimate known for the classical deterministic problem. Next, we discuss the *propagation of interface* and show that the thickness of the layer remains of order $\mathcal{O}(\varepsilon)$ as time passes, and that in the sharp interface limit, as $\varepsilon \to 0$, the law of motion of the interface is given by

$$V = (n-1)\kappa + c\dot{W}_t, \tag{1.2}$$

where V is the inward normal velocity, κ denotes the mean curvature, c is a positive constant and $\dot{W}_t$ is a white noise. The above equation was first derived in [25, 43] for a special class of ε-dependent initial data that already have well-developed transition layers. Our result confirms the validity of the same equation for rather general ε-independent initial data. Furthermore, we also show that the *profile of the solution* near the interface is well approximated by a traveling wave. This implies that the solution profile near the interface is quite robust and is not destroyed by the random noise, as long as the noise depends only on the time variable.

The singular limit of a stochastic Allen–Cahn equation of the form (1.1) was studied by Funaki in his pioneering work [25] for two space dimensions, and later by Weber [43] for general space dimensions $n \geq 2$. Our results improve the work of [25, 43] in three notable aspects. First, as mentioned above, our paper studies the emergence of steep transition layers (the *generation of interface*) at the very initial stage of evolution, which is not discussed in [25, 43]. Secondly, our $\mathcal{O}(\varepsilon)$ estimate of the thickness of layers is optimal and therefore, is considerably better than the order $\mathcal{O}(\varepsilon^\alpha)\,(0 < \alpha < 1)$ estimates presented in [25, 43]. Thirdly, we show the robustness of the solution profile around the interface in the presence of noise (*rigidity of profile*), a fact that has been totally unknown before.

Concerning results on the generation of interface, let us also mention the very recent papers [30, 31]. In [30], the author considers the one-dimensional case with space-time white noise and studies both the generation and motion of the interface, thus improving the work [24], which did not consider the generation of interface. However, as we shall explain in Subsection 1.2, the one-dimensional case is totally different from the multi-dimensional case as the curvature effect does not appear in the former. Therefore the problems treated in [24, 30] are different from the subject of the present paper. In [31], the authors considers a multi-dimensional problem under a space-time noise that is smooth in x. However [31] deals with only the generation of interface, thus the motion of interface under such a noise remains unknown.

We also refer to the influential theory of stochastic viscosity solutions of Lions and Souganidis which covers a large class of stochastic fully nonlinear partial differential equations with applications to phase transitions and propagation of fronts in the presence of noise; see for example the works [32, 33, 34, 35]. In [32], the authors introduced the notion of weak solutions (stochastic viscosity solutions) for parabolic, possibly degenerate, second-order stochastic pdes posed in $\mathbb{R}^N$. In particular [33, subsection 2.3] considers the specific case of the ε-dependent Allen-Cahn equation with the stochastic perturbation introduced by Funaki in [25], posed in the unbounded domain $\mathbb{R}^N$. It is supplemented with a general initial data u_0, not depending on ε and with not necessarily convex initial interface $\Gamma_0 := \{x \in \mathbb{R}^N : u_0(x) = 0\}$, see [33, (2.6) and (2.7)]. In the context of viscosity solutions, a proper approximation of the stochastic problem is proposed which, on the sharp interface limit, yields the stochastic motion by mean curvature and the limiting profile of u to ± 1. Notice however that our approach stands in the short-time existence of a solution to (1.2) as proposed in [25] and [19], see Section 2. This allows us to prove much finer properties of the convergence of (1.1) to (1.2), namely the optimal thickness estimate of the thin layers of solutions of (1.1) when ε is very small (see (3.3) of Theorem 3.1), and the proof of the robustness of the layer profile (see Theorem 3.4), neither of which can be obtained through the viscosity approach, see [1].

1.1. **Assumptions.** Let us state our standing assumptions in the present paper. The nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = a_\pm$. More specifically, we assume that

$$f \text{ is } C^2 \text{ and has exactly three zeros } a_- < a < a_+, \tag{1.3}$$

$$f'(a_\pm) < 0, \quad f'(a) > 0, \tag{1.4}$$

and

$$\int_{a_-}^{a_+} f(u)du = 0. \tag{1.5}$$

This last assumption (1.5) makes f a *balanced* bistable nonlinearity. We will use this assumption only in Section 5, where we study the propagation of interface. No such assumption is needed for the emergence of interface, which we discuss in Section 4.

Concerning the initial data, we assume that $u_0 \in C^2(\overline{\Omega})$, and define

$$C_0 := \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|\Delta u_0\|_{C^0(\overline{\Omega})}. \tag{1.6}$$

The initial interface is defined by

$$\Gamma_0 := \{x \in \Omega : u_0(x) = a\}. \tag{1.7}$$

We assume that $\Gamma_0 \subset\subset \Omega$ is a $C^{2,\alpha}$ $(0 < \alpha < 1)$ hypersurface without boundary and that

$$\nabla u_0(x) \cdot n(x) \neq 0 \text{ for any } x \in \Gamma_0, \tag{1.8}$$

where $n = n(x)$ denotes the outward unit normal vector to Γ_0 at x.

Let Ω_0 denote the region enclosed by Γ_0. Without loss of generality, we may assume that

$$u_0(x) < a \text{ for any } x \in \Omega_0 \text{ and } u_0(x) > a \text{ for any } x \in \Omega \setminus \overline{\Omega_0}. \tag{1.9}$$

As regards the perturbation term $\xi^\varepsilon(t)$, we shall consider two types of mild noises as specified below, following [25] and [43].

First type of noise (MN1)

Following Funaki [25], we consider a mild noise ξ^ε given in the form

$$\xi^\varepsilon(t) := \varepsilon^{-\gamma_1}\xi(\varepsilon^{-2\gamma_1}t), \quad t > 0, \tag{1.10}$$

for some

$$0 < \gamma_1 < \frac{1}{3}, \tag{1.11}$$

where $\xi(t) = \xi_t$ is a stochastic process in t that is stationary and strongly mixing. More specifically, let $F_{\xi_{t_1+\tau},\ldots,\xi_{t_k+\tau}}$ be the distribution function of the k random variables $\xi_{t_1+\tau},\ldots,\xi_{t_k+\tau}$, then the stochastic process ξ_t is called *stationary* if for all k, τ and for all $t_1,\ldots,t_k$

$$F_{\xi_{t_1+\tau},\ldots,\xi_{t_k+\tau}} = F_{\xi_{t_1},\ldots,\xi_{t_k}}.$$

Let $(\Omega_{prob},\mathcal{F},\mathbb{P})$ be the probability space where ξ_t is realized, with $\mathcal{F} := \sigma(\xi_r :\ 0 \le r < +\infty)$ the σ-algebra generated by ξ_r for $0 \le r < +\infty$, and $\mathbb{P}$ the probability measure. Then $\mathcal{F}_{s,t} := \sigma(\xi_r : s \le r \le t)$ is the subalgebra of $\mathcal{F}$ generated by ξ_r for $s \le r \le t$. We assume that the process ξ_t is strongly mixing in the following sense: the mixing rate $\rho(t)$ defined by

$$\rho(t) := \sup_{s\ge 0} \sup_{A\in\mathcal{F}_{s+t,\infty},\, B\in\mathcal{F}_{0,s}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|/\mathbb{P}(B), \quad t \ge 0,$$

satisfies

$$\int_0^\infty \rho(t)^{1/p}dt < +\infty \quad \text{for some } \ p > 3/2.$$

In Funaki [25], this last condition is used to derive some estimates that are uniform in ε; see the proof of Proposition 4.1 and Lemma 5.3 in [25].

Furthermore, it is assumed that $t \mapsto \xi(t)$ is C^1 almost surely,

$$|\xi(t)| \le M, \quad |\dot{\xi}(t)| \le M, \quad E[\xi(t)] = 0,$$

for some deterministic constant M, with $\dot{\xi} := \frac{d\xi}{dt}$. Obviously, the above implies that

$$t \mapsto \xi^\varepsilon(t) \text{ is } C^1 \text{ almost surely,}$$

and that

$$|\xi^\varepsilon(t)| \le M\varepsilon^{-\gamma_1}, \quad |\dot{\xi^\varepsilon}(t)| \le M\varepsilon^{-3\gamma_1}. \tag{1.12}$$

In Funaki [25], these conditions are used to justify the limit interface equation (1.17) as $\varepsilon \to 0$, but, as we shall see, the estimate (1.12) will also be fundamental for our analysis of the initial formation of layers (the generation of interface).

Notice that the coefficient $\varepsilon^{-\gamma_1}$ in the definition (1.10) implies that $\xi^\varepsilon(t)$ is unbounded as $\varepsilon \to 0$. As shown in [25], $\xi^\varepsilon(t)$ converges to an irregular white noise as $\varepsilon \to 0$ in a certain sense.

Second type of noise (MN2)

Following Weber [43], we define the mild noise $\xi^\varepsilon(t) = \xi^\varepsilon_t$ as the derivative of a mollified Brownian motion. More precisely, let $W(t)$, $t \ge 0$, be a Brownian motion defined on the space $(\Omega_{prob},\mathcal{F},\mathbb{P})$. (Here, as usual, the dependence of W on the sample points $\omega \in \Omega_{prob}$ is not shown explicitly.) For technical reasons, $W(t)$ is extended over $\mathbb{R}$ by considering an independent Brownian motion $\widetilde{W}(t)$, $t \ge 0$, and setting $W(t) = \widetilde{W}(-t)$ for $t < 0$. Then $W(t)$, $t \in \mathbb{R}$, is a Gaussian process, with independent stationary increments and a distinguished point $W(0) = 0$ almost surely. Also, let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a mollifying smooth and symmetric kernel, with $\rho = 0$ outside $[-1, 1]$ and $\int_\mathbb{R} \rho = 1$. The approximated Brownian motion $W^\varepsilon(t), t \ge 0$, is defined as usual by

$$W^\varepsilon(t) := W * \rho^\varepsilon(t) := \int_{-\infty}^{\infty} \rho^\varepsilon(t-s)W(s)ds, \tag{1.13}$$

where $\rho^\varepsilon(\tau) := \varepsilon^{-\gamma_2}\rho(\varepsilon^{-\gamma_2}\tau)$ for some constant γ_2 satisfying

$$0 < \gamma_2 < \frac{2}{3}. \tag{1.14}$$

Note that the Brownian motion for negative times is needed only in the expression (1.13), so only the negative times in $(-\varepsilon^{\gamma_2}, 0]$ will play a role. The constant γ_2 determines how quickly W^ε converges to the true integrated white noise as $\varepsilon \to 0$. Since $W(t)$ is Hölder continuous almost surely, $W^\varepsilon(t)$ is a smooth function of t almost surely. The noise $\xi^\varepsilon(t)$ is then defined as the derivative of $W^\varepsilon(t)$, that is,

$$\xi^\varepsilon(t) = \dot{W}^\varepsilon(t). \tag{1.15}$$

In [43, Propositions 1.2 and 1.3], the author derives estimates for $\xi^\varepsilon(t)$ and its derivative $\dot{\xi}^\varepsilon(t)$ in the form

$$|\xi^\varepsilon(t)| \le M\varepsilon^{-\tilde{\gamma}/2}, \qquad |\dot{\xi}^\varepsilon(t)| \le M\varepsilon^{-3\tilde{\gamma}/2} \qquad (\gamma_2 < {}^\forall\tilde{\gamma} < 2/3),$$

by using Lévy's well-known result on the modulus of continuity of Brownian motion:

$$\mathbb{P}\left[\limsup_{\delta\to 0} \frac{1}{g(\delta)} \max_{\substack{0\le s<t\le T\\ t-s\le\delta}} |W(t)-W(s)| = 1\right] = 1,$$

where the modulus of continuity is given by $g(\delta) = \sqrt{2\delta\log(\frac{1}{\delta})}$. Actually the very same argument as in [43] gives the following slightly more refined estimates, whose proof is omitted as it is straightforward—roughly speaking it suffices to set $\delta = \varepsilon^{\gamma_2}$ in $g(\delta)$.

Proposition 1.1 (Estimates of the noise term). *For any $T > 0$, there exist a non-random constant $M = M(T) > 0$ and (random) $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \le \varepsilon_0$ and all $0 \le t \le T$,*

$$|\xi^\varepsilon(t)| \le M\varepsilon^{-\gamma_2/2}|\log\varepsilon|^{1/2}, \quad |\dot{\xi}^\varepsilon(t)| \le M\varepsilon^{-3\gamma_2/2}|\log\varepsilon|^{1/2}. \tag{1.16}$$

This is an analogue of (1.12) and will be fundamental for our analysis of the emergence of interface.

1.2. **Deterministic and stochastic Allen–Cahn equations.** The (deterministic) Allen–Cahn equation was proposed in [4] as a model for the dynamics of interfaces in crystal structures in alloys. The same equation also appears as a model for various other problems, including population genetics and nerve conduction.

As far as the one-dimensional case is concerned, the behavior of the solution as $\varepsilon \to 0$ was analyzed in [13, 16]. After a very short time, the value of the solution becomes close to a_+ or a_- in most part of the domain, thus generating possibly many very steep transition layers. These well developed transition layers then start to move very slowly, and each time a pair of transition layers meet, the two layers annihilate each other, thus the number of layers decrease gradually. Although those collision-annihilation process takes place rather quickly, the motion of layers between the collisions is extremely slow, and the profile of the layers look nearly unchanged during those slow motion periods; in other words, the solution exhibits a metastable pattern. The situation is quite different in the multi-dimensional case, where such metastable patterns hardly appear because of the curvature effect on the motion of the interface. This curvature effect in higher dimensions is well illustrated by the sharp interface limit $\varepsilon \to 0$, where the motion of layers (sharp interfaces) is known to be governed by the mean curvature flow plus some driving forces. There is a large literature on the rigorous justification of this singular limit; we refer, among others, to [12], [14, 15], [37, 38], [2]

Stochastic systems of Allen–Cahn type have been analyzed in [20]. For the one-dimensional case, in [24], [11], the authors studied the stochastic Allen–Cahn equation with initial data close to a Heaviside function. They proved under an appropriate scaling that the solution stays close to this shape, while the random perturbation creates a dynamic for the single interface which is observed on a much faster time scale than in the deterministic case. This has been also studied in [44] via an invariant measure approach. The author therein, under certain assumptions, proves exponential convergence towards a curve of minimizers of the energy, and a concentration of the measure on configurations with precisely one jump. In [39], the authors studied the competition between some energy functional that is minimized for small noise strength, and they also investigate the entropy induced by a system of large size.

If the initial data involves more than one interfaces, it is believed that these interfaces also exhibit a random movement which is much quicker than in the deterministic case, while different interfaces should annihilate when they meet [21], and the limiting process is related to the Brownian (see [23] for formal arguments).

As far as the sharp interface limit of the stochastic Allen–Cahn equation (1.1) is concerned, we first mention the pioneering work of Funaki [25]: the law of motion of the limit interface is rigorously derived, for dimension $n = 2$ and convex initial interface Γ_0 and it is given by

$$V = (n-1)\kappa + c\dot{W}_t, \tag{1.17}$$

where V is the inward normal velocity of the inner interface Γ_t, κ is the mean curvature of Γ_t, $\dot{W}_t$ is the white noise in t (namely the singular limit of the mild noise as $\varepsilon \to 0$) and c denotes an identified constant. Note that this motion law was derived under the assumption that the initial data is well-prepared, in the sense that it depends on ε in such a way that it is very close to the formal asymptotics, i.e., $U_0\left(\frac{d(x,0)}{\varepsilon}\right)$, where $U_0(z)$ is the underlying one-dimensional travelling wave and $d(\cdot, 0)$ the signed distance function to Γ_0 [1]. Later, in [43], the classical result of [25] was extended to spatial dimensions greater than two without the restriction of initial convexity.

The multi-dimensional stochastic Allen–Cahn equation driven by a multiplicative noise is studied in [40]. This noise is non-smooth in time and smooth in space (finite sum of time-dependent Brownian motions, with coefficients deterministic functions of the spatial variables). The authors prove for ε-dependent initial data, the tightness of solutions for the sharp interface limit problem and show convergence to phase-indicator functions. The existence and properties of such a stochastic flow, was first established in [45], in the context of geometric measure theory. More precisely, in [45] an iterative scheme is constructed, and a sequence of sets with randomly perturbed boundaries is introduced. The analysis in [40] was based on energy estimates and is related to the construction of [45]. In [10], a stochastic Allen–Cahn equation is considered; the authors study its large deviation asymptotics in a joint sharp interface and small noise limit.

The space-time white noise driven Allen–Cahn equation is known to be ill-posed in space dimensions greater than one, [42], [18]. Therefore, in [27], a multi-dimensional stochastic Allen–Cahn equation with mollified additive white space-time noise is analyzed (finite sum of time-dependent Brownian motions with finite noise strength). For regular ε-independent initial data, it is shown that as the mollifier is removed, the solutions converge weakly to zero, independently of the initial condition. If the noise strength converges to zero at a sufficiently fast rate, then the solutions converge to those of the deterministic equation. A large deviation principle is discussed in [28].

Considering stochastic models where the Allen–Cahn operator appears, or stochastic sharp interface limit problems from phase separation, we also refer to the recent results of [5], [8], [7], [6]. More specifically, in [5], the mass conserving Allen–Cahn equation with noise was analyzed and the

[1] In this paper, a signed distance function $d(\cdot, t)$ to Γ_t is always negative in the region enclosed by Γ_t, and positive elsewhere.

stochastic dynamics of a droplet's motion along the boundary in dimension 2 were derived. In [8], the authors established the stochastic existence and investigated the regularity of solutions for the so-called Cahn-Hilliard/Allen–Cahn equation with space-time white noise; for the same problem, in [7], Malliavin calculus was applied for the proof of existence of a density for the stochastic solution, in dimension one. The stochastic sharp interface limit for the Cahn-Hilliard equation with space-time smooth in space noise has been presented in [6].

1.3. **Motivation for the current work.** Our work stands in the framework of [25] and [43], where the mild noises (MN1) and (MN2), of subsection 1.1, have been initially introduced. As mentioned before, in these works, it is shown that the sharp interface limit of (1.1) is motion driven by mean curvature plus an additional stochastic forcing term. This holds true for well-prepared initial data. However, whether or not this motion law is valid for a large class of initial data has never been studied. To answer this question, one has to study the *generation of interface* in details. We analyze first the solution's profile for short times, and show that layers of thickness $\mathcal{O}(\varepsilon)$ are rapidly formed. Then, considering later times, we prove that for rather general initial data the thickness of the layers remains of order $\mathcal{O}(\varepsilon)$, and we determine the shape of the solution $u^\varepsilon(x,t)$ *inside* the layers.

To do so, we shall rely on the results of [2] and [3] for the deterministic Allen–Cahn equation

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2}\left(f(u) - \varepsilon g^\varepsilon(x,t)\right). \tag{1.18}$$

The authors in [2] showed that, for a rather general class of initial data that are independent of ε, the solution $u^\varepsilon(x,t)$ of (1.18) develops a steep internal layer within a short time interval of $\mathcal{O}(\varepsilon^2 \ln \varepsilon)$. Consequently, $u^\varepsilon(x,t)$ lies between a pair of super- and sub-solutions u^+, u^- for $t_\varepsilon \le t \le T$, whose profiles are very close to the formal asymptotics of typical fronts and are located within the distance of $\mathcal{O}(\varepsilon)$ from each other. Since the fronts of both u^+, u^- move by the correct motion law with an error margin of $\mathcal{O}(\varepsilon)$, so does the front of u^ε. This indicates that the layers of $u^\varepsilon(x,t)$ move by motion by mean curvature plus an additional pressure term, and that the their thickness is $\mathcal{O}(\varepsilon)$. Recently, the authors in [3] have found a way to explore the profile of the solution $u^\varepsilon(x,t)$ *inside* these layers. More precisely, they have proved the validity of the principal term of the formal asymptotic expansions for rather general initial data.

Our present analysis of the singular limit of problem (1.1) reveals, in particular, that the profile of the solutions $u^\varepsilon(x,t)$ is not altered by the mild noise, for both the thickness of the layers (compare Theorem 3.1 with [2]) and the profile *inside* the layers (compare Theorem 3.4 with [3]). The main difference with the deterministic problem stands in a slight shift of the position of the layers which occurs in the very early times. This will be clarified in Section 4.

Let us underline that, while the perturbation term $g^\varepsilon(x,t)$ in (1.18) remains uniformly bounded as $\varepsilon \to 0$ [2], in the present paper, we allow the perturbation $\xi^\varepsilon(t)$ to become singular as $\varepsilon \to 0$, as can be seen in (1.12) and (1.16). We therefore need to modify our argument for the generation of interface (see Section 4) and then to use the stochastic approach of [25], [19], [43]. The latter is suitable for perturbations $\xi^\varepsilon(t)$ which behaves like white noise as $\varepsilon \to 0$, resulting to random dynamics in the limit, in contrast with [2].

2. On stochastic motion by mean curvature

Before stating our main results, we need to give a precise definition of the motion law of the form (1.17) for the limit interface. The interpretation of this motion law actually depends on the type of noise under consideration, namely the (MN1) type noise and the (MN2) type one mentioned earlier.

2.1. **Motion law for the (MN1) type noise.** The interpretation of the motion law (1.17) for this type of noise was clarified by Funaki [25]. In this subsection we will adopt his definition and first recall some of his results. Note that the interpretation of (1.17) in this sense holds as long as the random curve Γ_t remains strictly convex and does not touch the boundary $\partial\Omega$.

Let $c_0 > 0$ and $\alpha_0 > 0$ be given constants (which will be taken as in (5.3) and (3.4) in our context). A strictly convex curve Γ can be parametrized by $\theta \in [0, 2\pi)$ in terms of the Gauss map: the position x on Γ is denoted by $x(\theta)$ if the angle between a fixed direction and the outward normal $n(x)$ at x to Γ is θ. Denote by $\kappa = \kappa(\theta)$ the (mean) curvature of Γ at $x = x(\theta)$. Then the stochastic motion by mean curvature dynamics

$$V = \kappa + c_0\alpha_0\dot{W}_t$$

is defined through the nonlinear stochastic partial differential equation for $\kappa = \kappa(\theta, t)$:

$$\partial_t\kappa = \kappa^2\partial_{\theta\theta}\kappa + \kappa^3 + c_0\alpha_0\kappa^2 \circ \dot{W}_t, \quad 0 < t < \sigma, \theta \in [0, 2\pi), \tag{2.1}$$

where $\circ$ means the Stratonovich stochastic integral and $\sigma = \lim_{N\to\infty}\sigma_N$. Stopping times are defined by

$$\sigma_N := \inf\{t > 0,\ \bar{\kappa}_t > N \text{ or } \mathrm{dist}(\Gamma_t, \partial\Omega) < 1/N\}, \quad N > 0,$$

where $\bar{\kappa}_t = \max_{\theta\in[0,2\pi)}\max\{\kappa(\theta, t), \kappa^{-1}(\theta, t), |\partial_\theta\kappa(\theta, t)|\}$. Indeed, once the mean curvature $\kappa(\theta, t)$ is obtained via (2.1), one can determine $\Gamma_t = \{x_t(\theta) \in \mathbb{R}^2 \cong \mathbb{C},\ \theta \in [0, 2\pi)\}$ by formula [25, (1.10)] to which we refer for further details.

Also, we need to consider approximations of this motion as follows. Let γ_0^ε (which will be taken as in (4.5) in our context) be a $C^{2,\alpha}$ hypersurface which is a slight shift of Γ_0, in the sense that

$$\gamma_0^\varepsilon \to \Gamma_0 \text{ as } \varepsilon \to 0, \text{ in the } C^{2,\alpha} \text{ sense.}$$

Furthermore, we replace the forcing term $c_0\alpha_0\dot{W}_t$ by $-\frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon}$, where $\delta \mapsto c(\delta)$ (which will be taken as in (5.1) in our context) is smooth in a neighborhood of zero, satisfies $c(0) = 0$ and $\partial_\delta c(0) = -c_0$. We are therefore equipped with a family of hypersurfaces $(\gamma_t^\varepsilon)_{0\le t<\sigma^\varepsilon}$, starting from γ_0^ε and evolving with the law

$$V = \kappa - \frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon} \quad \text{on } \gamma_t^\varepsilon.$$

Here we have $\sigma^\varepsilon = \lim_{N\to\infty}\sigma_N^\varepsilon$, where

$$\sigma_N^\varepsilon := \inf\{t > 0, \quad \bar{\kappa}_t^\varepsilon > N \text{ or } \mathrm{dist}(\gamma_t^\varepsilon, \partial\Omega) < 1/N\}, \quad N > 0, \tag{2.2}$$

where

$$\bar{\kappa}_t^\varepsilon = \max_{\theta\in[0,2\pi)}\max\{\kappa^\varepsilon(\theta, t), (\kappa^\varepsilon)^{-1}(\theta, t), |\partial_\theta\kappa^\varepsilon(\theta, t)|\}, \tag{2.3}$$

with κ^ε the mean curvature of γ_t^ε.

Since $-\frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon} \sim c_0\xi^\varepsilon(t)$ as $\varepsilon \to 0$, and since $\xi^\varepsilon(t)$ converges to $\alpha_0\dot{W}_t$ in distribution sense, it is expected that the approximations (γ_t^ε) converge, in some sense, to (Γ_t). Using the martingale method such a convergence—see Corollary 3.2 for a precise statement—is proved in [25], when $\gamma_0^\varepsilon = \Gamma_0$. In particular, for all but countable many $N > 0$ we have $\sigma_N^\varepsilon \to \sigma_N$ as $\varepsilon \to 0$.

2.2. **Motion law for the (MN2) type noise.** The precise meaning of the motion law (1.17) in the context of the (MN2) type noise can be clarified by using the results of Dirr, Luckhaus and Novaga [19]. In this subsection we summarize their results and apply them to our problem. More precisely, we refer to [19, Theorem 3.1] for the existence result and to [19, Corollary 4.2] for the estimate of the deviation from the original problem, when the white noise is smoothly approximated and when the initial hypersurface is slightly shifted.

Let $c_0 > 0$ be a given constant (which will be taken as in (5.3) in our context). Since the initial hypersurface $\Gamma_0 = \partial\Omega_0$ is of class $C^{2,\alpha}$, there is a stopping time $\tau = \tau(\Gamma_0) = \tau(\omega, \Gamma_0)$ depending on the $C^{2,\alpha}$-norm of Γ_0, and a family of hypersurfaces $(\Gamma_t)_{0\leq t<\tau} = (\Gamma_t(\omega))_{0\leq t<\tau(\omega,\Gamma_0)}$ of class $C^{2,\alpha}$, such that, for any $X_0 \in \Gamma_0$, there is a process $X(\cdot)$ with $X(t) = X(t,\omega) \in \Gamma_t = \Gamma_t(\omega)$ for almost all $\omega \in \Omega_{prob}$ which solves the Itô equation

$$dX = \nu(X(t,\omega),t)(n-1)\kappa(X(t,\omega),t)dt + \nu(X(t,\omega),t)c_0 dW, \quad X(0) = X_0,$$

where $\kappa(y,t)$ and $\nu(y,t)$ are respectively the mean curvature and the inner normal at $y \in \Gamma_t$. This is the sense we adopt for the motion law

$$V = (n-1)\kappa + c_0\dot{W}_t,$$

or $dV = (n-1)\kappa dt + c_0 dW_t$, which we call the stochastic motion by mean curvature.

Also, we need to consider approximations of this motion as follows. Let γ_0^ε (which will be taken as in (4.5) in our context) be a $C^{2,\alpha}$ hypersurface which is a slight shift of Γ_0, in the sense that

$$\gamma_0^\varepsilon \to \Gamma_0 \text{ as } \varepsilon \to 0, \text{ in the } C^{2,\alpha} \text{ sense.}$$

Furthermore, we replace the forcing term $c_0\dot{W}_t$ by $-\frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon}$, where $\delta \mapsto c(\delta)$ (which will be taken as in (5.1) in our context) is smooth in a neighborhood of zero, satisfies $c(0) = 0$ and $\partial_\delta c(0) = -c_0$. We are therefore equipped with a family of hypersurfaces $(\gamma_t^\varepsilon)_{0\leq t<\tau^\varepsilon}$, starting from γ_0^ε and evolving with the law

$$V = (n-1)\kappa - \frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon} \quad \text{on } \gamma_t^\varepsilon.$$

From the definition of the noise $\xi^\varepsilon(t)$ as the derivative of an approximated Brownian motion $W^\varepsilon(t)$ (by convolution with a mollifier) and the above assumptions, we have—see [43, Lemma 3.3]—that, for any $T > 0$, the random functions $t \mapsto \int_0^t -\frac{c(\varepsilon\xi^\varepsilon(s))}{\varepsilon}ds$ converge almost surely to $t \mapsto c_0W(t)$ in $C^{0,\alpha}([0,T])$ for any $0 < \alpha < \frac{1}{2}$. This enables to quote [19, Corollary 4.2]: there is a time $T > 0$ such that

$$\sup_{0\leq t\leq T} \|d(t,x) - d^\varepsilon(t,x)\|_{C^{2,\alpha}} \to 0, \quad \text{as } \varepsilon \to 0, \tag{2.4}$$

where $d(\cdot,t)$, $d^\varepsilon(\cdot,t)$ denote the signed distance functions to Γ_t, γ_t^ε respectively.

3. Main results

Our first main result is to localize the transitions layers of the solution of the stochastic Allen–Cahn equation in a $\mathcal{O}(\varepsilon)$ neighborhood of a family of hypersurfaces (γ_t^ε), which is defined as follows. The initial hypersurface γ_0^ε is defined in (4.5) and is a slight shift of the initial interface Γ_0 defined in (1.7) (we hope that the reason for such a shift will become transparent for the reader in Section 4).

Let the family (γ_t^ε) evolve with the law of motion

$$V = (n-1)\kappa - \frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon} \quad \text{on } \gamma_t^\varepsilon, \tag{3.1}$$

where $c(\delta)$ is the speed of the bistable traveling wave $m(z;\delta)$ defined in (5.1). Recalling Section 2, if the noise is of the (MN1) type then this family is defined for $0 < t \leq \sigma_N^\varepsilon$, $N > 0$ arbitrary, whereas if the noise is of the (MN2) type this family is defined for $0 < t \leq \tau^\varepsilon$. In the latter case, let $T > 0$ be given as in (2.4). Also Ω_t^ε denotes the region enclosed by γ_t^ε.

Theorem 3.1 (Emergence and motion of $\mathcal{O}(\varepsilon)$ Allen–Cahn layers)**.** *Let the nonlinearity f and the initial data u_0 satisfy the assumptions of subsection 1.1, and the mild noise be of (MN1) or (MN2) type. In the former case, let $N > 0$ be given. Let $u^\varepsilon(x,t)$ be the solution of* (1.1)*. Let $\eta \in (0, \eta_0 := \min(a - a_-, a_+ - a))$ be arbitrary and define μ as the derivative of $f(u)$ at the unstable zero $u = a$, that is*

$$\mu = f'(a) > 0. \tag{3.2}$$

Then there exist positive constants ε_0 and C such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t^\varepsilon \leq t \leq \sigma_N^\varepsilon$—if noise is of (MN1) type—or all $t^\varepsilon \leq t \leq T$—if noise is of (MN2) type—where

$$t^\varepsilon := \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|, \text{ with } \mu_\varepsilon \to \mu \text{ as } \varepsilon \to 0,$$

we have

$$u^\varepsilon(x,t) \in \begin{cases} [a_- - \eta, a_+ + \eta] & \text{if } x \in \mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon) \\ [a_- - \eta, a_- + \eta] & \text{if } x \in \Omega_t^\varepsilon \setminus \mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon) \\ [a_+ - \eta, a_+ + \eta] & \text{if } x \in (\Omega \setminus \overline{\Omega_t^\varepsilon}) \setminus \mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon), \end{cases} \tag{3.3}$$

where $\mathcal{N}_r(\gamma_t^\varepsilon) := \{x \in \Omega : dist(x, \Gamma_t) < r\}$ denotes the r-neighborhood of γ_t^ε.

As mentioned in the introduction and clear from the above theorem, the deterministic $\mathcal{O}(\varepsilon)$ thickness of the layers of the solutions $u^\varepsilon(x,t)$—as estimated in [2]—is not altered by the mild noise.

The above theorem enables to generalize the convergence results of [25] and [43]—which are concerned with well-prepared initial data—to rather general data.

Corollary 3.2 (Extension of Funaki [25] to general data)**.** *Let the nonlinearity f and the initial data u_0 satisfy the assumptions of subsection 1.1. Let the mild noise be of (MN1) type. Assume further that $\xi^\varepsilon(0) = 0$. Let $u^\varepsilon(x,t)$ be the solution of* (1.1)*. Assume further that $n = 2$ and that Ω_0 is convex. Following subsection 2.1, let $(\Gamma_t)_{0 \leq t < \sigma := \lim_{N\to\infty}\sigma_N}$ evolve by*

$$V = \kappa + (c_0\alpha_0)\dot{W}_t,$$

with $c_0 > 0$ the constant defined in (5.3)*, and*

$$\alpha_0 := \sqrt{2\int_0^\infty E[\xi_0\xi_t]dt}. \tag{3.4}$$

Then the random motion of curves $(\gamma_t^\varepsilon)_{0 \leq t < \sigma^\varepsilon := \lim_{N\to\infty}\sigma_N^\varepsilon}$ defined in subsection 2.1 satisfies the following two conditions.

(i) *Let $N > 0$ be given. For $0 \le t < \sigma_N^\varepsilon$, let $x \mapsto \Phi^\varepsilon(x,t)$ be the step function with value a_- in the region enclosed by γ_t^ε and a_+ elsewhere. Then*

$$\sup_{t^\varepsilon \le t \le \sigma_N^\varepsilon} \|u^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to 0 \qquad \text{in probability, as } \varepsilon \to 0,$$

where t^ε is as in Theorem 3.1.

(ii) *γ_t^ε converges to Γ_t as $\varepsilon \to 0$ in the following sense: for any $T > 0$ and all but countable many $N \in \mathbb{R}^+$, the joint distribution of $(\sigma_N^\varepsilon, \Gamma^\varepsilon_{t \wedge \sigma_N^\varepsilon})$ on $\mathbb{R}^+ \times C([0,T], C([0,2\pi), \mathbb{R}^2))$ converges, as $\varepsilon \to 0$, to that of $(\sigma_N, \Gamma_{t\wedge\sigma_N})$.*

Corollary 3.3 (Extension of Weber [43] to general data)**.** *Let the nonlinearity f and the initial data u_0 satisfy the assumptions of subsection 1.1. Let the mild noise be of (MN2) type. Let $u^\varepsilon(x,t)$ be the solution of* (1.1)*. Following subsection 2.2, let $(\Gamma_t)_{0 \le t < \tau(\Gamma_0)}$ evolve by*

$$dV = (n-1)\kappa dt + c_0 dW_t,$$

with $c_0 > 0$ the constant defined in (5.3)*. For $0 \le t < \tau(\Gamma_0)$, let $x \mapsto \Phi(x,t)$ be the step function with value a_- in the region enclosed by Γ_t and a_+ elsewhere. Let $T > 0$ be as in* (2.4)*.*

Then

$$\sup_{t^\varepsilon \le t \le T} \|u^\varepsilon(\cdot,t) - \Phi(\cdot,t)\|_{L^2(\Omega)} \to 0 \qquad \text{almost surely, as } \varepsilon \to 0,$$

where t^ε is as in Theorem 3.1.

By performing formal asymptotic expansions, see [2, Section 2] for more details, it is suspected that, close to the limit interface Γ_t, the solution is approximated by

$$u^\varepsilon(x,t) \sim U_0\left(\frac{d(x,t)}{\varepsilon}\right) + \cdots, \tag{3.5}$$

with $d(\cdot,t)$ the signed distance function to Γ_t, and $U_0(z)$ the unique solution (whose existence is guaranteed by the integral condition (1.5)) of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0 \\ U_0(-\infty) = a_-\,, \quad U_0(0) = a\,, \quad U_0(\infty) = a_+\,. \end{cases} \tag{3.6}$$

This represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. As recently proved in [3] for the deterministic case, we are actually able to prove the validity of the first term of expansion (3.5) for the stochastic case under consideration.

Let us define the level surface of the solution u^ε

$$\Gamma_t^\varepsilon := \{x \in \Omega : \, u^\varepsilon(x,t) = a\} \tag{3.7}$$

and the *signed distance function associated with* Γ^ε by

$$\overline{d^\varepsilon}(x,t) := \begin{cases} -\mathrm{dist}(x, \Gamma_t^\varepsilon) & \text{if } u^\varepsilon(x,t) < a \\ \ \ \mathrm{dist}(x, \Gamma_t^\varepsilon) & \text{if } u^\varepsilon(x,t) > a\,. \end{cases} \tag{3.8}$$

Recall that if noise is of the (MN2) type, then $T > 0$ was defined in (2.4). To unify the notation in the following, if noise is of the (MN1) type, then, for a given $N > 0$, we select $0 < T < \sigma_N$ (see subsection 2.1). It therefore follows from (3.3) that $\Gamma_t^\varepsilon \subset \mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon)$ for all $t^\varepsilon \le t \le T$, so that

$$|\overline{d^\varepsilon}(x,t) - d^\varepsilon(x,t)| \le C\varepsilon \quad \forall (x,t) \in \overline{\Omega} \times [t^\varepsilon, T], 0 < \varepsilon << 1. \tag{3.9}$$

Theorem 3.4 (Profile in the layers)**.** *Let the assumptions of Theorem 3.1 hold. Fix $\rho > 1$ and $0 < T' < T$. Then*

(i) *If $\varepsilon > 0$ is small enough then, for any $t \in [\rho t^\varepsilon, T']$, the level set Γ_t^ε is a smooth hypersurface and can be expressed as a graph over γ_t^ε.*

(ii) *We have*

$$\lim_{\varepsilon\to 0} \sup_{\rho t^\varepsilon \le t \le T',\, x\in\bar\Omega} \left| u^\varepsilon(x,t) - U_0\left(\frac{\overline{d^\varepsilon}(x,t)}{\varepsilon}\right)\right| = 0\,, \tag{3.10}$$

where $\overline{d^\varepsilon}$ denotes the signed distance function associated with Γ^ε.

As mentioned in the introduction and clear from the above theorem, the deterministic profile of the solutions $u^\varepsilon(x,t)$ *inside* the layers—as explored in [3]—is not altered by the mild noise.

The rest of the paper is organized as follows. In Section 4, we prove the emergence of internal layers for the problem (1.1). In Section 5, we construct accurate sub- and super-solutions to study the motion of the layers that then takes place. The combination of these two studies is performed in Section 6 where we prove Theorem 3.1 which, using the results of [25], [19], [43], implies Corollaries 3.2 and 3.3. Last, we prove Theorem 3.4 in Section 7.

4. Rapid emergence of $\mathcal{O}(\varepsilon)$ layers

This section deals with the emergence of internal layers (or the generation of interface) which occurs very quickly. In other words, given a virtually arbitrary initial data, we prove that the solution $u^\varepsilon(x,t)$ quickly becomes close to $a_\pm$ in most part of Ω. In order to track the $\mathcal{O}(\varepsilon)$ thickness of the layers, we show that the generation occurs in a $\mathcal{O}(\varepsilon)$ neighborhood of some smooth hypersurface γ_0^ε, which itself lies in a $o(\varepsilon^{1-\gamma})$ neighborhood of the initial interface Γ_0, where γ is chosen such that

$$\begin{cases} 0 < \gamma_1 < \gamma < \frac{1}{3} \text{ if the noise is of the (MN1) type} \\ 0 < \frac{\gamma_2}{2} < \gamma < \frac{1}{3} \text{ if the noise is of the (MN2) type.} \end{cases}$$

The reason for such an initial drift is the following. For $0 \le t \le T$, the mean value theorem provides a $0 < \theta < 1$ such that

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \dot\xi^\varepsilon(\theta t)t = \xi^\varepsilon(0) + o(\varepsilon), \tag{4.1}$$

as long as $0 \le t \le \mathcal{O}(\varepsilon^2|\ln\varepsilon|)$, where we have used (1.12) under the noise assumption (MN1), and (1.16) under the noise assumption (MN2). Once the crucial observation (4.1) is made, the treatment of the $o(\varepsilon)$ term follows from the generation of interface property performed in [2, Section 4], whereas the $\xi^\varepsilon(0) = o(\varepsilon^{-\gamma})$ term explains the initial shift.

In order to take advantage of observation (4.1), we define

$$f^\varepsilon(u) := f(u) + \varepsilon\xi^\varepsilon(0). \tag{4.2}$$

In view of assumptions (1.3) and (1.4) on f, and since $\varepsilon\xi^\varepsilon(0) = o(\varepsilon^{1-\gamma}) \to 0$, we have, for $\varepsilon > 0$ small enough, that f^ε is still of the bistable type, in the sense that

$$f^\varepsilon \text{ has exactly three zeros } \; a_-^\varepsilon < a^\varepsilon < a_+^\varepsilon, \tag{4.3}$$

where $a_-^\varepsilon = a_- + o(\varepsilon^{1-\gamma})$, $a^\varepsilon = a + o(\varepsilon^{1-\gamma})$, $a_+^\varepsilon = a_+ + o(\varepsilon^{1-\gamma})$, and

$$\frac{d}{du} f^\varepsilon(a_\pm^\varepsilon) \to f'(a_\pm) < 0, \quad \mu_\varepsilon := \frac{d}{du} f^\varepsilon(a^\varepsilon) \to \mu = f'(a) > 0. \tag{4.4}$$

We now define

$$\gamma_0^\varepsilon := \{x \in \Omega : \ u_0(x) = a^\varepsilon\}\,, \tag{4.5}$$

which consists in a $o(\varepsilon^{1-\gamma})$ shift of the initial interface Γ_0 defined in (1.7). In view of assumptions in subsection 1.1, γ_0^ε is a smooth hypersurface without boundary and properties analogous to (1.8) and (1.9) hold true with obvious changes. In particular, thanks to the compactness of Γ_0, (1.8) is transferred into

$$\nabla u_0(x) \cdot n^\varepsilon(x) \geq \vartheta > 0 \ \text{ for any } x \in \gamma_0^\varepsilon, \tag{4.6}$$

for all $\varepsilon > 0$ small enough. We can now state our generation of interface result.

Theorem 4.1 (Emergence of $\mathcal{O}(\varepsilon)$ layers around γ_0^ε)**.** *Let the nonlinearity f and the initial data u_0 satisfy the assumptions of subsection 1.1. Let the mild noise be of (MN1) or (MN2) type. Let $u^\varepsilon(x,t)$ be the solution of (1.1). Let $\eta \in (0, \eta_0 := \min(a - a_-, a_+ - a))$ be arbitrary.*

Then there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,

(i) for all $x \in \Omega$,

$$a_- - \eta \leq u^\varepsilon(x, \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|) \leq a_+ + \eta, \tag{4.7}$$

(ii) for all $x \in \Omega$ such that $|u_0(x) - a^\varepsilon| \geq M_0\varepsilon$, we have that

$$\text{if } u_0(x) \geq a^\varepsilon + M_0\varepsilon \text{ then } u^\varepsilon(x, \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|) \geq a_+ - \eta, \tag{4.8}$$

$$\text{if } u_0(x) \leq a^\varepsilon - M_0\varepsilon \text{ then } u^\varepsilon(x, \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|) \leq a_- + \eta. \tag{4.9}$$

Proof. In view of the crucial observation (4.1) and definition (4.2), the Allen–Cahn equation (1.1) is recast (for small enough times)

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2}\left(f^\varepsilon(u) - \varepsilon g^\varepsilon(t)\right), \quad 0 < t \leq \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|, \quad x \in \Omega,$$

where the perturbation term

$$g^\varepsilon(t) := -\xi^\varepsilon(t) + \xi^\varepsilon(0),$$

satisfies $\|g^\varepsilon\|_{L^\infty(0,\mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|)} = o(\varepsilon)$, as $\varepsilon \to 0$. Moreover, using (1.12) under the noise assumption (MN1), and (1.16) under the noise assumption (MN2), we get that, in any case,

$$\|\dot g^\varepsilon\|_{L^\infty(0,\mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|)} = \mathcal{O}\left(\varepsilon^{-1}\right), \quad \text{as } \varepsilon \to 0.$$

After writing the problem in such a form and as far as the perturbation term is concerned, we are in the footsteps of the Allen–Cahn equation (P^ε) studied in [2], since the above estimate corresponds to assumption (1.3) in [2].

On the other hand, we need to handle the following minor change: f in [2] is replaced by f^ε in our setting. This difference implies that a in [2] is replaced by a^ε and is the reason why the generation occurs around γ_0^ε (and not around Γ_0). Nevertheless, it is completely transparent that f^ε is still of the bistable type *uniformly with respect to small* $\varepsilon > 0$ (this property is used in order to derive certain estimates). More precisely, (4.3) and (4.4) correspond to assumption (1.1) in [2], uniformly with respect to small $\varepsilon > 0$. Similarly, the non degeneracy assumption (4.6), when crossing the initial interface γ_0^ε, is uniform with respect to small $\varepsilon > 0$ and corresponds to assumption (1.10) in [2].

We can then construct the analogous of the sub- and supersolutions of [2, Section 4], namely

$$w^\pm(x,t) = Y^\varepsilon\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C(e^{\tilde\mu_\varepsilon \frac{t}{\varepsilon^2}} - 1); \pm\varepsilon\right),$$

where $C > 0$ is a large constant, $\tilde{\mu}_\varepsilon$ is a very small perturbation of μ_ε, and $Y^\varepsilon(\tau, \xi; \delta)$ is the solution of the Cauchy problem

$$\begin{cases} Y^\varepsilon_\tau(\tau, \xi; \delta) = f^\varepsilon(Y^\varepsilon(\tau, \xi; \delta)) + \delta & \text{for } \tau > 0 \\ Y^\varepsilon(0, \xi; \delta) = \xi. \end{cases}$$

Notice that, in this very early stage of emergence of the layers, the above sub- and supersolutions are obtained by considering only the nonlinear reaction term, that is diffusion is neglected.

For the aforementioned reasons, we can reproduce the lengthy arguments of [2, Section 4] to prove Theorem 4.1, which is nothing else that the analogous of [2, Theorem 3.1] taking into account the change $f \leftarrow f^\varepsilon$. □

5. Propagation of $\mathcal{O}(\varepsilon)$ layers

In this section, we construct a pair of sub- and supersolutions whose role is to capture in an $\mathcal{O}(\varepsilon)$ sandwich the layers of the solution $u^\varepsilon(x, t)$, while they are propagating. In order to proceed to the aforementioned construction, we need to define first properly some traveling waves and a signed distance function used in the definition of this pair.

5.1. **Some traveling waves.** For $\delta_0 > 0$ small enough and any $|\delta| \leq \delta_0$, the function $u \mapsto f(u) + \delta$ is still of bistable type, and we denote by

$$a_-(\delta) = a_- + \mathcal{O}(\delta) < a(\delta) = a + \mathcal{O}(\delta) < a_+(\delta) = a_+ + \mathcal{O}(\delta),$$

its three zeros.

Let $c(\delta)$, $m(z; \delta)$ be the speed and the profile of the unique traveling wave associated with the one dimensional problem

$$\partial_t v = v_{zz} + f(v) + \delta, \ \ t > 0, \ z \in \mathbb{R}.$$

In other words, we have

$$\begin{aligned} & m_{zz}(z; \delta) + c(\delta) m_z(z; \delta) + f(m(z; \delta)) + \delta = 0, \quad z \in \mathbb{R}, \\ & m(-\infty; \delta) = a_-(\delta), \quad m(0; \delta) = a(\delta), \quad m(+\infty; \delta) = a_+(\delta). \end{aligned} \tag{5.1}$$

Notice in particular that the assumption of balanced nonlinearity (1.5) implies $c(0) = 0$. Moreover, the following estimates are well-known (see in [17], [25] or [43]).

Lemma 5.1 (Estimates on traveling waves)**.** *There exist constants $\delta_0 > 0$, $C > 0$, $\lambda > 0$ such that, for all $|\delta| \leq \delta_0$,*

$$\begin{aligned} & 0 < a_+(\delta) - m(z; \delta) \leq C e^{-\lambda |z|}, \quad z \geq 0, \\ & 0 < m(z; \delta) - a_-(\delta) \leq C e^{-\lambda |z|}, \quad z \leq 0, \\ & 0 < m_z(z; \delta) \leq C e^{-\lambda |z|}, \quad z \in \mathbb{R}, \\ & |m_{zz}(z; \delta)| \leq C e^{-\lambda |z|}, \quad z \in \mathbb{R}, \\ & |m_\delta(z; \delta)| \leq C, \quad z \in \mathbb{R}, \end{aligned} \tag{5.2}$$

and

$$\partial_\delta c(0) = -c_0 := -\frac{a_+ - a_-}{\displaystyle\int_{a_-}^{a_+} \sqrt{2F(u)}\, du} < 0, \quad F(u) := \int_u^{a_+} f(z)\, dz. \tag{5.3}$$

5.2. **Signed distance functions.** We recall that the family of hypersurfaces (γ_t^ε) follows the law (3.1) with initial data γ_0^ε defined in (4.5). If the noise is of the (MN1) type then it follows from (2.2) and (2.3) that, up to reducing ε_0 if necessary,

$$\mathcal{K} := \sup_{0<\varepsilon<\varepsilon_0} \sup_{0\leq t\leq \sigma_N^\varepsilon} \sup_{y\in\gamma_t^\varepsilon} \sup_{1\leq i\leq n-1} |\kappa_i^\varepsilon(y,t)| < \infty,$$

with $\kappa_i^\varepsilon(y,t)$ the i-th principal curvature of γ_t^ε at point y. On the other hand, if the noise is of the (MN2) type then it follows from (2.4) that, up to reducing ε_0 if necessary,

$$\mathcal{K} := \sup_{0<\varepsilon<\varepsilon_0} \sup_{0\leq t\leq T} \sup_{y\in\gamma_t^\varepsilon} \sup_{1\leq i\leq n-1} |\kappa_i^\varepsilon(y,t)| < \infty.$$

In the sequel we unify the notations by letting $\mathcal{T} = \sigma_N^\varepsilon$, $\mathcal{T} = T$ if the noise is of the (MN1) type, (MN2) type respectively.

Let Ω_t^ε denote the region enclosed by γ_t^ε. We then define the associated signed distance function by

$$\tilde{d}^\varepsilon(x,t) := \begin{cases} -\mathrm{dist}(x,\gamma_t^\varepsilon) & \text{for } x\in\Omega_t^\varepsilon,\\ +\mathrm{dist}(x,\gamma_t^\varepsilon) & \text{for } x\in\Omega\setminus\overline{\Omega_t^\varepsilon}. \end{cases} \tag{5.4}$$

For $d_0>0$, choose an increasing function $\varphi\in C^\infty(\mathbb{R})$ satisfying

$$\varphi(s) = \begin{cases} -2d_0 & \text{if } \ s\leq -2d_0,\\ s & \text{if } \ |s|\leq d_0,\\ 2d_0 & \text{if } \ s\geq 2d_0. \end{cases}$$

If d_0 is sufficiently small, then, for any $0<\varepsilon<\varepsilon_0$,

$$d^\varepsilon(x,t) := \varphi(\tilde{d}^\varepsilon(x,t))$$

is smooth in $\Omega\times(0,\mathcal{T})$, satisfies $d^\varepsilon(x,t)=0$ for $x\in\gamma_t^\varepsilon$,

$$|\nabla d^\varepsilon(x,t)| = 1 \quad \text{ in } \{(x,t):\,|d^\varepsilon(x,t)|<d_0\}. \tag{5.5}$$

Also, since the inward normal velocity V and the mean curvature κ are equal to $\partial_t d^\varepsilon$ and $\frac{\Delta d^\varepsilon}{n-1}$, equation (3.1) is recast as

$$\partial_t d^\varepsilon(y,t) = \Delta d^\varepsilon(y,t) - \frac{c(\varepsilon\xi^\varepsilon(t))}{\varepsilon} \quad \text{ on } \{(y,t):\, y\in\gamma_t^\varepsilon\}. \tag{5.6}$$

5.3. **An $\mathcal{O}(\varepsilon)$-sandwich of the layers.** Equipped with the above material, we are now in the position to construct sub-and supersolutions for equation (1.1) in the form

$$u_\varepsilon^\pm(x,t) := m\left(\frac{d^\varepsilon(x,t)\pm\varepsilon p(t)}{\varepsilon};\varepsilon\xi^\varepsilon(t)\right) \pm q(t), \tag{5.7}$$

where

$$p(t) := -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \quad q(t) := \sigma\big(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\big), \tag{5.8}$$

where β, σ, K and L are positive constants to be chosen. Notice that $q=\sigma\varepsilon^2 p_t$. Notice also that, initially, the vertical shift $p(0)$ is $\mathcal{O}(1)$ but, as soon as $t>0$, $p(t)$ becomes $\mathcal{O}(\varepsilon^2)$. Furthermore, it is clear from the definition of $u_\varepsilon^\pm$ that, as soon as $t>0$, $\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = a_-$, respectively a_+, if $x\in\Omega_t^\varepsilon$, respectively $x\in\Omega\setminus\overline{\Omega_t^\varepsilon}$.

Proposition 5.2 (Sub- and supersolutions for the propagation)**.** *Choose $\beta > 0$ and $\sigma > 0$ appropriately. Then for any $K > 1$, there exist constants $\varepsilon_0 > 0$ and $L > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions $(u_\varepsilon^-, u_\varepsilon^+)$ are a pair of sub- and super-solutions for equation* (1.1) *in the domain $\Omega \times (0, \mathcal{T})$, that is*

$$\mathcal{L}u_\varepsilon^+ := \partial_t u_\varepsilon^+ - \Delta u_\varepsilon^+ - \frac{1}{\varepsilon^2} f(u_\varepsilon^+) - \frac{1}{\varepsilon}\xi^\varepsilon(t) \geq 0, \quad \mathcal{L}u_\varepsilon^- \leq 0,$$

in $\Omega \times (0, \mathcal{T})$.

Proof. We only give the proof of the inequality for u_ε^+, since the one for u_ε^- follows the same argument. In the sequel, m and its derivatives are evaluated at

$$(z^*; \delta^*) := \left(\frac{d^\varepsilon(x,t) + \varepsilon p(t)}{\varepsilon}; \varepsilon \xi^\varepsilon(t) \right)$$

which belongs to $\mathbb{R} \times (-\delta_0, \delta_0)$ if $\varepsilon > 0$ is small enough. Straightforward computations combined with

$$f(m+q) = f(m) + q f'(m) + \frac{1}{2} q^2 f''(\theta), \qquad \text{for some } m < \theta = \theta(x,t) < u_\varepsilon^+,$$

and equation (5.1) yield $\mathcal{L}u_\varepsilon^+ = E_1 + E_2 + E_3 + E_4$, with

$$\begin{aligned} E_1 &= -\frac{1}{\varepsilon^2} q \left(f'(m) + \frac{1}{2} q f''(\theta) \right) + m_z p_t + q_t \\ E_2 &= (1 - |\nabla d^\varepsilon|^2) \frac{m_{zz}}{\varepsilon^2} \\ E_3 &= \left(\partial_t d^\varepsilon(x,t) - \Delta d^\varepsilon(x,t) + \frac{c(\varepsilon \xi^\varepsilon(t))}{\varepsilon} \right) \frac{m_z}{\varepsilon} \\ E_4 &= \varepsilon \dot{\xi}^\varepsilon(t) m_\delta. \end{aligned}$$

Let us first present some useful inequalities. By assumption (1.4), there are $b > 0$, $\rho > 0$ such that

$$f'(m(z;\delta)) \leq -\rho \qquad \text{if} \quad m(z;\delta) \in [a_- - b, a_- + b] \cup [a_+ - b, a_+ + b]. \tag{5.9}$$

On the other hand, since the region $\{(z;\delta) \in \mathbb{R} \times (-\delta_0, \delta_0) : m(z;\delta) \in [a_- + b, \, a_+ - b] \}$ is compact, there is $a_1 > 0$ such that

$$m_z(z;\delta) \geq a_1 \qquad \text{if} \quad m(z;\delta) \in [a_- + b, \, a_+ - b]. \tag{5.10}$$

We now select

$$\beta = \frac{\rho}{4}, \quad 0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2), \tag{5.11}$$

where

$$\sigma_0 := \frac{a_1}{\rho + \|f'\|_{L^\infty(a_- - 1, a_+ + 1)}}, \quad \sigma_1 := \frac{1}{2(\beta + 1)}, \quad \sigma_2 := \frac{4\beta}{\|f''\|_{L^\infty(a_- - 1, a_+ + 1)}(\beta + 1)}.$$

Combining (5.9), (5.10) and $0 < \sigma \leq \sigma_0$, we obtain

$$m_z(z;\delta) - \sigma f'(m(z;\delta)) \geq \sigma\rho, \quad \forall (z;\delta) \in \mathbb{R} \times (-\delta_0, \delta_0). \tag{5.12}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $\mathcal{L}u_\varepsilon^+ \geq 0$ provided that the constants ε_0 and L are appropriately chosen. We go on under the following assumption (to be checked at the end)

$$\varepsilon_0^2 L e^{L\mathcal{T}} \leq 1. \tag{5.13}$$

Then, given any $\varepsilon \in (0, \varepsilon_0)$, we have, since $\sigma \leq \sigma_1$, $0 \leq q(t) \leq \frac{1}{2}$, that

$$a_- - 1 \leq u_\varepsilon^\pm(x,t) \leq a_+ + 1. \tag{5.14}$$

Using the expressions for p and q, the "favorable" term E_1 is recast as

$$E_1 = \frac{\beta}{\varepsilon^2}\, e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2 \sigma L),$$

where

$$I = m_z(z^*;\delta^*) - \sigma f'(m(z^*;\delta^*)) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (5.12), (5.14) and (5.13), we have $I \geq \sigma\rho - \frac{\sigma^2}{2}\|f''\|_{L^\infty(a_- -1, a_+ +1)}(\beta + 1)$. Since $0 < \sigma \leq \sigma_2$, we obtain $I \geq 2\sigma\beta$. Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt} \geq 2\sigma\beta L e^{Lt}.$$

Next, in view of (5.5), $E_2 = 0$ in the region $|d^\varepsilon(x,t)| \leq d_0$. Next we consider the region where $|d^\varepsilon(x,t)| \geq d_0$. We deduce from Lemma 5.1 that

$$|E_2| \leq \frac{C}{\varepsilon^2} e^{-\lambda|d^\varepsilon(x,t) + \varepsilon p(t)|/\varepsilon} \leq \frac{C}{\varepsilon^2} e^{-\lambda(d_0/\varepsilon - p(t))}.$$

We remark that $0 < K - 1 \leq p \leq e^{L\mathcal{T}} + K$. Consequently, if we assume (to be checked at the end)

$$e^{L\mathcal{T}} + K \leq \frac{d_0}{2\varepsilon_0}, \tag{5.15}$$

then $\dfrac{d_0}{\varepsilon} - p(t) \geq \dfrac{d_0}{2\varepsilon}$, so that $|E_2| \leq \frac{C}{\varepsilon^2} e^{-\lambda d_0/(2\varepsilon)} = \mathcal{O}(1)$, as $\varepsilon \to 0$.

Let us now turn to the term E_3. In the region where $|d^\varepsilon(x,t)| \geq \min(d_0, \frac{1}{2\mathcal{K}}) > 0$ (away from the interface), argument similar as those for E_2 yield $|E_3| = \mathcal{O}(1)$ as $\varepsilon \to 0$ (thanks to the exponential decay of the wave). In the region where $|d^\varepsilon(x,t)| \leq \min(d_0, \frac{1}{2\mathcal{K}})$, let us pick a $y \in \gamma_t^\varepsilon$ such that $|d^\varepsilon(x,t)| = dist(x,y)$. In view of (5.6) and $\partial_t d^\varepsilon(x,t) = \partial_t d^\varepsilon(y,t)$ we get

$$E_3 = (\Delta d^\varepsilon(y,t) - \Delta d^\varepsilon(x,t))\, \frac{m_z}{\varepsilon}.$$

But it follows from [26, Lemma 14.17] that

$$\begin{aligned} |\Delta d^\varepsilon(y,t) - \Delta d^\varepsilon(x,t)| &= \left| \sum_{i=1}^{n-1} \kappa_i^\varepsilon(y,t) - \sum_{i=1}^{n-1} \frac{\kappa_i^\varepsilon(y,t)}{1 - d^\varepsilon(x,t)\kappa_i^\varepsilon(y,t)} \right| \\ &\leq |d^\varepsilon(x,t)| \sum_{i=1}^{n-1} \frac{(\kappa_i^\varepsilon)^2(y,t)}{|1 - d^\varepsilon(x,t)\kappa_i^\varepsilon(y,t)|} \\ &\leq 2|d^\varepsilon(x,t)| \sum_{i=1}^{n-1} (\kappa_i^\varepsilon)^2(y,t) \end{aligned}$$

since $|d^\varepsilon(x,t)| \leq \frac{1}{2\mathcal{K}}$, and $|\kappa_i^\varepsilon(y,t)| \leq \mathcal{K}$. As a result we have $|E_3| \leq 2(n-1)\mathcal{K}^2|d^\varepsilon(x,t)| =: C|d^\varepsilon(x,t)|$, so that

$$\begin{aligned}
|E_3| &\leq C\frac{|d^\varepsilon(x,t)|}{\varepsilon} m_z\left(\frac{d^\varepsilon(x,t)+\varepsilon p(t)}{\varepsilon}; \varepsilon\xi^\varepsilon(t)\right)\\
&\leq C \sup_{z\in\mathbb{R},|\delta|\leq\delta_0} |zm_z(z;\delta)| + C\varepsilon|p(t)| \sup_{z\in\mathbb{R},|\delta|\leq\delta_0} |m_z(z;\delta)|\\
&\leq C_3 + C_3'(e^{Lt}+K),
\end{aligned}$$

for some constants $C_3>0$, $C_3'>0$ and where we have used Lemma 5.1.

Last, it follows from (1.12), (1.16) and Lemma 5.1 that $|E_4| \to 0$, as $\varepsilon \to 0$, uniformly in $\Omega \times (0,\mathcal{T})$.

Putting the above estimates all together, we arrive at

$$\mathcal{L}u_\varepsilon^+ \geq (2\sigma\beta L - C_3')e^{Lt} - \mathcal{O}(1)$$

which is nonnegative, if $L>0$ is sufficiently large, and $\varepsilon_0>0$ sufficiently small to validate assumptions (5.13) and (5.15). The theorem is proved. □

6. Description of the $\mathcal{O}(\varepsilon)$ layers and their convergence

6.1. **Proof of Theorem 3.1.** Let $\eta \in (0,\eta_0)$ be given. Let us select $\beta>0$ and $\sigma>0$ that satisfy (5.11)—so that Proposition 5.2 is available—and $\beta\sigma \leq \eta/3$. By the emergence of the layers property, we are equipped with small $\varepsilon_0>0$ and a $M_0>0$ such that (4.7), (4.8), (4.9) hold with $\beta\sigma/2$ playing the role of η. On the other hand, in view of (4.6), there is $M_1>0$ such that we have the following correspondence

$$\begin{array}{llll}
\text{if} & d^\varepsilon(x,0) \geq \ M_1\varepsilon & \text{then} & u_0(x) \geq a^\varepsilon + M_0\varepsilon\\
\text{if} & d^\varepsilon(x,0) \leq -M_1\varepsilon & \text{then} & u_0(x) \leq a^\varepsilon - M_0\varepsilon,
\end{array} \tag{6.1}$$

where we recall that $d^\varepsilon(x,0)$ denotes the signed distance function associated with the hypersurface $\gamma_0^\varepsilon := \{x : u_0(x) = a^\varepsilon\}$. Now we define functions $H^+(x), H^-(x)$ by

$$H^+(x) = \begin{cases} a_+ + \sigma\beta/2 & \text{if } d^\varepsilon(x,0) \geq -M_1\varepsilon\\ a_- + \sigma\beta/2 & \text{if } d^\varepsilon(x,0) < -M_1\varepsilon,\end{cases}$$
$$H^-(x) = \begin{cases} a_+ - \sigma\beta/2 & \text{if } d^\varepsilon(x,0) \geq \ M_1\varepsilon\\ a_- - \sigma\beta/2 & \text{if } d^\varepsilon(x,0) < \ M_1\varepsilon.\end{cases}$$

Then from the above observations we see that, after a very short time $\mathcal{O}(\varepsilon^2|\ln\varepsilon|)$, we have an $\mathcal{O}(\varepsilon)$ sandwich of the layers, namely

$$H^-(x) \leq u^\varepsilon(x, \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|) \leq H^+(x) \qquad \text{for } x\in\Omega. \tag{6.2}$$

We now would like to use the sub and supersolutions (5.7) for the propagation described at Section 5. Observe that

$$u_\varepsilon^\pm(x,0) = m\left(\frac{d^\varepsilon(x,0)\pm K}{\varepsilon}; \varepsilon\xi^\varepsilon(0)\right) \pm \sigma(\beta+\varepsilon^2 L),$$

so that it follows from $\varepsilon\xi^\varepsilon(0) = \mathcal{O}(\varepsilon^{1-\gamma}) \to 0$ and Lemma 5.1 on traveling waves $m(z;\delta)$ that we can select $K >> M_1$ so that

$$u_\varepsilon^-(x,0) \leq H^-(x) \leq u^\varepsilon(x,\mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|) \leq H^+(x) \leq u_\varepsilon^+(x,0) \qquad \text{for } x\in\Omega.$$

Let us now choose $\varepsilon_0 > 0$ and $L > 0$ so that Proposition 5.2 applies. It therefore follows from the comparison principle that

$$u_\varepsilon^-(x,t) \leq u^\varepsilon(x,t+t^\varepsilon) \leq u_\varepsilon^+(x,t) \quad \text{for } x \in \Omega,\ 0 \leq t \leq \mathcal{T} - t^\varepsilon, \tag{6.3}$$

where $t^\varepsilon = \mu_\varepsilon^{-1}\varepsilon^2|\ln\varepsilon|$.

To conclude, in view of $\varepsilon\xi^\varepsilon(t) = \mathcal{O}(\varepsilon^{1-\gamma}) \to 0$ and Lemma 5.1 on traveling waves, we can select $\varepsilon_0 > 0$ small enough and $C > 0$ large enough so that, for all $\varepsilon \in (0, \varepsilon_0)$, all $0 \leq t \leq \mathcal{T} - t^\varepsilon$,

$$m(C - e^{L\mathcal{T}} - K; \varepsilon\xi^\varepsilon(t)) \geq a_+ - \frac{\eta}{2} \quad \text{and} \quad m(-C + e^{L\mathcal{T}} + K; \varepsilon\xi^\varepsilon(t)) \leq a_- + \frac{\eta}{2}. \tag{6.4}$$

Using inequalities (6.3), expressions (5.7) for $u_\varepsilon^\pm$, estimates (6.4) and $\sigma\beta \leq \eta/3$ we then see that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $0 \leq t \leq \mathcal{T} - t_\varepsilon$, we have

$$\begin{array}{llll} \text{if} & d^\varepsilon(x,t) \geq \ C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \geq a_+ - \eta \\ \text{if} & d^\varepsilon(x,t) \leq -C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \leq a_- + \eta, \end{array} \tag{6.5}$$

and $u^\varepsilon(x, t+t^\varepsilon) \in [a_- - \eta, a_+ + \eta]$, which completes the proof of Theorem 3.1. □

6.2. **Proof of Corollary 3.2.** Let us observe that the simplifying assumption $\xi^\varepsilon(0) = 0$ enables to get rid of the initial small drift which happens during the emergence of the layers. Precisely, in view of (4.1) and (4.5), γ_0^ε is nothing else than Γ_0. As a result, the approximated (deterministic) γ_t^ε involve perturbations of the speed but not of the initial data. This enables (see the end of subsection 2.1) to reproduce the arguments of [25] to derive Corollary 3.2 from our Theorem 3.1.

Notice that, if $\xi^\varepsilon(0) \neq 0$, then one needs to derive an analogous of (2.4) (available in the Weber's context) in the Funaki's context. We think that, following [25], this can be performed but this is beyond the scope of the present paper so we decided to avoid this situation.

6.3. **Proof of Corollary 3.3.** Combining Theorem 3.1 and estimate (2.4), we get Corollary 3.3 by reproducing the arguments of [43, Proof of Theorem 1.1].

7. Profile in the layers

Equipped with Theorem 3.1, we can now prove the validity of the first term of the asymptotic expansions *inside* the layers, namely Theorem 3.4. The proof consists in using the stretched variables, a blow-up argument and the result of [9], as performed in the deterministic case [3].

Before going further, we recall that a solution of an evolution equation is called *eternal* (or an *entire* solution) if it is defined for all positive and negative time. We follow this terminology to refer to a solution $w(z,\tau)$ of

$$w_\tau = \Delta_z w + f(w)\,, \quad z \in \mathbb{R}^n,\ \tau \in \mathbb{R}\,. \tag{7.1}$$

Stationary solutions and travelling waves are examples of eternal solutions. We quote below a result of Berestycki and Hamel [9] asserting that "any planar-like eternal solution is actually a planar wave". More precisely, the following holds (for $z \in \mathbb{R}^n$ we write $z = (z^{(1)}, \cdots, z^{(n)})$).

Lemma 7.1 ([9, Theorem 3.1])**.** *Let $w(z,\tau)$ be an eternal bounded solution of* (7.1) *satisfying*

$$\liminf_{z^{(n)}\to\infty} \inf_{z'\in\mathbb{R}^{n-1},\tau\in\mathbb{R}} w(z,\tau) > a\,, \quad \limsup_{z^{(n)}\to-\infty} \sup_{z'\in\mathbb{R}^{n-1},\tau\in\mathbb{R}} w(z,\tau) < a\,, \tag{7.2}$$

where $z' := (z^{(1)}, \cdots, z^{(n-1)})$. Then there exists a constant $z^* \in \mathbb{R}$ such that

$$w(z,\tau) = U_0(z^{(n)} - z^*)\,, \quad z \in \mathbb{R}^n\,,\ \tau \in \mathbb{R}\,.$$

7.1. **Proof of (ii) in Theorem 3.4.** Let $\rho > 1$ and $0 < T' < T$ be given. Assume by contradiction that (3.10) does not hold. Then there is $\eta > 0$ and sequences $\varepsilon_k \downarrow 0$, $t_k \in [\rho t^{\varepsilon_k}, T']$, $x_k \in \bar{\Omega}$ $(k = 1, 2, ...)$ such that

$$\left| u^{\varepsilon_k}(x_k, t_k) - U_0\left(\frac{\overline{d^{\varepsilon_k}}(x_k, t_k)}{\varepsilon_k}\right)\right| \geq 2\eta \,. \tag{7.3}$$

In view of (3.3), (3.9) and $U_0(\pm\infty) = a_\pm$, for (7.3) to hold it is necessary to have

$$d^\varepsilon(x_k, t_k) = \mathcal{O}(\varepsilon_k)\,, \qquad \text{as } k \to \infty\,. \tag{7.4}$$

Recall that $d^\varepsilon(\cdot, t)$ denotes the signed distance function to γ_t^ε as defined in subsection 5.2, whereas $\overline{d^\varepsilon}(\cdot, t)$ denotes that to Γ_t^ε defined in (3.8).

If $u^{\varepsilon_k}(x_k, t_k) = a$, then this would mean that $x_k \in \Gamma_{t_k}^{\varepsilon_k}$, in which case the left-hand side of (7.3) would be 0 (since $U_0(0) = a$), which is impossible. Hence $u^{\varepsilon_k}(x_k, t_k) \neq a$. By extracting a subsequence if necessary, we may assume without loss of generality that $u^{\varepsilon_k}(x_k, t_k) - a$ has a constant sign for $k = 0, 1, 2, \ldots$, say

$$u^{\varepsilon_k}(x_k, t_k) > a \quad (k = 0, 1, 2, \ldots), \tag{7.5}$$

which then implies that $\overline{d^{\varepsilon_k}}(x_k, t_k) > 0$ $(k = 0, 1, 2, \ldots)$. Since the mean curvature of γ_t^ε is uniformly bounded for $0 \leq t \leq T'$, $0 < \varepsilon << 1$, there is a small $\delta > 0$ such that each x in a δ-tubular neighborhood of γ_t^ε has a unique orthogonal projection on γ_t^ε. Since the sequence (x_k) remains very close to $\gamma_{t_k}^{\varepsilon_k}$ by (7.4), each x_k (with sufficiently large k) has a unique orthogonal projection $p_k = p^{\varepsilon_k}(x_k, t_k) \in \gamma_{t_k}^{\varepsilon_k}$. Let y_k be a point on $\Gamma_{t_k}^{\varepsilon_k}$ that has the smallest distance from x_k. If such a point is not unique, we choose one such point arbitrarily. Then we have

$$u^{\varepsilon_k}(y_k, t_k) = a \quad (k = 0, 1, 2, \ldots), \tag{7.6}$$

$$\overline{d^{\varepsilon_k}}(x_k, t_k) = \|x_k - y_k\|\,, \tag{7.7}$$

$$u^{\varepsilon_k}(x, t_k) > a \qquad \text{if } \ \|x - x_k\| < \|y_k - x_k\|\,. \tag{7.8}$$

$$x_k - p_k \perp \gamma_{t_k}^{\varepsilon_k} \quad \text{at } \ p_k \in \gamma_{t_k}^{\varepsilon_k}\,,$$

Furthermore, (7.4) and (3.9) imply

$$\|x_k - p_k\| = \mathcal{O}(\varepsilon_k)\,, \quad \|y_k - p_k\| = \mathcal{O}(\varepsilon_k) \quad (k = 0, 1, 2, \ldots). \tag{7.9}$$

We now rescale the solution u^ε around (p_k, t_k) and define

$$w^k(z, \tau) := u^{\varepsilon_k}(p_k + \varepsilon_k \mathcal{R}_k z, t_k + \varepsilon_k^2 \tau)\,, \tag{7.10}$$

where $\mathcal{R}_k$ is a matrix in $SO(n, \mathbb{R})$ that rotates the $z^{(n)}$ axis onto the normal at $p_k \in \gamma_{t_k}^{\varepsilon_k}$, that is,

$$\mathcal{R}_k : (0, \ldots, 0, 1)^T \mapsto n^{\varepsilon_k}(p_k, t_k)\,,$$

where $(\ )^T$ denotes a transposed vector and $n^\varepsilon(p, t)$ the outward normal unit vector at $p \in \gamma_t^\varepsilon$. Since γ_t^ε (hence the points p_k) is uniformly separated from $\partial\Omega$ by some positive distance, there exists $c > 0$ such that w^k is defined (at least) on the box

$$B^k := \left\{ (z, \tau) \in \mathbb{R}^n \times \mathbb{R} \,:\, \|z\| \leq \frac{c}{\varepsilon_k}, \quad -(\rho - 1)\mu_{\varepsilon_k}^{-1} |\ln \varepsilon_k| \leq \tau \leq \frac{T - T'}{\varepsilon_k^2} \right\}\,,$$

where we recall that $\mu_\varepsilon \to \mu = f'(a) > 0$ as $\varepsilon \to 0$. Since u^ε satisfies the equation in (1.1), we see that w^k satisfies

$$w^k_\tau = \Delta_z w^k + f(w^k) + \varepsilon_k \xi^{\varepsilon_k}(t_k + \varepsilon_k^2 \tau) \quad \text{in } B^k \,. \tag{7.11}$$

Moreover, if $(z,\tau) \in B^k$ then $t^{\varepsilon_k} \le t_k + \varepsilon_k^2 \tau \le T$. Therefore (3.3) implies

$$\begin{cases} d^{\varepsilon_k}(p_k + \varepsilon_k \mathcal{R}_k z, t_k + \varepsilon_k^2 \tau) \le -C\varepsilon_k & \Rightarrow \quad w^k(z,\tau) \le a_- + \eta\,, \\ d^{\varepsilon_k}(p_k + \varepsilon_k \mathcal{R}_k z, t_k + \varepsilon_k^2 \tau) \ge C\varepsilon_k & \Rightarrow \quad w^k(z,\tau) \ge a_+ - \eta\,, \end{cases} \tag{7.12}$$

as long as $(z,\tau) \in B^k$. Now we recall that the rotation by $\mathcal{R}_k$ of the $z^{(n)}$ axis is normal to $\gamma^{\varepsilon_k}_{t_k}$ at p_k, and that the mean curvature of γ^ε_t is uniformly bounded for $0 \le t \le T'$, $0 < \varepsilon << 1$. Also the normal speed of γ^ε_t, given by $V = (n-1)\kappa - \frac{c(\varepsilon \xi^\varepsilon_t)}{\varepsilon}$, is $\mathcal{O}(\varepsilon^{-\gamma'})$ for some $0 < \gamma' < \frac{1}{3}$ in view of $c(\delta) = -c_0\delta + \mathcal{O}(\delta^2)$ as $\delta \to 0$, and (1.12) (if (MN1) noise) or Proposition 1.1 (if (MN2) noise). As a result $d^\varepsilon(x,t)$ satisfies

$$|d^\varepsilon(x,t) - d^\varepsilon(x,t')| \le \frac{\tilde{C}}{\varepsilon^{\gamma'}}|t - t'|, \quad 0 \le t, t' \le T', 0 < \varepsilon << 1,$$

for some $\tilde{C} > 0$. From these observations and (7.12), we see that there exists a constant $K > 0$, which is independent of k, such that

$$z^{(n)} \le -K \Rightarrow w^k(z,\tau) \le a_- + \eta\,, \quad z^{(n)} \ge K \Rightarrow w^k(z,\tau) \ge a_+ - \eta\,, \tag{7.13}$$

for all $(z,\tau) \in B^k$ with $\|z\| \le \sqrt{1/\varepsilon_k}$ and $|\tau| \le 1/({\varepsilon_k}^{1-\gamma'})$.

Now, since w^k solves (7.11), the uniform (w.r.t. $k \ge 0$) boundedness of w^k and standard parabolic estimates, along with the derivative bounds on $\tau \mapsto \varepsilon_k \xi^{\varepsilon_k}(t_k + \varepsilon_k^2 \tau)$ (see (1.12) or Proposition 1.1), imply that w^k is uniformly bounded in $C^{2+\gamma, 1+\frac{\gamma}{2}}_{loc}(B^1)$. We can therefore extract from (w^k) a subsequence that converges to some w in $C^{2,1}_{loc}(B^1)$. By repeating this on all B^k, we can find a subsequence of (w^k) that converges to some w in $C^{2,1}_{loc}(\mathbb{R}^n \times \mathbb{R})$ (note that $\cup_{k \ge 0} B^k = \mathbb{R}^n \times \mathbb{R}$). Passing to the limit in (7.11) yields

$$w_\tau = \Delta_z w + f(w) \quad \text{on } \mathbb{R}^n \times \mathbb{R}\,.$$

Hence we have constructed an eternal solution $w(z,\tau)$ which—in view of (7.13)—satisfies (7.2). Lemma 7.1 then implies that

$$w(z,\tau) = U_0(z^{(n)} - z^*) \tag{7.14}$$

for some $z^* \in \mathbb{R}$.

Now we define sequences of points (z_k), $(\tilde{z}_k)$ by

$$z_k := \frac{1}{\varepsilon_k} \mathcal{R}_k^{-1}(x_k - p_k)\,, \quad \tilde{z}_k := \frac{1}{\varepsilon_k} \mathcal{R}_k^{-1}(y_k - p_k)\,.$$

By (7.9), these sequences are bounded, so we may assume without loss of generality that they converge:

$$z_k \to z_\infty\,, \quad \tilde{z}_k \to \tilde{z}_\infty\,, \quad \text{as} \quad k \to \infty\,.$$

By the definition of the z coordinates, z_∞ must lie on the $z^{(n)}$ axis, that is,

$$z_\infty = (0, \ldots, 0, z^{(n)}_\infty)^T\,.$$

It follows from (7.6) and (7.8) that

$$w(\tilde{z}_\infty, 0) = a\,, \qquad w(z, 0) \ge a \;\text{ if }\; \|z - z_\infty\| \le \|\tilde{z}_\infty - z_\infty\|\,. \tag{7.15}$$

Note that by (7.14), the level set $w(z,0)=a$ coincides with the hyperplane $z^{(n)}=z^*$, and recall that ${U_0}'>0$. Therefore, in view of (7.14) and (7.15), we have either $\tilde z_\infty = z_\infty$, or that the ball of radius $\|\tilde z_\infty - z_\infty\|$ centered at z_∞ is tangential to the hyperplane $z^{(n)}=z^*$ at $\tilde z_\infty$. This implies that $\tilde z_\infty$, as well as z_∞, must also lie on the $z^{(n)}$ axis. Therefore

$$\tilde z_\infty = (0,\dots,0,z^*)^T\,,$$

and the inequality $w(z_\infty,0)\geq a$ implies that $z_\infty^{(n)}\geq z^*$. On the other hand equality (7.7) implies $\overline{d^{\varepsilon_k}}(x_k,t_k)/\varepsilon_k = \|x_k-y_k\|/\varepsilon_k = \|z_k-\tilde z_k\| \to \|z_\infty - \tilde z_\infty\| = z_\infty^{(n)}-z^*$. The assumption (7.3) then yields

$$\begin{aligned} 0 &= \left|w(z_\infty,0)-U_0(z_\infty^{(n)}-z^*)\right| \\ &= \left|\lim_{k\to\infty} u^{\varepsilon_k}(x_k,t_k) - U_0\Big(\lim_{k\to\infty}\frac{\overline{d^{\varepsilon_k}}(x_k,t_k)}{\varepsilon_k}\Big)\right| \\ &\geq 2\eta\,. \end{aligned}$$

This contradiction proves statement (ii) of Theorem 3.1. □

7.2. **Proof of (i) in Theorem 3.4.** The proof of (i) below uses an argument similar to the proof of Corollary 4.8 in [36]. Fix $\rho>1$ and $0<T'<T$. For a given $\eta\in(0,\min(a-a_-,a_+-a))$ define $\varepsilon_0>0$ and $C>0$ as in Theorem 3.1. Then we claim that

$$\liminf_{\varepsilon\to 0}\ \inf_{x\in\mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon),\,\rho t^\varepsilon\leq t\leq T'} \nabla u^\varepsilon(x,t)\cdot n^\varepsilon(p(x,t),t)>0\,, \tag{7.16}$$

where $n^\varepsilon(p,t)$ denotes the outward unit normal vector at $p\in\gamma_t^\varepsilon$.

Indeed, assume by contradiction that there exist sequences $\varepsilon_k\downarrow 0$, $t_k\in[\rho t^{\varepsilon_k},T']$, $x_k\in\mathcal{N}_{C\varepsilon_k}(\gamma_{t_k}^{\varepsilon_k})$ $(k=1,2,...)$ such that

$$\nabla u^{\varepsilon_k}(x_k,t_k)\cdot n^{\varepsilon_k}(p_k,t_k)\leq 0\,,$$

where $p_k=p(x_k,t_k)$. By rescaling around (p_k,t_k) and using arguments similar to those in the proof of (ii), one can find a point z_∞ with $|z_\infty^{(n)}|\leq C$ such that

$${U_0}'(z_\infty^{(n)})\leq 0\,,$$

which contradicts to the fact that ${U_0}'>0$ and establishes (7.16). Since, in view of Theorem 3.1, $\Gamma_t^\varepsilon\subset\mathcal{N}_{C\varepsilon}(\gamma_t^\varepsilon)$, the estimate (7.16) implies that $\nabla u^\varepsilon(x,t)\neq 0$ for all $x\in\Gamma_t^\varepsilon$; hence by the implicit function theorem, Γ_t^ε is a smooth hypersurface in a neighborhood of any point on it. The fact that Γ_t^ε can be expressed as a graph over γ_t^ε also follows from (7.16). This proves the statement (i) of Theorem 3.1. □

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