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MATHEMATICAL ANALYSIS AND FINITE ELEMENT STRATEGY FOR 3D NUMERICAL SIMULATION OF NAVIER STOKES EQUATIONS IN THIN DOMAINS.

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Abstract. *In this lecture, we will particularly analyze the effect of the shallowness on the Navier Stokes equations, together with anisotropic eddy viscosities. We will derive as an* asymptotic *model the* hydrostatic approximation *of the Navier Stokes equations. We will present a stable mixed 3D-FEM discretization, which allows for the computation of the 3D-velocity as a whole. Let us emphasize that* vertical velocities *are not post-processed as in most circulation models, but computed simultaneously.*

1 INTRODUCTION

Almost all geophysical fluids share a *spatial* feature, namely their depth is very small compared to their horizontal width. This is true for the atmosphere, as well as for oceans, seas, lakes, . . . and puddles as the following table shows:

Fluid Domain	Horizontal Width	Depth	Aspect Ratio
Atmosphere above Europe	5000 km	10 km	0.2 %
Northern Atlantic	5000 km	5 km	0.1%
Lake of Neuchâtel	40 km	100 m	0.25%
Street Puddle	1 m	2.5 mm	0.25%

Table 1: the "Puddle Law".

Therefore the use of an asymptotical model is natural.
Another important *time* feature is the effect of the earth rotation. Indeed the Coriolis effect must be taken into account in numerical simulations for time intervals greater than a few hours. We shall not venture in giving a state of the art of the mathematical theory of geophysical fluids. Let us only mention the pioneer work done by O. Besson and coworkers [5, 6] on the mathematical analysis of the hydrostatic hypothesis, and the fundamental work done by J.L. Lions, R. Temam and Wang [12, 13], on the climatological scale, including the temperature, salinity and density phenomena. For a survey of the mathematical analysis of geophysical fluid dynamics, we refer to the recent book by R. Lewandowski [10]. We shall focus on the hydrostatic approximation, which assumes that the pressure in the fluid increases linearly *w.r.t.* the depth.

2 MODELLING OF FLUID FLOW IN THIN DOMAINS

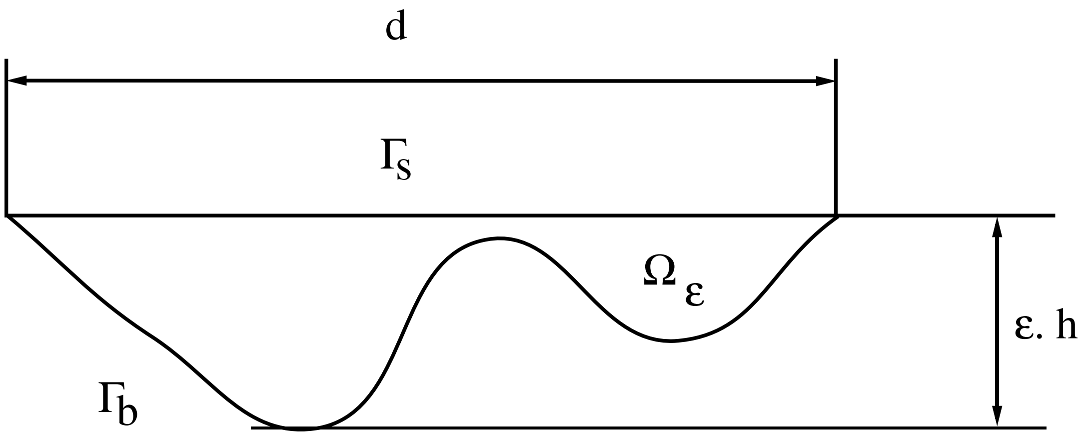

Figure 1: the Lake of Neuchâtel.

2.1 The Equations Governing the Flow.

These are the incompressible Navier-Stokes equations, with an anisotropic turbulent viscosity and the Coriolis effect.

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = \Delta_\nu \mathbf{v} - 2\omega\wedge\mathbf{v} - \nabla P + \mathbf{g} \quad \text{in } \Omega_\epsilon \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega_\epsilon \tag{2}$$

In Eq. (1), Δ_ν is defined by $\nu_1\frac{\partial^2}{\partial x_1{}^2} + \nu_2\frac{\partial^2}{\partial x_2{}^2} + \nu_z\frac{\partial^2}{\partial z^2}$. $\mathbf{v} = (u_1, u_2, w)$ designates the fluid velocity, $\omega = (0, 0, f)$ the angular earth rotation speed, $\nu = (\nu_1, \nu_2, \nu_z)$ the anisotropic eddy kinematical viscosity. In the following, we set $p = P - gz$ the hydrodynamical pressure.

2.2 Forces Acting on the Fluid and Boundary Conditions.

On the bottom of the lake, we assume the classical no-slip condition.

$$\mathbf{v} = 0 \quad \text{on } \Gamma_b \tag{3}$$

On the surface, we assume traction by the wind. τ_1 and τ_2 denote the shear stresses induced by the wind on the fluid surface.

$$\nu_z\frac{\partial u_1}{\partial z} = \tau_1, \quad \nu_z\frac{\partial u_2}{\partial z} = \tau_2, \quad w = 0 \quad \text{on } \Gamma_s \tag{4}$$

The fluid is initially at rest.

$$\mathbf{v}(., t=0) = 0 \quad \text{in } \Omega_\epsilon. \tag{5}$$

2.3 Scaling.

As fig. 1 pictures it, ϵ relates to the aspect ratio.

$$\Omega_\epsilon = \{\mathbf{x} = (x_1, x_2, z);\ (x_1, x_2) \in\ \Gamma_s,\ -\epsilon\cdot h(x_1, x_2) < z < 0\}$$

It is then natural to perform the geometrical and cinematical scaling:

$$z = \epsilon\cdot x_3,\ w = \epsilon\cdot u_3$$

$$\Omega = \{\mathbf{x} = (x_1, x_2, x_3);\ (x_1, x_2) \in\ \Gamma_s,\ -h(x_1, x_2) < x_3 < 0\}.$$

For simple dimensional analysis, as well as deeper physical and mathematical reasons, see [5]-[4], it is *judicious* to scale also the mechanical quantities:

$$\nu_z = \epsilon^2\cdot\nu_3,\ \tau_i = \epsilon\cdot\theta_i,\ i = 1, 2.$$

The scaled problem is now

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + \mathbf{u}\cdot\nabla u_1 - \Delta_\nu u_1 - f u_2 + \frac{\partial p}{\partial x_1} &= 0 \quad \text{in } \Omega && (6)\\
\frac{\partial u_2}{\partial t} + \mathbf{u}\cdot\nabla u_2 - \Delta_\nu u_2 + f u_1 + \frac{\partial p}{\partial x_2} &= 0 \quad \text{in } \Omega && (7)\\
\epsilon^2 \,\{\frac{\partial u_3}{\partial t} + \mathbf{u}\cdot\nabla u_3 - \Delta_\nu u_3\} + \frac{\partial p}{\partial x_3} &= 0 \quad \text{in } \Omega && (8)\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega && (9)\\
\mathbf{u} &= 0 \quad \text{on } \Gamma_b && (10)\\
\nu_3 \frac{\partial u_1}{\partial x_3} = \theta_1, \quad \nu_3 \frac{\partial u_2}{\partial x_3} = \theta_2, \quad u_3 &= 0 \quad \text{on } \Gamma_s && (11)\\
\mathbf{u}(.,\mathbf{t}=\mathbf{0}) &= 0 \quad \text{in } \Omega && (12)
\end{aligned}$$

Eq. (6)-(12) are *anisotropic* Navier-Stokes equations. They possess at least one weak solution, built in the same way as for the usual Navier-Stokes ones, see [11].

3 ASYMPTOTICAL ANALYSIS OF THE HYDROSTATIC APPROXIMATION

Let the aspect ratio ϵ go to zero. Passing to the limit *formally* we obtain

3.1 The Limit Model

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + \mathbf{u}\cdot\nabla u_1 - \Delta_\nu u_1 - f u_2 + \frac{\partial p}{\partial x_1} &= 0 \quad \text{in } \Omega && (13)\\
\frac{\partial u_2}{\partial t} + \mathbf{u}\cdot\nabla u_2 - \Delta_\nu u_2 + f u_1 + \frac{\partial p}{\partial x_2} &= 0 \quad \text{in } \Omega && (14)\\
\frac{\partial p}{\partial x_3} &= 0 \quad \text{in } \Omega && (15)\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega && (16)\\
u_1 = u_2 = u_3 n_3 &= 0 \quad \text{on } \Gamma_b && (17)\\
\nu_3 \frac{\partial u_1}{\partial x_3} = \theta_1, \quad \nu_3 \frac{\partial u_2}{\partial x_3} = \theta_2, \quad u_3 &= 0 \quad \text{on } \Gamma_s && (18)\\
u_1(.,t=0) = u_2(.,t=0) &= 0 \quad \text{in } \Omega && (19)
\end{aligned}$$

Remark. The problem degenerates for the vertical velocity, which disappears from Eq. (15) and is present only in the incompressibility condition (16). This explains why the boundary condition (17) is weaker for u_3, which belongs to an anisotropic Sobolev space (see section below). Moreover, as there is no time derivative for u_3, an initial condition on u_3 does not make sense, hence the initial condition (19) holds only for horizontal velocities. ■

3.2 Weak Formulation.

To analyze the equations (13)-(19), we need some anisotropic Sobolev spaces.

$$H_b^1(\Omega) = \left\{\varphi \in H^1(\Omega);\ \varphi = 0 \text{ on } \Gamma_b\right\}$$

$$H(\partial_3, \Omega) = \left\{\varphi \in L^2(\Omega);\ \partial_3\,\varphi \in L^2(\Omega)\right\}$$

$$\|\varphi\|^2_{H(\partial_3,\Omega)} = \|\varphi\|^2_{L^2(\Omega)} + \|\partial_3\,\varphi\|^2_{L^2(\Omega)}$$

$$H_0(\partial_3, \Omega) = \{\varphi \in H(\partial_3, \Omega);\ \varphi\, n_3 \ = \ 0 \ \text{ on } \partial\Omega\}$$

$$\mathbf{W} = \left\{\mathbf{u} \in H_b^1 \times H_b^1 \times H_0(\partial_3);\ \operatorname{div}\mathbf{u} = 0\right\}$$

T designates a fixed positive duration. We make the natural assumption that the wind energy is finite:

$$\theta_1,\ \theta_2 \in L^2(0, T; H^{-1/2}(\Gamma_s))$$

The weak formulation of Eq. (13)-(19) reads

Find $\mathbf{u} \in L^2(0, T; \mathbf{W})$, with $u_1, u_2 \in L^\infty(0, T; L^2)$, such that

$$\int_0^T (u_1, \frac{\partial v_1}{\partial t})\,\mathrm{dt} + \int_0^T (u_1, \mathbf{u}\cdot\nabla v_1)\,\mathrm{dt} + f\cdot\int_0^T (u_2, v_1)\,\mathrm{dt} - \int_0^T (\nabla_\nu u_1, \nabla_\nu v_1)\,\mathrm{dt}$$

$$+ \int_0^T (u_2, \frac{\partial v_2}{\partial t})\,\mathrm{dt} + \int_0^T (u_2, \mathbf{u}\cdot\nabla v_2)\,\mathrm{dt} - f\cdot\int_0^T (u_1, v_2)\,\mathrm{dt} - \int_0^T (\nabla_\nu u_2, \nabla_\nu v_2)\,\mathrm{dt}$$

$$= \int_0^T < \theta_1, v_1 > \ \mathrm{dt} + \int_0^T < \theta_2, v_2 > \ \mathrm{dt}$$

$$\forall\,\mathbf{v} = (v_1, v_2, v_3) \text{ regular enough, with } \mathbf{v}(T) = 0.$$

Remark. The weak form is naturally built by integration by parts from (13)-(19). ■

3.3 Existence and Convergence Results.

Theorem 1 *For $\theta_1, \theta_2 \in L^2(0, T; H^{-1/2}(\Gamma_s))$, there exists a weak solution of the Navier Stokes Hydrostatic equations $\mathbf{u} \in L^2(0, T; \mathbf{W})$, with u_1, u_2 in $L^\infty(0, T; L^2)$, obtained as a limit of weak solutions $\mathbf{u}^\epsilon$ of the anisotropic Navier Stokes equations when the aspect ratio ϵ goes to zero.*

For the proof of this theorem, we refer to [3, 4].
Remark. This result gives a mathematical justification of the hydrostatic approximation, which is in current use in geophysics, see [14],[15], [12]. ■

Remark. One major difficulty of the proof is the lack of regularity of the vertical velocity, which is determined only by the incompressibility condition (16). ■

Remark. The nonlinear stationary case was solved in [5], the linear stationary (Stokes) was studied in [6] and [1, 2]. The time-dependent linear case was solved in [2]. ■

4 FINITE ELEMENTS FOR SHALLOW FLUIDS

4.1 Geophysical flow are quasi-horizontal but nevertheless 3D!

Circulation models are very often based on a barotropic-baroclinic splitting. First the horizontal velocities and the surface pressure are computed, then with the help of the incompressibility equation the vertical velocity is recovered. As this is a first order equation[1], the vertical velocities are not so safely computed. This way of treating differently the prognostic variable (u, v) and the diagnostic w has a long history. Indeed the vertical velocity is usually very small compared to the horizontal one, a difference of one order of magnitude due to the small aspect ratio (hydrostatic limit) and even two if the Rossby number is small enough (geostrophic limit). Therefore circulation model were designed bearing in mind this fact, and were basically, say 2.5D. To sum up, w was considered as a noise ..., so its reliable computation was not considered as relevant. However there is a need to compute precisely the vertical velocities in order to be able to simulate very delicate phenomena (like e.g. “El Niño ”).

For the above reasons, we advocate to compute simultaneously u, v, w and p by a mixed finite element method.

4.2 Stability conditions for mixed FEM.

As is well known the careless discretization of the incompressibility constraint may lead to spurious pressures (the doomed checkerboard pattern). To prevent this, the finite elements chosen for velocity, pressure must fulfill the Brezzi-Babuška condition, see Girault-Raviart [8]. Here there is another constraint on the pressure, the hydrostatic hypothesis, which may lead to spurious vertical velocities, therefore the finite element discretization must fulfill a supplementary stability condition,see [1, 2]

$$\sup_{p_h} \frac{\int_\Omega \partial_3 u_{3h}\, p_h}{\|p_h\|_{L^2}} \geq C\|\partial_3 u_{3h}\|_{L^2}. \tag{20}$$

In addition, these two constraints are in competition, so the choice of a clean element is delicate. The following $P_2 - P_1 - P_{0.5}$ prismatic element meets the required conditions, see fig.2. A peculiar feature of this element is that the pressure is locally hydrostatic, i.e. does not depend on x_3. Note that the lateral edges must be vertical (see [2] for details).

4.3 A projection method for the hydrostatic Navier-Stokes equations.

The previous element, though stable, cannot be used in a standard conjugate gradient or Uzawa algorithm, because the hydrostatic Stokes problem is not positive definite, that is its Schur complement is only semidefinite.

One way to circumvent this, is to come back to the original anisotropic Navier-Stokes equations, and use a *projection method* [9]. So one strategy [7] is to

[1] moreover with two boundary conditions at the bottom and at the surface ...

- keep the full third momentum equation instead of the hydrostatic approximation
- use a projection method

The great advantage being that while the computation remains fully $3D$, there are much less degrees of freedom than with the full $P_2 - P_1$ Taylor-Hood element, so the computational load for computing w is marginal. Numerical tests have been performed on various swiss lakes by O.Besson [7] and we refer to his lecture in this conference for more details and numerical results. He observed that a $P_2 - P_1 - P_1$ element can also be successfully used in this approach, though it violates condition (20), but still fulfills Brezzi's condition. Let us finally remark that, due to the previous convergence analysis, keeping the full third momentum equation can be viewed as a natural penalization of the hydrostatic equation.

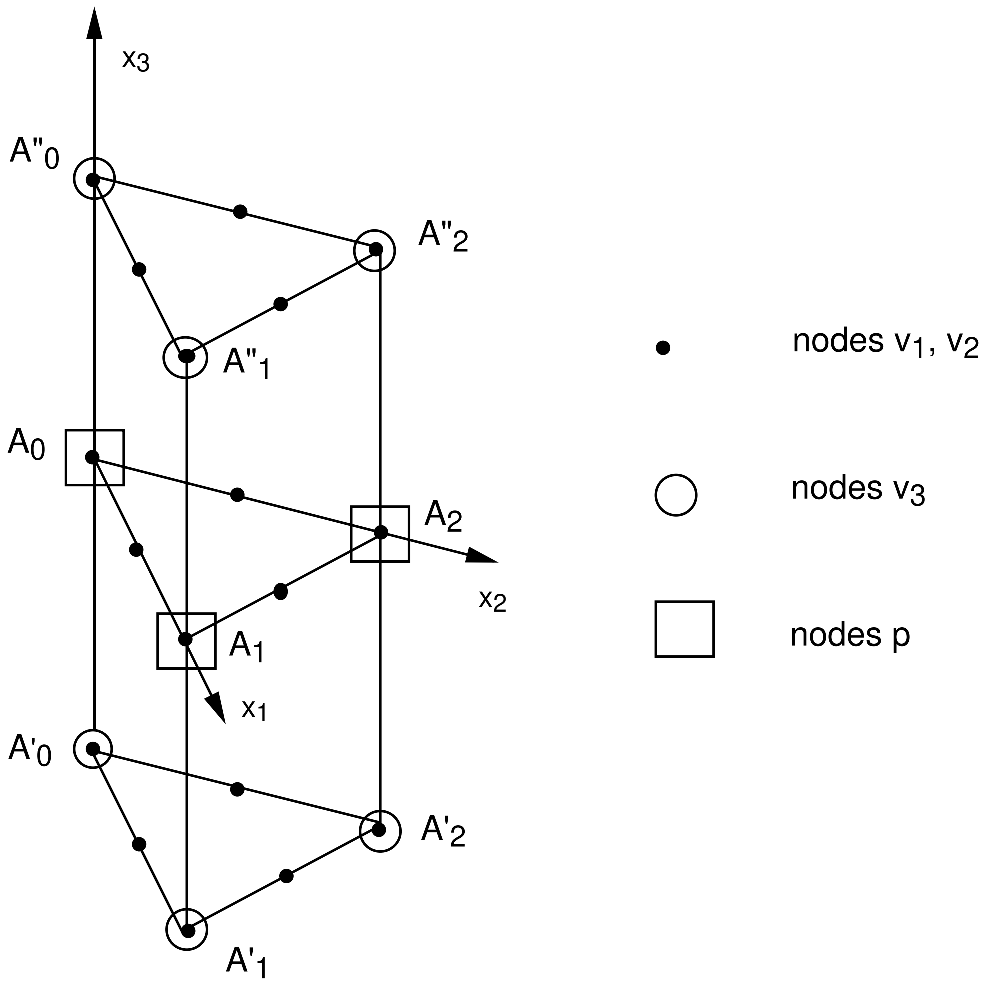

Figure 2: the Hydrostatic $P_2 - P_1 - P_{0.5}$ Finite Element

REFERENCES

[1] P. Azérad, "Analyse et approximation du problème de Stokes dans un bassin peu profond", *C.R. Acad. Sci. Paris,* **318**, 53–58 (94)

[2] P. Azérad, *Analyse des équations de Navier Stokes et de l'équation de transport*, Thèse de Doctorat ès sciences, Université de Neuchâtel, 1996. http://gala.univ-perp.fr/~azerad/

[3] P. Azérad and F. Guillén, "Equations de Navier-Stokes en bassin peu profond : l'approximation hydrostatique", *C.R. Acad. Sci. Paris,* **329**, 961–966 (1999).

[4] P. Azérad and F. Guillén, "Mathematical justification of the hydrostatic approximation in geophysical fluid dynamics", *preprint.*

[5] O. Besson and M. R. Laydi, "Some estimates for the anisotropic Navier-Stokes equations and for the hydrostatic approximation", *M2AN - Mod. Math. Ana. Num.,* **26**, 855–865 (1992).

[6] O. Besson, M. R. Laydi, and R. Touzani, "Un modèle asymptotique en océanographie", *C.R. Acad. Sci. Paris,* **310**, 661–665 (1990).

[7] O. Besson, "Finite Element solution of Navier-Stokes equations in shallow domains", *Preprint.*

[8] V. Girault and P. A. Raviart, *Finite Element for Navier-Stokes equations*, Springer, 1986.

[9] J. L. Guermond, "Some practical implementations of projection methods for Navier-Stokes equations", *M2AN-Model. Math. Anal. Num.,* **30**, 637–667 (1996).

[10] R. Lewandowski, *Analyse Mathématique et Océanographie*, Masson, 1997.

[11] J. L. Lions, *Quelques Methodes de Résolution des Problèmes aux limites non linéaires*, Dunod, 1969.

[12] J. L. Lions, R. Temam and S. Wang, "Geostrophic asymptotics of the primitive equations of the atmosphere" *Topol. Methods Nonlinear Anal.,* **4**, 253–287 (1994).

[13] J. L. Lions, R. Temam and S. Wang, "On the Equations of the Large Scale Ocean" *Nonlinearity,* **5**, 1007–1053 (1992).

[14] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, 1987.

[15] R. K. Zeytounian, *Modélisation Asymptotique en Mécanique des Fluides newtoniens*, Springer-Verlag, 1994.