# On a stochastic partial differential equation with non-local diffusion.

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**Abstract.** In this paper, we prove existence, uniqueness and regularity for a class of stochastic partial differential equations with a fractional Laplacian driven by a space-time white noise in dimension one. The equation we consider may also include a reaction term.

**Keywords:** Fractional derivative operator, stochastic partial differential equation, space–time white noise, nonlocal diffusion, Fourier transform

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## 1. Introduction and general framework

In recent years, fractional calculus has received a great deal of attention. Equations involving fractional derivatives and fractional Laplacians have been studied by various authors (see, e.g. Podlubny [15] and references therein). In probability theory, fractional calculus has been extensively used in the study of fractional Brownian motions. In this work we consider a stochastic partial differential equations where the standard Laplacian operator is replaced by a fractional one.

Let $\lambda > 0$. We consider the fractional Laplacian $\Delta_\lambda = -(-\frac{1}{4\pi^2}\Delta)^{\lambda/2} = -(-\frac{1}{4\pi^2}\frac{\partial^2}{\partial x^2})^{\lambda/2}$, the symmetric fractional derivative of order $\lambda$ on $\mathbb{R}$. This is a non-local operator defined via the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(\Delta_\lambda v)(\xi) = -|\xi|^\lambda \mathcal{F}(v)(\xi).$$

There are other integral representations, which will not be used hereafter, e.g. for $1 < \lambda < 2$,

$$\Delta_\lambda v(x) = K \int_{\mathbb{R}} \{v(x+y) - v(x) - \nabla v(x) \cdot y\} \frac{dy}{|y|^{1+\lambda}}, \tag{1.1}$$

for some positive constant $K = K_\lambda$, see [7], which identifies it as the infinitesimal generator for the symmetric $\lambda$-stable Lévy process (see,

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e.g., Itô [8], Stroock [16], Komatsu [9], Dawson and Gorostiza [5]).

Let $W = \{W(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ be a Brownian sheet on a complete probability space $(\Omega, \mathcal{G}, P)$. That is, $W$ is a zero-mean Gaussian random field with covariance function

$$E(W(t,x)W(s,y)) = \frac{1}{2}(s \wedge t)\left(|x| + |y| - |x-y|\right),$$

$x, y \in \mathbb{R}$, $s, t \in [0,T]$. Then, for each $t \in [0,T]$, we define a filtration

$$\mathcal{G}_t^0 = \sigma\left(W(s,x), s \in [0,t], x \in \mathbb{R}\right), \quad \mathcal{G}_t = \mathcal{G}_t^0 \vee \mathcal{N},$$

where $\mathcal{N}$ is the $\sigma$-field generated by sets with $P$-outer measure zero.

The family of $\sigma$-fields $\{\mathcal{G}_t, 0 \leq t \leq T\}$ constitutes a stochastic basis on the probability space $(\Omega, \mathcal{G}, P)$. Let $\mathcal{P}$ the corresponding predictable $\sigma$-field on $\Omega \times [0,T] \times \mathbb{R}$. The stochastic integral with respect to the Brownian sheet is explained in Cairoli et al. [3] or Walsh [17].

We focus on the following parabolic stochastic partial differential equation, driven by space–time white noise in one space dimension on $[0,T] \times \mathbb{R}$

$$\textbf{(E)} \quad \frac{\partial u}{\partial t}(t,x) = \Delta_\lambda u(t,x) + b\left(t,x,u\left(t,x\right)\right) + \sigma\left(t,x,u\left(t,x\right)\right)\dot{W}(t,x),$$

with initial condition $u(0,x) = u_0(x)$ $\mathcal{G}_0$-measurable and satisfying some conditions that will be specified later. The process $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}$ is the generalized (distribution) derivative of the Brownian sheet. The properties of $\dot{W}$ are described in Walsh [17].

In principle one can think of a wide variety of random forcing terms. White noise in time and space is very often a candidate. Main motivations behind this choice are central limit type theorems and the insufficient knowledge of the neglected effects or external disturbances.

Evolution problems involving fractional Laplace operator have long been extensively studied in mathematical and physical literature. In the latter, this type of models has been motivated by fractal (anomalous) diffusion related to the Lévy flights (see, e.g., Stroock [16], Bardos et al. [1], Dawson and Gorostiza [5], Metzler and Klafter [12], Mann and Woyczynski [11]). In fact, in various physical phenomena in statistical mechanics, the anomalous diffusive terms can be nonlocal and fractal, i.e. represented by a fractional power of the Laplacian.

Equation **(E)** is a generalization of the classical stochastic heat equation where $\lambda = 2$ (see, e.g., Walsh [17], Pardoux [14] and the references quoted therein). In those papers, the authors prove existence and uniqueness of the mild solution in the space interval $[0,1]$. The proof relies strongly on properties of the explicit Green kernel associated to the operator $\frac{\partial^2}{\partial x^2}$ in bounded space interval with Dirichlet boundary conditions. In the present paper, we consider the above class of equations in the whole line, instead of a bounded interval, for the space variable. The main properties of the semigroup generated by the fractional Laplacian can be derived by Fourier transform techniques.

Consider the fundamental solution $G_\lambda(t,x)$, associated to the equation **(E)** on $[0,T]\times\mathbb{R}$ i.e. the convolution kernel of the Lévy semigroup $\exp(t\Delta_\lambda)$ in $\mathbb{R}$.

Using Fourier transform, we easily see that $G_\lambda(t,x)$ is given by :

$$G_\lambda(t,x) = \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda})(x) = \int_{\mathbb{R}} e^{2i\pi x\xi} e^{-t|\xi|^\lambda} d\xi = \mathcal{F}(e^{-t|\cdot|^\lambda})(x).$$

For $\lambda \in ]0,2]$, the most important property of $G_\lambda$ is its *nonnegativity* (see Lévy [10] or Droniou et al. [6] for a quick proof ).

Throughout this work we consider solutions to the spde **(E)** in the mild sense, following Walsh [17], given by the following definition (which is formally equivalent to Duhamel's principle or the variation of parameters formula):

DEFINITION 1.1. *A stochastic process $u : \Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$, which is jointly measurable and $\mathcal{G}_t$-adapted, is said to be a (stochastically) mild solution to the stochastic equation* **(E)** *with initial condition $u_0$ if there exists a martingale measure $W$, defined on $\Omega$, such that a.s. for almost all $t\in[0,T], x\in\mathbb{R}$,*

$$\begin{aligned} u(t,x) \; &= \; G_\lambda(t,\cdot) * u_0(x) + \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y) b(s,y,u(s,y)) dyds \\ &\quad + \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\sigma(s,y,u(s,y)) W(dy,ds), \quad (1.2) \end{aligned}$$

*where the last integral is an Itô stochastic integral.*

We assume that the reaction term $b$ and the white-noise amplitude $\sigma$ are continuous functions on $[0,T]\times\mathbb{R}\times\mathbb{R}$ and satisfy the following growth and Lipschitz-Hölder conditions:

$(\mathbf{H_0})$
For all $T>0$, there exists a constant $C=C(T)>0$, such that for all $0\leq t\leq T, x\in \mathbb{R}$ and $u,v\in\mathbb{R}$,

$$|b(t,x,u)|+|\sigma(t,x,u)|\leq C(1+|u|), \tag{1.3}$$

$$|\sigma(t,x,u)-\sigma(t,x,v)|\leq C\,|\,u-v\,|, \tag{1.4}$$

$$|b(s,x,u)-b(t,y,v)|\leq C\left(|\,t-s\,|^{\frac{\lambda-1}{2\lambda}}+|\,x-y\,|^{\frac{\lambda-1}{2}}+|\,u-v\,|\right). \tag{1.5}$$

REMARK 1.1. *Since $1\leq\lambda\leq 2$, hypothesis (1.5) can be replaced by the more stringent one*

$$|b(s,x,u)-b(t,y,v)|\leq C\left(|\,t-s\,|+|\,x-y\,|+|\,u-v\,|\right). \tag{1.6}$$

We shall also need some hypotheses on the initial condition $u_0$ :

$(\mathbf{H_1.1})$ $\sup_{x\in\mathbb{R}} E(|u_0(x)|^p)<\infty$, $\forall p\in[1,+\infty[$.

$(\mathbf{H_1.2})$ $\exists\rho\in(0,1)$, $\forall z\in\mathbb{R}$, $\forall p\in[1,+\infty[$, $\exists C_p>0$

$$\sup_{y\in\mathbb{R}} E|u_0(y+z)-u_0(y)|^p\leq C_p|z|^{\rho p}.$$

Let us recall some well-known properties (see e.g. Droniou, Gallouët and Vovelle [6] pp. 501-502) of the Green kernel $G_\lambda(t,x)$ which will be used later on.

LEMMA 1.1. *Let $\lambda\in]0,2]$. The convolution kernel $G_\lambda$ satisfies the following properties:*
*(**a**) For any $t\in]0,+\infty[$ and $x\in\mathbb{R}$,*

$$G_\lambda(t,x)\geq 0 \quad\text{and}\quad \int_{\mathbb{R}} G_\lambda(t,x)dx=1.$$

*(**b**) (self similarity) For any $t\in\mathbb{R}_+$ and $x\in\mathbb{R}$*

$$G_\lambda(t,x)=t^{-\frac{1}{\lambda}}G_\lambda(1,t^{-\frac{1}{\lambda}}x),$$

*(**c**) $G_\lambda$ is $C^\infty$ on $]0,+\infty[\times\mathbb{R}$ and, for $m\geq 0$, there exists $C_m>0$ such that for any $t\in\mathbb{R}_+$ and $x\in\mathbb{R}$*

$$|\,\partial_x^m G_\lambda(t,x)\,|\leq \frac{1}{t^{(1+m)/\lambda}}\frac{C_m}{(1+t^{-2/\lambda}|x|^2)}.$$

*(**d**) For any* $(s,t) \in ]0,\infty[\times]0,\infty[$

$$G_\lambda(s,\cdot) * G_\lambda(t,\cdot) = G_\lambda(s+t,\cdot).$$

*(**e**)* $\int_0^T dt \int_{\mathbb{R}} dx\, G_\lambda(t,x)^\alpha < \infty$ *iff* $1/2 < \alpha < 1+\lambda$.

In this paper, in order to define the stochastic integral, we restrict ourselves to the case $\lambda \in ]1,2]$ : we must take $\lambda \le 2$ to have $G_\lambda$ positive and we have to take $\lambda > 1$ in order that $\int_0^T \int_{\mathbb{R}} G_\lambda(t,x)^2\, dtdx < \infty$, by lemma 1.1 (**e**).

Inessential constants will be denoted generically by $C$, even if they vary from line to line.

The paper is organized as follows. In section 2, we prove existence and uniqueness of the solution. In section 3 we prove Hölder continuity of the solution in space and time. A Gronwall-type improved inequality and an Hölder inequality frequently used in the paper are collected in the appendix.

## 2. Existence and Uniqueness of the solution

The main result of this section is the following:

THEOREM 2.1. *Let* $\lambda \in ]1,2]$. *Suppose that the hypothesis* $(\mathbf{H_0})$ *and* $(\mathbf{H_1.1})$ *hold. Then there exists a unique solution* $u(t,x)$ *to* **(E)** *such that: for any* $T>0$ *and* $p \ge 1$,

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}} E(|u(t,x)|^p) \le C_p < \infty. \tag{2.1}$$

**Proof.** The proof of the existence can be done by the usual Picard iteration procedure. That is, we define recursively

$$u^0(t,x) = \int_{\mathbb{R}} G_\lambda(t,x-y)u_0(y)\,dy,$$

$$\begin{aligned} u^{n+1}(t,x) &= u^0(t,x) + \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)\sigma(s,y,u^n(s,y))W(dy,ds) \\ &\quad + \int_0^t\int_{\mathbb{R}} G_\lambda(t-s,x-y)b(s,y,u^n(s,y))dyds, \end{aligned} \tag{2.2}$$

for all $n \geq 0$. We start by proving that given $t > 0$, $2 \leq p < \infty$,

$$\sup_{n\geq 0}\ \sup_{0\leq s\leq t}\ \sup_{x\in\mathbb{R}} E(|u^n(s,x)|^p) \leq C < +\infty, \tag{2.3}$$

where $C$ is a constant depending on $p, t$, the supremum norm of $u_0$ and the Lipschitz constants of $\sigma$ and $b$. Indeed,

$$E(\left|u^{n+1}(t,x)\right|^p) \leq C\left\{E(|u^0(t,x)|^p) + E(|A_n(t,x)|^p) + E(|B_n(t,x)|^p)\right\}, \tag{2.4}$$

where $A_n(t,x)$ is the second term in (2.2) and $B_n(t,x)$ is the third term in the right-hand side of the same equation.

By Jensen's inequality

$$|u^0(s,x)|^p \leq \left(\int_{\mathbb{R}} G_\lambda(s,x-y)|u_0(y)|^p dy\right).$$

Taking expectation and applying Fubini's theorem we obtain :

$$E(|u^0(s,x)|^p) \leq \sup_{y\in\mathbb{R}} E(|u_0(y)|^p) \int_{\mathbb{R}} dy\, G_\lambda(s,x-y) \leq \sup_{y\in\mathbb{R}} E(|u_0(y)|^p).$$

Now as $(\mathbf{H_1.1})$ holds, we get :

$$\sup_{0\leq s\leq T}\ \sup_{x\in\mathbb{R}} E(|u^0(s,x)|^p) \leq C < \infty, \tag{2.5}$$

for some positive constant $C$.

Burkholder's inequality yields, for any $p \geq 2$

$$E(|A_n(t,x)|)^p \leq CE\left(\int_0^t\int_{\mathbb{R}} G_\lambda^2(t-s,x-y)\sigma^2(s,y,u^n(s,y))\ dyds\right)^{p/2}.$$

Set

$$\nu_t = \int_0^t\int_{\mathbb{R}} G_\lambda^2(t-s,x-y)dyds,$$

Since $\lambda > 1$, $\nu_t \leq \int_0^T\int_{\mathbb{R}} G_\lambda^2(t-s,x-y)dyds < \infty$ by lemma 1.1($\mathbf{e}$). Consider

$$J(t-s) = \int_{\mathbb{R}} G_\lambda^2(t-s,y)dy. \tag{2.6}$$

Due to the scaling property (see lemma 1.1 ($\mathbf{b}$) ), one easily checks that

$$J(t-s) = C(t-s)^{-1/\lambda}. \tag{2.7}$$

Because of the linear growth of $\sigma$ (1.3), Hölder's inequality (4.1) applied with$f = \sigma^2(s,y,u^n(s,y))$, $h = G^2_\lambda(t-s,x-y)$ and $q = p/2$ implies

$$E(|A_n(t,x)|)^p \;\le\; C\,\nu_t^{\frac{p}{2}-1} E\left(\int_0^t\!\!\int_{\mathbb{R}} G^2_\lambda(t-s,x-y)\sigma^p(s,y,u^n(s,y))\,dyds\right)$$
$$\le\; C\;\left(\int_0^t (1+\sup_{y\in\mathbb{R}} E(|u^n(s,y)|^p)\left(\int_{\mathbb{R}} G^2_\lambda(t-s,x-y)dy\right)ds\right).$$
$$E(|A_n(t,x)|)^p \;\le\; C\int_0^t \left(1+\sup_{0\le s\le t}\sup_{y\in\mathbb{R}} E(|u^n(s,y)|^p)\right) J(t-s)\,ds. \tag{2.8}$$

The linear growth assumption on $b$ (1.3) and Hölder's inequality applied to integrals with respect to the measure $G_\lambda(t-s,x-y)dsdy$ implies

$$E(|B_n(t,x)|^p) \le C\int_0^t \left(1+\sup_{0\le s\le t}\sup_{y\in\mathbb{R}} E(|u^n(s,y)|^p)\right) ds. \tag{2.9}$$

Collecting (2.4),(2.5),(2.8),(2.9) and (2.7) we conclude that

$$\begin{aligned}
&E(\left|u^{n+1}(t,x)\right|^p)\\
&\le C\left(E(|u^0(t,x)|^p) + \int_0^t \left(1+\sup_{y\in\mathbb{R}} E(|u^n(s,y)|^p)\right)(J(t-s)+1)ds\right)\\
&\le C\left(1+\int_0^t (t-s)^{-\frac{1}{\lambda}}\sup_{0\le s\le t}\sup_{y\in\mathbb{R}} E(|u^n(s,y)|^p)\,ds\right).
\end{aligned}$$

Thus by lemma 4.2 (see appendix) we obtain (2.3).

In order to prove that $(u_n(t,x),\, n\ge 0)$ converges in $L^p$, let $n \ge 0$, $0\le t\le T$ and set

$$M_n(t) = \sup_{0\le s\le t}\sup_{x\in\mathbb{R}} E(|u^{n+1}(s,x) - u^n(s,x)|^p).$$

Using the Lipschitz property of $\sigma$ and $b$, a similar computation implies

$$M_n(t) \le C\int_0^t ds\, M_{n-1}(s)(J(t-s)+1).$$

Moreover, owing to (2.3) we have $\sup_{0\le t\le T} M_0(t) < \infty$. Therefore, by lemma 4.2 the sequences $(u_n(t,x),\, n\ge 0)$ converges in $L^p(\Omega,\mathcal{G},P)$, uniformly in $x\in\mathbb{R}$ and $0\le t\le T$, to a limit $u(t,x)$. It is easy to see that $u(t,x)$ satisfies (1.2), (2.1) which proves the existence of a solution. Following the same approach as in Walsh [17], we can prove that the process $(u(t,x),\, t \geq 0, x \in \mathbb{R})$ has a jointly measurable version which is continuous in $L^p$ and fulfills (1.2). Uniqueness of the solution is checked by standard arguments. $\square$

## 3. Hölder continuity of the solution

In this section we analyze the path regularity of $u(t,x)$. The next result extends and improves similar estimates known for the stochastic heat equation (corresponding to the case $\lambda = 2$).

THEOREM 3.1. *Let $\lambda \in ]1,2]$. Suppose that $(\mathbf{H_0})$, $(\mathbf{H_1.1})$ and $(\mathbf{H_1.2})$ are satisfied. Then, $\omega$-almost surely, the function $(t,x) \longmapsto u\,(t,x)\,(\omega)$ belongs to Hölder space $\mathcal{C}^{\alpha,\beta}\,([0,T] \times \mathbb{R})$ for $0 < \alpha < (\frac{\rho}{\lambda} \wedge \frac{\lambda-1}{2\lambda})$ and $0 < \beta < (\rho \wedge \frac{\lambda-1}{2})$.*

**Proof.** Fix $T > 0$, $0 < h < 1$ and $p \in ]1, 1/\rho[$. We show first that

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} E(\mid u(t+h,x) - u(t,x) \mid^p) \leq C\, h^{\alpha p}, \tag{3.1}$$

for any $0 < \alpha < (\frac{\rho}{\lambda} \wedge \frac{\lambda-1}{2\lambda})$.

Indeed, we have

$$E(\mid u(t+h,x) - u(t,x) \mid^p) \leq C \sum_{i=1}^{4} I_i(t,h,x), \tag{3.2}$$

where

$$\begin{aligned}
I_1(t,h,x) &= E\left|\int_{\mathbb{R}} (G_\lambda(t+h,x-y) - G_\lambda(t,x-y))u_0(y)dy\right|^p, \\
I_2(t,h,x) &= E\left(\left|\int_0^t \int_{\mathbb{R}} [G_\lambda(t+h-s,x-y) - G_\lambda(t-s,x-y)]\right.\right. \\
&\qquad\qquad \left.\left. \times\, \sigma(s,y,u(s,y))W(dy,ds)\right|^p\right), \\
I_3(t,h,x) &= E\left(\left|\int_t^{t+h}\!\!\int_{\mathbb{R}} G_\lambda(t+h-s,x-y)\sigma(s,y,u(s,y))W(dy,ds)\right|^p\right), \\
I_4(t,h,x) &= E\left(\left|\int_0^{t+h} ds \int_{\mathbb{R}} dy\, G_\lambda(t+h-s,x-y)b(s,y,u(s,y))\right.\right. \\
&\qquad \left.\left. - \int_0^t ds \int_{\mathbb{R}} dy\, G_\lambda(t-s,x-y)b(s,y,u(s,y))\right|^p\right).
\end{aligned}$$

Using the semigroup property of the convolution kernel $G_\lambda$,

$$G_\lambda(t+h,x-y) = \int_{\mathbb{R}} G_\lambda(t,x-y-z)\, G_\lambda(h,z)\, dz.$$

Hence

$$I_1(t,h,x) = E\left( \left| \int_{\mathbb{R}} G_\lambda(h,z) \left( \int_{\mathbb{R}} G_\lambda(t,x-y)\,(u_0(y-z)-u_0(y))\,dy \right) dz \right|^p \right).$$

With Hölder's inequality (4.1), the assumption ($\mathbf{H_1.2}$) and Fubini's theorem we obtain

$$\begin{aligned} I_1(t,h,x) \quad &\le \int_{\mathbb{R}} G_\lambda(h,z) \sup_{y\in\mathbb{R}} E|u_0(y-z)-u_0(y)|^p\,dz \\ &\le C \int_{\mathbb{R}} G_\lambda(h,z)\,|z|^{\rho p}\,dz. \end{aligned} \tag{3.3}$$

Now, due to the self-similarity property (see lemma 1.1 **b** )

$$\begin{aligned} \int_{\mathbb{R}} G_\lambda(h,z)\,|z|^{\rho p}\,dz &= \int_{\mathbb{R}} h^{-1/\lambda} G_\lambda(1,h^{-1/\lambda}\,z)\,|z|^{\rho p}\,dz \\ &= h^{\frac{\rho p}{\lambda}} \int_{\mathbb{R}} G_\lambda(1,y)|y|^{\rho p}\,dy. \end{aligned}$$

Using the fact that $G_\lambda(1,y) \le \frac{C}{1+y^2}$ (see lemma 1.1 **c**), and that $\rho p < 1$ we obtain that

$$\int_{\mathbb{R}} G_\lambda(1,y)|y|^{\rho p}\,dy < \infty.$$

Therefore we have proved that

$$I_1(t,h,x) \;\le\; C\,h^{\frac{\rho p}{\lambda}}. \tag{3.4}$$

Burkholder's and Hölder's inequalities applied to integrals with respect to the measure $[G_\lambda(t+h-s,x-y)-G_\lambda(t-s,x-y)]^2 dsdy$, the growth assumption on $\sigma$ (1.3) and (2.1) yield the following bound on $I_2$.

$$\begin{aligned} I_2(t,h,x) &\le C\left(1+\sup_{0\le s\le t}\sup_{y\in\mathbb{R}} E(|u(s,y)|^p)\right) \\ &\quad\times \left( \left| \int_0^t \int_{\mathbb{R}} [G_\lambda(t+h-s,x-y)-G_\lambda(t-s,x-y)]^2 dsdy \right|^{p/2} \right) \\ &\le C \left( \int_0^t\!\!\int_{\mathbb{R}} \left( \mathcal{F}(e^{-(t+h-s)|\cdot|^\lambda})(y) - \mathcal{F}(e^{-(t-s)|\cdot|^\lambda})(y) \right)^2 dsdy \right)^{p/2}. \end{aligned}$$

Therefore, using Plancherel identity one easily checks that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}} \left(\mathcal{F}(e^{-(t+h-s)|\cdot|^\lambda}) - \mathcal{F}(e^{-(t-s)|\cdot|^\lambda})\right)^2 (y)\, dsdy \\
&\qquad = \int_0^t \int_{\mathbb{R}} \left(e^{-(t+h-s)|y|^\lambda} - e^{-(t-s)|y|^\lambda}\right)^2 dsdy \\
&\qquad = \int_0^t\int_{\mathbb{R}} e^{-2(t-s)|y|^\lambda} \left(e^{-h|y|^\lambda} - 1\right)^2 ds\, dy.
\end{aligned}$$

Decomposing the integral on $\mathbb{R}$ into integrals on $\{|y| > 1\}$ and its complementary set, we have

$$I_2(t,h,x) \leq C\left(I_{2,1}(t,h,x) + I_{2,2}(t,h,x)\right)$$

where

$$\begin{aligned}
I_{2,1}(t,h,x) &= \left(\int_0^t\int_{|y|\leq 1} e^{-2(t-s)|y|^\lambda} \left(e^{-h|y|^\lambda} - 1\right)^2 dsdy\right)^{p/2}, \\
I_{2,2}(t,h,x) &= \left(\int_0^t\int_{|y|> 1} e^{-2(t-s)|y|^\lambda} \left(e^{-h|y|^\lambda} - 1\right)^2 dsdy\right)^{p/2}.
\end{aligned}$$

Then by the mean value theorem,

$$\begin{aligned}
\int_0^t\int_{|y|\leq 1} e^{-2(t-s)|y|^\lambda} \left(e^{-h|y|^\lambda} - 1\right)^2 dsdy &\leq \int_0^T\int_{|y|\leq 1} e^{-2(t-s)|y|^\lambda} h^2 dsdy \\
&\leq Ch^2.
\end{aligned}$$

On the set $\{|y| > 1\}$, let $0 < \alpha < \frac{\lambda-1}{2\lambda}$, then the same argument as above implies

$$\begin{aligned}
&\int_0^t\int_{|y|>1} e^{-2(t-s)|y|^\lambda} \left(e^{-h|y|^\lambda} - 1\right)^2 dsdy \\
&\quad = \int_0^t\int_{|y|>1} e^{-2(t-s)|y|^\lambda} \left(1 - e^{-h|y|^\lambda}\right)^{2\alpha} \left(1 - e^{-h|y|^\lambda}\right)^{2-2\alpha} dsdy \\
&\quad \leq C \int_0^\infty\int_{|y|>1} e^{-2s|y|^\lambda} |h|^{2\alpha} |y|^{2\lambda\alpha} dsdy \\
&\quad \leq C \int_{|y|>1} |h|^{2\alpha} |y|^{2\lambda\alpha} |y|^{-\lambda} dy \\
&\quad \leq C\, h^{2\alpha} \int_{|y|>1} |y|^{\lambda(2\alpha-1)} dy \leq Ch^{2\alpha}.
\end{aligned}$$

Consequently, for $0 < \alpha < \frac{\lambda-1}{2\lambda}$, we have proved that

$$I_{2,1}(t,h,x) \le C\, h^p,$$

$$I_{2,2}(t,h,x) \le C\, h^{\alpha p}.$$

Since $0 < \alpha < \frac{\lambda-1}{2\lambda} < 1$, $\forall \lambda \in ]1,2]$, we obtain

$$I_2(t,h,x) \le C\, h^{\alpha p}. \tag{3.5}$$

As before, Burkholder's and Hölder's inequalities applied to integrals with respect to the measure $G_\lambda^2(t+h-s, x-y)dsdy$, the growth assumption on $\sigma$ (1.3) and (2.1) yield

$$\begin{aligned} I_3(t,h,x) \;&\le\; C \left(1 + \sup_{0\le s\le t} \sup_{x\in \mathbb{R}} E(|u(s,x)|^p)\right) \\ &\quad \times \left(\int_t^{t+h} \int_{\mathbb{R}} G_\lambda^2(t+h-s, x-y)\, dsdy\right)^{p/2}. \end{aligned}$$

Recalling from (2.7) that

$$\int_{\mathbb{R}} G_\lambda^2(t+h-s, x-y)\, dy = J(t+h-s) = C(t+h-s)^{-1/\lambda}$$

we compute $\displaystyle\int_t^{t+h} (t+h-s)^{-1/\lambda} ds = C\, h^{\frac{\lambda-1}{\lambda}}$.
Thus

$$I_3(t,h,x) \le C\, h^{\frac{p(\lambda-1)}{2\lambda}}. \tag{3.6}$$

A change of variable yields

$$I_4(t,h,x) \le C\,(I_{4,1}(t,h,x) + I_{4,2}(t,h,x))$$

with

$$\begin{aligned} I_{4,1}(t,h,x) \;&=\; E\left(\left|\int_0^h ds \int_{\mathbb{R}} dy\, G_\lambda(t+h-s, x-y) b(s,y,u(s,y))\right|^p\right), \\ I_{4,2}(t,h,x) \;&=\; E\left(\left|\int_0^t ds \int_{\mathbb{R}} dy\, G_\lambda(t-s, x-y)\right.\right. \\ &\qquad \left.\left. \times \left(b\,(s+h, y, u(s+h,y)) - b\,(s,y,u(s,y))\right)\right|^p\right). \end{aligned}$$

Applying Hölder's inequality (4.1) to integrals with respect to the measure $G_\lambda(t+h-s, x-y)\, dsdy$, the growth assumption on $b$ (1.3) and

(2.1) we get

$$\begin{aligned} I_{4,1}(t,h,x) \; &\leq \; C\left(1+\sup_{0\leq s\leq t}\sup_{x\in\mathbb{R}} E(|u(s,x)|^p)\right) \\ &\qquad \times \left(\int_0^h ds\int_{\mathbb{R}} dy\, G_\lambda(t+h-s,x-y)\right)^p . \end{aligned}$$

Since $\int_{\mathbb{R}} G_\lambda(t+h-s,x-y)\,dy = 1$, we obtain

$$I_{4,1}(t,h,x) \leq C h^p. \tag{3.7}$$

Again Hölder's inequality applied to integral w.r.t. the measure $G_\lambda(t-s,x-y)\,dsdy$, Fubini's theorem and the Hölder-Lipschitz property of $b$ (1.5) imply

$$\begin{aligned} I_{4,2}(t,h,x) \; &\leq \; C\left(\int_0^t\left(h^{(\frac{\lambda-1}{2\lambda})\,p}+\sup_{y\in\mathbb{R}} E(|u(s+h,y)-u(s,y)|^p)\right)\,ds\right) \\ &\qquad \times \left(\int_0^T\int_{\mathbb{R}} G_\lambda(t-s,x-y)\,dsdy\right). \end{aligned}$$

Hence

$$I_{4,2}(t,h,x) \leq C\left(h^{(\frac{\lambda-1}{2\lambda})\,p}+\int_0^t\sup_{y\in\mathbb{R}} E(|u(s+h,y)-u(s,y)|^p)\,ds\right). \tag{3.8}$$

Then, putting together (3.2)-(3.8) we obtain for $0<\alpha<\frac{\lambda-1}{2\lambda}$

$$\begin{aligned} \sup_{x\in\mathbb{R}} E(|u(t+h,x)-u(t,x)|^p) \; &\leq \; C\, h^{p\min(\frac{\rho}{\lambda},\alpha)} \\ &\quad + \; C\int_0^t\sup_{x\in\mathbb{R}} E(|u(s+h,x)-u(s,x)|^p)ds. \end{aligned}$$

Finally, the estimates (3.1) follows from standard Gronwall's Lemma. Consider now the increments in the space variable. We want to check that for any $T>0$, $p\in[2,\infty)$, $x\in\mathbb{R}$, $z$ in a compact set $K$ of $\mathbb{R}$ and $\beta\in(0,\rho\wedge(\frac{\lambda-1}{2}))$,

$$\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}} E(|\,u(t,x+z)-u(t,x)\,|^p) \leq C\, z^{\beta p}, \tag{3.9}$$

We write

$$E(|\,u(t,x+z)-u(t,x)\,|^p) \leq C\sum_{i=1}^{3} J_i(t,z,x), \tag{3.10}$$

with

$$\begin{aligned}
J_1(t,z,x) &= E\left|\int_{\mathbb{R}}(G_\lambda(t,x+z-y)-G_\lambda(t,x-y))u_0(y)dy\right|^p,\\
J_2(t,z,x) &= E\left(\left|\int_0^t\!\!\int_{\mathbb{R}}[G_\lambda(t-s,x+z-y)-G_\lambda(t-s,x-y)]\right.\right.\\
&\qquad\left.\left.\times\sigma(s,y,u(s,y))W(dy,ds)\right|^p\right),\\
J_3(t,z,x) &= E\left(\left|\int_0^t ds\int_{\mathbb{R}}dy\,[G_\lambda(t-s,x+z-y)-G_\lambda(t-s,x-y)]\right.\right.\\
&\qquad\left.\left.\times\, b(s,y,u(s,y))\right|^p\right).
\end{aligned}$$

In the remainder of the proof we are going to establish separate upper bounds for $J_1, J_2$ and $J_3$.

A change of variable gives immediately

$$J_1(t,z,x)=E\left|\int_{\mathbb{R}}G_\lambda(t,x-y)\,(u_0(y+z)-u_0(y))\;dy\right|^p.$$

Applying again Hölder's inequality (4.1) to integral w.r.t. the measure $G_\lambda(t,x-y)\,dy$, the assumption $(\mathbf{H_1.2})$ and Fubini's theorem we obtain

$$\begin{aligned}
J_1(t,z,x) &\le C\left(\int_{\mathbb{R}}G_\lambda(t,x-y)\sup_{y\in\mathbb{R}}E(|u_0(y+z)-u_0(y)|^p)\,dy\right)\\
&\le C\left(\int_{\mathbb{R}}G_\lambda(t,x-y)|z|^{\rho p}\,dy\right)\le C\,|z|^{\rho p}.
\end{aligned}$$

Burkholder's inequality and Hölder's inequality (4.1) applied to integrals w.r.t. $[G_\lambda(t-s,x+z-y)-G_\lambda(t-s,x-y)]^2\,dsdy$, the linear growth assumption on $\sigma$ (1.3) and (2.1) imply

$$\begin{aligned}
J_2(t,z,x) &\le C\left(1+\sup_{0\le t\le T}\sup_{x\in\mathbb{R}}E(|u(t,x)|^p)\right)\\
&\quad\times\left|\int_0^t ds\int_{\mathbb{R}}dy\;\,|G_\lambda(t-s,x+z-y)-G_\lambda(t-s,x-y)|^2\right|^{p/2}\\
&\le C\left(\int_0^t\int_{\mathbb{R}}\left|\mathcal{F}(e^{-2\pi iz\cdot}e^{-(t-s)|\cdot|^\lambda})-\mathcal{F}(e^{-(t-s)|\cdot|^\lambda})\right|^2(x-y)\right)^{p/2}\\
&\le C\left(\int_0^t ds\int_{\mathbb{R}}dy\left|e^{-2\pi izy}e^{-(t-s)|\,y\,|^\lambda}-e^{-(t-s)|\,y\,|^\lambda}\right|^2\right)^{p/2}\\
&\le C\,(J_{2,1}(t,z,x)+J_{2,2}(t,z,x)),
\end{aligned}$$

where we have used the property that $\mathcal{F}\left(f(x)\right)(\xi+a)=\mathcal{F}\left(e^{-2i\pi ax}f(x)\right)(\xi)$ and the Plancherel identity and denote

$$\begin{aligned}
J_{2,1}(t,z,x) &= \left(\int_0^t ds\int_{|y|\le 1} dy\left|e^{-2\pi izy}e^{-(t-s)|y|^\lambda}-e^{-(t-s)|y|^\lambda}\right|^2\right)^{p/2},\\
J_{2,2}(t,z,x) &= \left(\int_0^t ds\int_{|y|>1} dy\left|e^{-2\pi izy}e^{-(t-s)|y|^\lambda}-e^{-(t-s)|y|^\lambda}\right|^2\right)^{p/2}.
\end{aligned}$$

We therefore have, by the mean value theorem

$$J_{2,1}(t,z,x)\le C|z|^p. \tag{3.11}$$

On the other hand, for any $0<\beta<\frac{\lambda-1}{2}$

$$\begin{aligned}
&J_{2,2}(t,z,x)\\
&=\left(\int_0^t\int_{|y|>1} e^{-(t-s)|y|^\lambda}|e^{-2\pi izy}-1|^{2\beta}|e^{-2\pi izy}-1|^{2-2\beta}dsdy\right)^{p/2}\\
&\le C\left(\int_0^t ds\int_{|y|>1} dy\, e^{-(t-s)|y|^\lambda}|y|^{2\beta}|z|^{2\beta}\right)^{p/2}\\
&\le C|z|^{2\beta}\left(\int_{|y|>1} dy\,|y|^{2\beta}\int_0^t ds\ e^{-(t-s)|y|^\lambda}\right)^{p/2}\\
&\le C|z|^{\beta p}\int_{|y|>1}\frac{dy}{|y|^{\lambda-2\beta}}\le C|z|^{\beta p}.
\end{aligned}\tag{3.12}$$

Finally, by a change of variable, the Hölder-Lipschitz property of $b$ (1.5) and Hölder's inequality,

$$\begin{aligned}
J_3(t,z,x) \;\le\; & E\left(\left|\int_0^t ds\int_{\mathbb{R}} dy\, G_\lambda(t-s,x-y)\right.\right.\\
&\qquad\left.\left.\times\,[b(s,y+z,u(s,y+z))-b(s,y,u(s,y))]\right|^p\right)\\
\le\; & C\left(z^{(\frac{\lambda-1}{2})p}+\int_0^t\sup_{y\in\mathbb{R}}E(|u(s,y+z)-u(s,y)|^p)\,ds\right)
\end{aligned}\tag{3.13}$$

Then (3.9) follows from (3.10)-(3.13) and Gronwall's lemma. The Hölder continuity in the time and space variables results from Kolmogorov criterion. □

REMARK 3.1. *Hypothesis* $(\mathbf{H_1.2})$ *is useful to have Hölder continuity up to time 0. If we discard* $(\mathbf{H_1.2})$ *and assume instead that*

$$(\mathbf{H_1.3}) \quad E\left(\int_{\mathbb{R}} |u_0(y)|\, dy\right)^p < \infty$$

*then* $\omega$*-almost surely, the function* $(t,x) \longmapsto u(t,x)(\omega)$ *belongs to* $\mathcal{C}^{\alpha,\beta}([\epsilon,T]\times\mathbb{R})$ *for* $0<\alpha<\frac{\lambda-1}{2\lambda}$ *and* $0<\beta<\frac{\lambda-1}{2}$*, for any* $\epsilon>0$*.*

Indeed, we slightly modify the preceding proof to bound

$$I_1(t,h,x) = E\left|\int_{\mathbb{R}} (G_\lambda(t+h,x-y) - G_\lambda(t,x-y))u_0(y)dy\right|^p,$$

and

$$J_1(t,z,x) = E\left|\int_{\mathbb{R}} (G_\lambda(t,x+z-y) - G_\lambda(t,x-y))u_0(y)dy\right|^p.$$

First we bound

$$I_1(t,h,x) \le \sup_{z\in\mathbb{R}} |G_\lambda(t+h,z) - G_\lambda(t,z)|^p \cdot E\left(\int_{\mathbb{R}} |u_0(y)|\, dy\right)^p.$$

The following estimates are elementary

$$G_\lambda(t+h,z) - G_\lambda(t,z) = \int_{\mathbb{R}} e^{2i\pi z\xi}(e^{-(t+h)|\xi|^\lambda} - e^{-t|\xi|^\lambda})\, d\xi,$$

$$|\, G_\lambda(t+h,z) - G_\lambda(t,z)| \le \int_{\mathbb{R}} e^{-t|\xi|^\lambda}|\, e^{-h|\xi|^\lambda} - 1|\, d\xi,$$

$$|\, G_\lambda(t+h,z) - G_\lambda(t,z)| \le h \int_{\mathbb{R}} e^{-\epsilon|\xi|^\lambda}|\xi|^\lambda\, d\xi = C\, h.$$

Hence

$$I_1(t,h,x) \le C\, h^p.$$

As for the space increments, we bound

$$G_\lambda(t,x+z) - G_\lambda(t,x) = \int_{\mathbb{R}} e^{2i\pi x\xi}e^{-t|\xi|^\lambda}(e^{2i\pi z\xi} - 1)\, d\xi,$$

$$|\, G_\lambda(t,x+z) - G_\lambda(t,x)| \le |z| \int_{\mathbb{R}} e^{-\epsilon|\xi|^\lambda} 2\pi|\xi|\, d\xi = C\,|z|,$$

Hence

$$J_1(t,z,x) \le C\, z^p.$$

The rest of the proof is the same as for theorem 3.1. □

## 4. Appendix

LEMMA 4.1. ( Hölder). *Let $f, h$ be two functions defined on* $\mathbb{R}$ *and* $\mu$ *a positive measure such that* $f \cdot h \in L^1(\mu)$. *Then, for all* $q > 1$, *we have:*

$$\left| \int f \cdot |h| d\mu \right|^q \leq \left( \int |f|^q \cdot |h| d\mu \right) \left( \int |h| d\mu \right)^{q-1}. \tag{4.1}$$

The following elementary Lemma is an extension of Gronwall's Lemma akin to lemma 3.3 established in Walsh [17], see [4] for an elegant proof.

LEMMA 4.2. *Let* $\theta > 0$. *Let* $(f_n, n \in \mathbb{N})$ *be a sequence of non-negative functions on* $[0,T]$ *and* $\alpha, \beta$ *be non-negative real numbers such that for* $0 \leq t \leq T$, $n \geq 1$

$$f_n(t) \leq \alpha + \int_0^t \beta\, f_{n-1}(s)(t-s)^{\theta-1} ds. \tag{4.2}$$

*If* $\sup_{0 \leq t \leq T} f_0(t) = M$, *then for* $n \geq 1$, $\sup_{n \geq 0} \sup_{0 \leq t \leq T} f_n(t) < \infty$, *and if* $\alpha = 0$, *then* $\sum_{n \geq 0} f_n(t)$ *converges uniformly on* $[0,T]$.

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