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Quasi-Stability of the Primary Flow in a Cone and Plate Viscometer

Pascal Azerad and Eberhard Bänsch

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Abstract. We investigate the flow between a shallow rotating cone and a stationary plate. This cone and plate device is used in rheometry, haemostasis as well as in food industry to study the properties of the flow w.r.t. shear stress. Physical experiments and formal computations show that close to the apex the flow is approximately azimuthal and the shear-stress is constant within the device, the quality of the approximation being controlled essentially by the single parameter $Re\,\epsilon^2$, where Re is the Reynolds number and ϵ the thinness of the cone-plate gap. We establish this fact by means of rigorous energy estimates and numerical simulations. Surprisingly enough, this approximation is valid though the primary flow is not itself a solution of the Navier–Stokes equations, and it does not even fulfill the correct boundary conditions, which are in this particular case discontinuous along a line, thus not allowing for a usual Leray solution. To overcome this difficulty we construct a suitable *corrector*.

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1. Introduction

The cone-and-plate apparatus (CPA) consists of a shallow rotating cone on top of a stationary plate, both surrounded by a circular cylinder, see Fig. 1.1. Liquid is flowing inside the device. When the angle α between the cone and plate is very small and the rotational speed ω is low, the flow is basically azimuthal, i.e. streamlines are concentric circles, the velocity profile is linear and the shear-rate is constant, like in Couette flow. Thus the most well-known motivation for the use of CPA has been a viscometer, mainly used for very viscous or viscoelastic fluids. Recently, it has been used to study fine properties of biological fluids under *shear stress.* For instance in haemostasis [14], CPA is used as a flow chamber to investigate shear-induced platelet aggregation (SIPA) and interactions between platelets and the vessel wall in blood flow. According to [11], SIPA has been implicated as a potential mechanism underlying thrombotic events occurring in blood vessels in which shear stresses are inordinately large, such as may be present in the vicinity of a stenosed coronary artery. CPA is also used in food industry to control the viscosity of some food products.

In fact the flow pattern is only approximately azimuthal: as the angle α and rotational rate ω increase, the fluid near the cone experiences an increasing centrifugal force that promotes radial fluid motion towards the periphery of the device. Thus a radial *secondary flow* develops, streamlines turn into spirals, until the onset of turbulence. This phenomenon was apparently first observed in [4], then numerically computed in [5]. The appearing of secondary flow was clearly identified in [9] where it is shown to be controlled by a single parameter $Re\,\epsilon^2$, with $\epsilon = \tan\alpha$ and Re the Reynolds number. In the latter paper, the authors considered an infinite domain, so that boundary effects were neglected and the analysis relied on formal asymptotic expansions.

From a mathematical view point CPA has been much less studied than Couette flow between cylinders, see [3, 7]. Indeed the cone-plate geometry gives rise to a quite different picture: whereas in Couette flow there is an exact analytical solution that becomes unstable, in CPA flow no analytical solution is available but only an approximate one, which is stable, at least in the small gap limit. In our paper, we give a precise mathematical proof of the fact that the flow is approximately azimuthal. One difficulty of the proof stems from the poor regularity of the boundary condition. Due to a discontinuity located at the intersection of the cylinder and the cone, the solution cannot belong to $W^{1,2}(\Omega)$. Navier–Stokes equations with poorly regular boundary conditions have been investigated by D. Serre [10]. The idea is to subtract from the solution a suitable corrector, so that the singular part be removed. Here we construct a corrector, still azimuthal, which will slightly perturb the basic flow, essentially only in the vicinity of the singularity. Moreover, we give estimates for these approximations which allow us to analyze the nonlinear stability of the primary flow.

The paper is organized as follows. In Section 2 we perform the natural scalings. Some useful notations are collected in Section 3 and the formal asymptotical analysis leading to the basic (*primary*) flow is performed in Section 4. We investigate the a priori singularity of the solution, introduce a suitable corrector and derive a suitable weak formulation in Section 5. We state the main results in Section 6, which we prove by means of careful energy estimates. Finally, in Section 7, we show numerical simulations confirming the abstract results and revealing further properties of the flow in this apparatus.

2. Problem setting and scaling

Let Ω_ϵ be the domain filled by the fluid, see Fig. 1.1, which we suppose to be Newtonian, of kinematic viscosity ν and constant density ρ. The small parameter

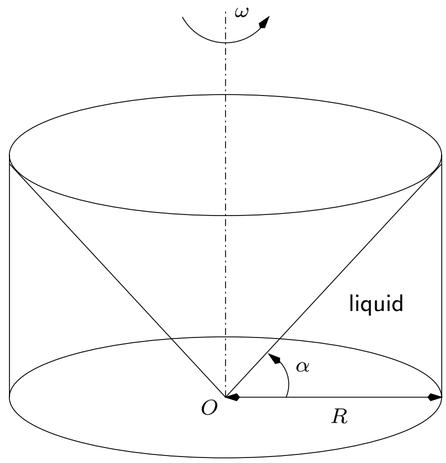

FIG. 1.1. Cone and Plate device, with angle α magnified.

is $\epsilon = \tan\alpha$. The radius of the outer cylinder is denoted by R. Let O be the cone apex, Oy_3 its axis, Oy_1y_2 the plane in a rectangular Cartesian coordinate system (y_1, y_2, y_3) and corresponding basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

The incompressible Navier–Stokes equations with no-slip Dirichlet boundary condition read: Given an initial function $\mathbf{U}_0$, find a velocity field $\mathbf{U} = (U_1, U_2, U_3)$ and a pressure field P, such that for $\tau > 0$

$$\partial_\tau \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} - \nu\Delta\mathbf{U} + \frac{1}{\rho}\nabla P = 0 \qquad \text{in } \Omega_\epsilon, \tag{2.1a}$$

$$\operatorname{div}\mathbf{U} = 0 \qquad \text{in } \Omega_\epsilon, \tag{2.1b}$$

$$\mathbf{U} = 0 \qquad \text{on } ((y_3 = 0) \cup (r = R)), \tag{2.1c}$$

$$\mathbf{U} = r\omega\cdot\mathbf{e}_\theta \qquad \text{on } (y_3 = r\cdot\epsilon), \tag{2.1d}$$

$$\mathbf{U}(\cdot, 0) = \mathbf{U}_0 \qquad \text{in } \Omega_\epsilon. \tag{2.1e}$$

Here and in the following, ω is the angular velocity of the cone, r the distance to the vertical axis and $\mathbf{e}_\theta$ the azimuthal direction, see also below. In order to bring (2.1a)–(2.1e) into non dimensional form and to make the domain independent of ϵ, we set

$$x_1 = \frac{y_1}{R}, \quad x_2 = \frac{y_2}{R}, \quad x_3 = \frac{y_3}{R\epsilon}, \quad t = \omega\tau$$

so that $\Omega = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3;\ 0 < r < 1,\ 0 < x_3 < r \text{ with } r = \sqrt{x_1^2 + x_2^2}\Big\}$ is the new fixed domain. Define the different parts of the boundary

$$\Gamma_1 := \Big\{x \in \partial\Omega;\ r = x_3\Big\}, \Gamma_2 := \Big\{x \in \partial\Omega;\ x_3 = 0\Big\}, \Gamma_3 := \Big\{x \in \partial\Omega;\ r = 1\Big\}.$$

The corresponding kinematic scaling is

$$u_1 = \frac{U_1}{R\omega}, \quad u_2 = \frac{U_2}{R\omega}, \quad u_3 = \frac{U_3}{R\omega\epsilon}, \quad p = \frac{P}{R^2\omega^2\rho}, \tag{2.2}$$

so that $\mathbf{u} = (u_1, u_2, u_3)$ is the new unknown velocity and p the new pressure. As corresponding Reynolds number we define

$$Re = R^2\omega/\nu.$$

Other choices for typical length and velocity are possible, see Section 6.

With the above considerations, problem (2.1a)–(2.1e) transforms into the following anisotropic Navier–Stokes equations: Find $\mathbf{u} = (u_1, u_2, u_3)$ and p, such that for $t > 0$

$$\partial_t u_i + \mathbf{u} \cdot \nabla u_i - \frac{1}{Re}\Delta_\epsilon u_i + \partial_i p = 0 \quad \text{in } \Omega,\ i = 1, 2 \tag{2.3a}$$

$$\epsilon^2 \left(\partial_t u_3 + \mathbf{u} \cdot \nabla u_3 - \frac{1}{Re}\Delta_\epsilon u_3\right) + \partial_3 p = 0 \quad \text{in } \Omega, \tag{2.3b}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.3c}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{2.3d}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{2.3e}$$

where Δ_ϵ is an anisotropic Laplacian, defined by

$$\Delta_\epsilon := \sum_{k=1}^{2} \frac{\partial^2}{\partial x_k^2} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial x_3^2},$$

and the boundary data is collected in

$$\mathbf{g} := \begin{cases} r \cdot \mathbf{e}_\theta & \text{on } \Gamma_1, \\ 0 & \text{on } \Gamma_2, \\ 0 & \text{on } \Gamma_3. \end{cases}$$

3. Notations and auxiliaries

Hereafter, $\|\cdot\| = \|\cdot\|_\Omega$, and for $G \subseteq \Omega$, $\|w\|_G = (\int_G |w|^2)^{1/2}$ denotes the usual $L^2(G)$-norm for scalar as well as vector- and matrix-valued functions on G, $(\cdot,\cdot)$ is the L^2-inner product. To avoid confusion, vector-valued functions will always be denoted with boldface characters.

For $0 < r_0 < 1$ define $\Omega_{r_0} := \{x \in \Omega;\ r < r_0\}$.

We will need the following function spaces.

$$\mathcal{V} = \{\boldsymbol{\phi} \in C_0^\infty(\Omega); \quad \operatorname{div} \boldsymbol{\phi} = 0\}.$$

$\mathbf{V}$ (resp. $\mathbf{H}$) is defined as the closure of $\mathcal{V}$ in $H_0^1(\Omega)$ (resp. $L^2(\Omega)$).

We shall frequently use cylindrical coordinates (r, θ, x_3). We recall some formulas, which will be used below. The cylindrical coordinates are defined by

$$r = \sqrt{x_1^2 + x_2^2},\ x_1 = r\cos\theta,\ x_2 = r\sin\theta$$

with the corresponding basis

$$\mathbf{e}_r = \frac{1}{r}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2), \quad \mathbf{e}_\theta = \frac{1}{r}(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2), \quad \mathbf{e}_3.$$

Thus a 3D vector field $\mathbf{w}$ can be decomposed as

$$\mathbf{w} = w_r\mathbf{e}_r + w_\theta\mathbf{e}_\theta + w_3\mathbf{e}_3.$$

Furthermore, define the horizontal part $\mathbf{v}_H$ of a 3D vector field $\mathbf{v}$ by

$$\mathbf{v}_H := v_1\mathbf{e}_1 + v_2\mathbf{e}_2$$

Likewise $\nabla_H := \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2$ and

$$|||\mathbf{w}|||_G^2 := \|\mathbf{w}_H\|_G^2 + \epsilon^2\|w_3\|_G^2.$$

In the sequel we will use the same notation for a function f depending on the Euclidean basis or cylindrical coordinates: $f(x_1, x_2, x_3) = f(r, \theta, x_3)$. This abuse of notation will not lead to confusion.

The following differential operators transform like:

$$\nabla = \mathbf{e}_r\partial_r + \frac{1}{r}\mathbf{e}_\theta\partial_\theta + \mathbf{e}_3\partial_3, \quad \partial_\theta\mathbf{e}_\theta = -\mathbf{e}_r, \quad \partial_\theta\mathbf{e}_r = \mathbf{e}_\theta,$$

$$\operatorname{div}\mathbf{w} = \frac{1}{r}\partial_r(rw_r) + \frac{1}{r}\partial_\theta w_\theta + \partial_3 w_3,$$

$$\Delta\, w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial x_3^2},$$

$$\Delta\,\mathbf{w} = \left(\Delta w_r - \frac{2}{r^2}\partial_\theta w_\theta - \frac{w_r}{r^2}\right)\cdot\mathbf{e}_r + \left(\Delta w_\theta + \frac{2}{r^2}\partial_\theta w_r - \frac{w_\theta}{r^2}\right)\cdot\mathbf{e}_\theta + \Delta w_3\cdot\mathbf{e}_3.$$

4. Formal asymptotics

Multiplying (2.3a) by ϵ^2, sending ϵ to zero and then formally equating (2.3a) yields

$$\frac{\partial^2\mathbf{u}_H}{\partial x_3^2} = 0,$$

so that $\mathbf{u}_H$ is linear in x_3:

$$\mathbf{u}_H = \mathbf{A}_H(x_1, x_2)x_3 + \mathbf{B}_H(x_1, x_2).$$

Taking into account the boundary conditions on Γ_1 and Γ_2 we formally get

$$u_1 = x_3(-x_2/r), \quad u_2 = x_3(x_1/r).$$

Plugging this in the incompressibility equation (2.3c) gives

$$\partial_3 u_3 = 0,$$

which, with the boundary condition on Γ_2 allows us to derive

$$u_3 = 0.$$

Hence the velocity field is completely characterized within the device:

$$\mathbf{u} = x_3 \cdot \mathbf{e}_\theta \tag{4.1}$$

This is what is called the *primary flow* and will be denoted by $\bar{\mathbf{u}}$.

Remark 4.1. The primary flow has only a θ-component (swirl), i.e. is purely azimuthal. Its modulus depends on x_3 only. It satisfies the no-slip boundary conditions on the cone Γ_1 and on the plane Γ_2 but violates the boundary condition at the outer cylinder Γ_3.

In the physical variables, the primary flow is

$$\bar{\mathbf{U}} = \frac{\omega y_3}{\epsilon} \cdot \mathbf{e}_\theta.$$

The resulting shear stress has constant magnitude:

$$\sigma_{\theta\, y_3}(\bar{\mathbf{U}}) = \rho\nu \partial_{y_3} \bar{\mathbf{U}} = \rho\nu \frac{\omega}{\epsilon} \cdot \mathbf{e}_\theta. \tag{4.2}$$

This basic flow is indeed observed in physical experiments and the shear stress (4.2) is used to measure fluid viscosity, see [9], provided that $Re\,\epsilon^2$ is small enough. Now, the surprising fact is that the primary flow is not a solution of the Navier–Stokes equations (2.3). Indeed, we compute

$$-\frac{1}{Re}\Delta_\epsilon \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} = \frac{1}{Re}\frac{x_3}{r^2} \cdot \mathbf{e}_\theta - \frac{x_3^2}{r} \cdot \mathbf{e}_r. \tag{4.3}$$

It is easy to see that (4.3) is not a gradient, so the above term cannot be balanced by a pressure gradient alone. Furthermore, if we take the limit of (2.3a)–(2.3b) in the sense that we set $\mathbf{u} = \bar{\mathbf{u}}$ and then formally send $\epsilon \to 0$ we get for the pressure

$$\partial_r p = \frac{x_3^2}{r}, \tag{4.4}$$

$$\partial_\theta p = -\frac{1}{Re}\frac{x_3}{r}, \tag{4.5}$$

$$\partial_3 p = 0. \tag{4.6}$$

Of course (4.4)–(4.6) cannot hold simultaneously.

For the same reason, as

$$-\frac{1}{Re}\Delta_\epsilon \bar{\mathbf{u}} = \frac{1}{Re}\frac{x_3}{r^2} \cdot \mathbf{e}_\theta, \tag{4.7}$$

neither is the primary flow solution of Stokes equations.

5. A suitable corrector and weak formulation

The boundary data $\mathbf{g}$ obviously exhibits a discontinuity at the line $\gamma := \{x\,;\, r = 1, x_3 = 1\}$, which excludes $W^{1,2}(\Omega)$ regularity of the solution. This means that the usual energy inequality makes no sense. In order to cure this, we shall construct a "corrector" $\mathbf{u}^\star = \phi(r, x_3) \cdot \mathbf{e}_\theta$ defined by

$$-\Delta_\epsilon \mathbf{u}^\star = 0 \qquad \text{in } \Omega, \tag{5.1a}$$

$$\mathbf{u}^\star = \mathbf{g} - \bar{\mathbf{u}} \qquad \text{on } \partial\Omega. \tag{5.1b}$$

Remark 5.1. Since $\mathbf{u}^\star$ is azimuthal, it satisfies

$$\operatorname{div} \mathbf{u}^\star = 0.$$

Hence, the corrector $\mathbf{u}^\star$ is simply the Stokes flow, with zero pressure gradient, which cancels the primary flow $\bar{\mathbf{u}}$ at the outer cylinder boundary.

The corrector $\mathbf{u}^\star$ is determined by the scalar function ϕ which in turn is given by

$$\mathcal{L}_\epsilon \phi = 0 \qquad \text{in } \Omega, \tag{5.2a}$$

$$\phi = g^\star \qquad \text{on } \partial\Omega \tag{5.2b}$$

with

$$g^\star := \begin{cases} 0 & \text{on } \Gamma_1, \\ 0 & \text{on } \Gamma_2, \\ -x_3 & \text{on } \Gamma_3, \end{cases}$$

and

$$\mathcal{L}_\epsilon \psi := -\Delta_\epsilon \psi + \frac{1}{r^2}\psi.$$

Again the boundary data does not belong to $W^{1/2,2}(\partial\Omega)$, so that one cannot simply use the standard energy method. Instead, we first give an existence result from classical potential theory and then prove pointwise and local energy estimates in Lemma 5.4.

Let us first recall ([6], Section 6.3) the concept of subsolutions for the elliptic equation $\mathcal{L}_\epsilon \eta = 0$.

Definition 5.2. A function $\psi \in C^0(\Omega)$ is a *subsolution* if and only if for every ball $B \Subset \Omega$ and every solution η of $\mathcal{L}_\epsilon \eta = 0$ in B, the inequality $\psi \le \eta$ on ∂B implies also $\psi \le \eta$ in B.

By the classical maximum principle (inside a ball $B \Subset \Omega$, the singularity at $r = 0$ plays no role), a function ψ belonging to $C^2(\Omega)$ such that $\mathcal{L}_\epsilon \psi \le 0$ is a subsolution.

Lemma 5.3. *There are functions* $\mathbf{u}^\star$, $\phi = \phi(r, x_3) \in C^2(\Omega)$, *solutions of* (5.1) *and* (5.2) *respectively, where the boundary conditions* (5.1b), (5.2b) *hold in the sense that*

$$\mathbf{u}^\star(x_n) \to \mathbf{g}(x) - \bar{\mathbf{u}}(x), \quad \phi(x_n) \to g^\star(x)$$

for all $x_n \to x$, *with* $x \in \partial\Omega \backslash \gamma$.

Proof. Applying Perron's method to problem (5.2) we get a solution $\phi \in C^2(\Omega)$ by setting

$$\phi(x) := \sup\{\psi(x) \mid \psi \text{ subsolution, } \psi \leq g^\star \text{ on } \partial\Omega\}.$$

Since the full proof is lengthy, we omit the details here. The method follows exactly the same steps as for subharmonic functions. Let us emphasize that the operator $\mathcal{L}_\epsilon \psi$ is strictly elliptic, and that the zero-order term $\dfrac{1}{r^2}\psi$, though unbounded, is positive. For the reader's convenience, we refer to [6], pp. 102–106 for a more complete account.
Since the domain Ω satisfies the *exterior sphere condition*, the boundary values are attained continuously for all points where $g^\star$ is continuous. Furthermore, for symmetry reasons, the solution does not depend on θ. Finally, setting $\mathbf{u}^\star := \phi \cdot \mathbf{e}_\theta$ we obtain a solution of (5.1). □

We now establish some estimates for the corrector.

Lemma 5.4. *Let* ϕ *be the solution of* (5.2) *from Lemma 5.3 and* $0 < r_0 < 1$. *Then we have the following properties:*

i) $-1 \leq \phi \leq 0$ *in* Ω, $\quad \|\phi\|_\infty = \|\mathbf{u}^\star\|_\infty = 1$,
ii) $|\phi(r, x_3)| \leq C\epsilon^2 r$ *for* $0 < r \leq r_0, \quad 0 \leq x_3 \leq r$,
iii) $\|\mathbf{u}^\star\|_{\Omega_{r_0}} = \|\phi\|_{\Omega_{r_0}} \leq C\epsilon^2$,
iv) $\|\partial_3 \phi\|_{\Omega_{r_0}} \leq C\epsilon^3$,

where in ii)–iv) *the constants* C *depend on* r_0: $C = C(r_0)$.

Proof. i) Since $-1 \leq g^\star \leq 0$, the specific construction of $\phi := \sup\{\psi(x) \mid \psi$ subsolution, $\psi \leq g^\star$ on $\partial\Omega\}$ allows to deduce the first inequality. The rest of i) is obvious.

ii) We shall construct a function $\psi = \psi(r, x_3)$ which is a subsolution. For this define

$$h(r) := \begin{cases} \delta r, & \text{for } 0 \leq r \leq r_0, \\ a(r - r_0)^3 + \delta r, & \text{for } r_0 \leq r \leq 1, \end{cases}$$

with some small $0 < \delta < 1$ to be determined below and $a = \frac{1-\delta}{(1-r_0)^3}$.

The function h fulfills:

$$h \in C^2([0,1]), \quad h(0) = 0, \quad h(1) = 1.$$

Now define $\psi(r, x_3) := -h(r)\omega(x_3)$ with $\omega(x_3) := \sin(\frac{\pi x_3}{2})$.

Clearly, $\psi \le g^\star$ on $\partial\Omega$. Furthermore we have

$$(\mathcal{L}_\epsilon \psi)(r, x_3) = \left(h''(r) + \frac{h'(r)}{r} - \frac{h(r)}{r^2} - \left(\frac{\pi}{2\epsilon}\right)^2 h(r) \right) \omega(x_3).$$

For $0 < r < r_0$ this gives

$$(\mathcal{L}_\epsilon \psi)(r, x_3) = \left(\frac{\delta}{r} - \frac{\delta}{r} - \left(\frac{\pi}{2\epsilon}\right)^2 \delta r \right) \omega(x_3) \le 0.$$

We want to choose δ such that $\mathcal{L}_\epsilon \psi \le 0$ also for $r > r_0$.
Now, for $r_0 \le r \le 1$ we have:

$$h''(r) + \frac{h'(r)}{r} = 6a(r - r_0) + \frac{3a(r - r_0)^2 + \delta}{r} \le 6a(1 - r_0) + \frac{3a(1 - r_0)^2}{r_0} + \frac{\delta}{r}.$$

On the other hand

$$\frac{h(r)}{r^2} + \left(\frac{\pi}{2\epsilon}\right)^2 h(r) = \left(a(r - r_0)^3 + \delta r\right) \left(\frac{1}{r^2} + \left(\frac{\pi}{2\epsilon}\right)^2 \right) \ge \frac{\delta}{r} + \delta r_0 \left(\frac{\pi}{2\epsilon}\right)^2.$$

If we choose $\delta = \dfrac{12(1 + r_0)\epsilon^2}{\pi^2 r_0^2 (1 - r_0)^2 + 12(1 + r_0)\epsilon^2}$ we easily verify, recalling $a = \dfrac{1 - \delta}{(1 - r_0)^3}$, that

$$6a(1 - r_0) + \frac{3a(1 - r_0)^2}{r_0} \le \delta r_0 (\frac{\pi}{2\epsilon})^2.$$

Thus also for $r_0 < r < 1$ we have

$$\mathcal{L}_\epsilon \psi \le 0 \quad \text{in } \Omega, \qquad \psi \le g^\star \le 0 \quad \text{on } \partial\Omega.$$

Again by definition of Perron's solution we have

$$-\epsilon^2 C\, r \le -h(r) \le \psi(x) \le \phi(x) \le 0 \quad \text{in } \Omega \tag{5.3}$$

for $0 < r \le r_0$ and with $C = \dfrac{12(1 + r_0)}{\pi^2 r_0^2 (1 - r_0)^2}$.

iii) follows now easily with the help of ii):

$$\|\phi\|^2_{\Omega_{r_0}} \le \epsilon^4 2\pi C^2 \int_0^{r_0} \int_0^{r} r^3 \, dx_3 dr = \epsilon^4 C^2 \frac{2\pi}{5} r_0^5,$$

C the constant from ii).

iv) Let $\eta \in C^\infty([0, 1])$ be a cut-off function fulfilling

$$0 \le \eta \le 1, \qquad \eta(r) \equiv 1 \quad \text{for } r \in [0, r_0], \qquad \eta(r) \equiv 0 \quad \text{for } r \in [r_1, 1],$$

$$|\eta'(r)| \le \frac{2}{r_1 - r_0} \quad \text{for } r \in [0, 1]$$

with some r_1, $r_0 < r_1 < 1$, e.g. let us define $r_1 := (1+r_0)/2$. Multiplying $\mathcal{L}_\epsilon \phi = 0$ by $\eta^2\phi$ and integrating by parts we get

$$\underbrace{\int_0^{r_1}\int_0^{r} r\eta^2\left\{|\partial_r\phi|^2 + \frac{1}{\epsilon^2}|\partial_3\phi|^2 + \frac{1}{r^2}|\phi|^2\right\}\, dx_3 dr}_{(I)} + 2\int_0^{r_1}\int_0^{r} r\partial_r\phi(\partial_r\eta)\eta\phi\, dx_3 dr = 0.$$

To see that integrating by parts is possible, we recall that ϕ is smooth in $\bar{\Omega}_{r_0}\backslash\{0\}$, see [6], and use some additional arguments for the origin.

From this we estimate

$$(I) \leq \frac{1}{2}\int_0^{r_1}\int_0^{r} r\eta^2|\partial_r\phi|^2 + 2\int_0^{r_1}\int_0^{r} r\phi^2(\partial_r\eta)^2.$$

Absorbing the first term on the right-hand side in (I) we get

$$(I) \leq 4\int_0^{r_1}\int_0^{r} r\phi^2(\partial_r\eta)^2 \leq C(r_0)\epsilon^4,$$

where we have used (5.3) to bound $|\phi|^2$ by $|\phi|^2 \leq C\epsilon^4$. This yields

$$\frac{1}{\epsilon^2}\|\partial_3\phi\|^2_{\Omega_{r_0}} \leq (I) \leq C(r_0)\epsilon^4,$$

which implies

$$\|\partial_3\phi\|_{\Omega_{r_0}} \leq C(r_0)\epsilon^3. \tag{5.4}$$

□

We now define the *corrected* primary flow

$$\tilde{\mathbf{u}} = \bar{\mathbf{u}} + \mathbf{u}^\star. \tag{5.5}$$

Let us decompose $\mathbf{u}$ into $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{v}$, where $\mathbf{v}$ has now homogeneous boundary data $\mathbf{v}_{|\partial\Omega} = 0$. Note that $u_3 = v_3$. Testing the nonlinear terms in the horizontal momentum equation (2.3a) with a test function $\boldsymbol{\phi} \in \mathcal{V}$ we get

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{u}_H, \boldsymbol{\phi}) &= ((\tilde{\mathbf{u}}+\mathbf{v})\cdot\nabla(\tilde{\mathbf{u}}+\mathbf{v}_H), \boldsymbol{\phi}_H)\\ &= (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}, \boldsymbol{\phi}_H) + (\tilde{\mathbf{u}}\cdot\nabla\mathbf{v}_H, \boldsymbol{\phi}_H) + (\mathbf{v}\cdot\nabla\tilde{\mathbf{u}}, \boldsymbol{\phi}_H) + (\mathbf{v}\cdot\nabla\mathbf{v}_H, \boldsymbol{\phi}_H)\\ &= (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}, \boldsymbol{\phi}_H) + (\tilde{\mathbf{u}}\cdot\nabla\mathbf{v}_H, \boldsymbol{\phi}_H) - (\mathbf{v}\cdot\nabla\boldsymbol{\phi}_H, \tilde{\mathbf{u}}) + (\mathbf{v}\cdot\nabla\mathbf{v}_H, \boldsymbol{\phi}_H),\end{aligned}$$

where the last step follows from integration by parts and observing that $\operatorname{div}\mathbf{v} = 0$. The nonlinear term in the axial momentum equation (2.3b) can be treated as:

$$(\mathbf{u}\cdot\nabla u_3\mathbf{e}_3, \boldsymbol{\phi}) = ((\tilde{\mathbf{u}}+\mathbf{v})\cdot\nabla v_3, \varphi_3) = (\tilde{\mathbf{u}}\cdot\nabla v_3, \varphi_3) + (\mathbf{v}\cdot\nabla v_3, \varphi_3).$$

Now we define the following forms:

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\psi},\boldsymbol{\phi}) &:= \frac{1}{Re}\Big((\nabla_H\boldsymbol{\psi}_H,\nabla_H\boldsymbol{\phi}_H)+\epsilon^2(\nabla_H\psi_3,\nabla_H\varphi_3)\Big)\\
&\quad+\frac{1}{Re\,\epsilon^2}\Big((\partial_3\boldsymbol{\psi}_H,\partial_3\boldsymbol{\phi}_H)+\epsilon^2(\partial_3\psi_3,\partial_3\varphi_3)\Big)\\
&\quad+(\tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\psi}_H,\boldsymbol{\phi}_H)-(\boldsymbol{\psi}\cdot\nabla\boldsymbol{\phi}_H,\tilde{\mathbf{u}})+\epsilon^2(\tilde{\mathbf{u}}\cdot\nabla\psi_3,\varphi_3),\\
\mathbf{b}(\mathbf{w},\boldsymbol{\psi},\boldsymbol{\phi}) &:= (\mathbf{w}\cdot\nabla\boldsymbol{\psi}_H,\boldsymbol{\phi}_H)+\epsilon^2(\mathbf{w}\cdot\nabla\psi_3,\varphi_3),\\
\mathbf{l}(\boldsymbol{\phi}) &:= \frac{1}{Re}(\Delta_\epsilon\tilde{\mathbf{u}},\boldsymbol{\phi}_H)-(\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}},\boldsymbol{\phi}_H).
\end{aligned}$$

The next lemma states that these forms are well defined for functions in $\mathbf{V}$.

Lemma 5.5. $\mathbf{a},\mathbf{b},\mathbf{l}$ *are continuous bilinear, trilinear and linear forms on* $\mathbf{V}$*, respectively.*

Proof. For $\mathbf{a}(\cdot,\cdot)$ the assertion follows from the estimate

$$\begin{aligned}
|\mathbf{a}(\boldsymbol{\psi},\boldsymbol{\phi})| &\le \max\big(\frac{1}{Re\,\epsilon^2},\frac{1}{Re},\frac{\epsilon^2}{Re}\big)\,\|\nabla\boldsymbol{\psi}\|\cdot\|\nabla\boldsymbol{\phi}\|\\
&\quad+(1+\epsilon^2)\|\tilde{\mathbf{u}}\|_\infty\|\nabla\boldsymbol{\psi}\|\cdot\|\boldsymbol{\phi}\|+\|\tilde{\mathbf{u}}\|_\infty\|\nabla\boldsymbol{\phi}\|\cdot\|\boldsymbol{\psi}\|.
\end{aligned}$$

For $\mathbf{b}(\cdot,\cdot,\cdot)$ we conclude in the usual way:

$$\begin{aligned}
|\mathbf{b}(\mathbf{w},\boldsymbol{\psi},\boldsymbol{\phi})| &\le (1+\epsilon^2)\,\|\mathbf{w}\|_{L^4}\|\nabla\boldsymbol{\psi}\|\cdot\|\boldsymbol{\phi}\|_{L^4}\\
&\le C(1+\epsilon^2)\,\|\mathbf{w}\|_{H^1}\|\boldsymbol{\psi}\|_{H^1}\|\boldsymbol{\phi}\|_{H^1}.
\end{aligned}$$

The estimate for $\mathbf{l}(\cdot)$ is a little bit more involved. Let us start with $(\Delta_\epsilon\tilde{\mathbf{u}},\boldsymbol{\phi}_H)$.

$$\begin{aligned}
|(\Delta_\epsilon\tilde{\mathbf{u}},\boldsymbol{\phi}_H)| &= |(\Delta_\epsilon(\bar{\mathbf{u}}+\mathbf{u}^\star),\boldsymbol{\phi}_H)| = |(\Delta_\epsilon\bar{\mathbf{u}},\boldsymbol{\phi}_H)|\\
&\le \|\boldsymbol{\phi}_\theta\|\cdot\|\frac{x_3}{r^2}\| \le \underbrace{\sqrt{\frac{2\pi}{3}}}_{C_1}\|\boldsymbol{\phi}_H\|,
\end{aligned} \tag{5.6}$$

because of (4.7). It remains to bound $(\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}},\boldsymbol{\phi}_H)$.

$$|((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}},\boldsymbol{\phi}_H)| \le \|\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}\|\cdot\|\boldsymbol{\phi}_H\|.$$

Recall that

$$(\mathbf{a}\cdot\nabla) = a_r\partial_r + a_\theta\frac{\partial}{r\partial\theta}+a_3\partial_3.$$

As $\tilde{\mathbf{u}}$ is purely azimuthal,

$$(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} = \tilde{u}_\theta\frac{\partial}{r\partial\theta}\tilde{\mathbf{u}} = -\frac{\tilde{u}_\theta^2}{r}\cdot\mathbf{e}_r.$$

Using the triangle inequality and the definition of $\tilde{\mathbf{u}}$ we see

$$|\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}|^2 \leq 8 \left(\frac{|\bar{\mathbf{u}}|^4}{r^2} + \frac{|\mathbf{u}^\star|^4}{r^2} \right) = 8 \Bigg(\underbrace{\frac{x_3^4}{r^2}}_{\leq 1 \text{ in } \Omega} + \frac{|\phi|^4}{r^2} \Bigg).$$

To estimate the last term on the right-hand side we split the integration over Ω in a part Ω_{r_0} and the complement $\Omega \backslash \Omega_{r_0}$ with, say, $r_0 = 1/2$.

From Lemma 5.4 ii) it follows:

$$\left\| \frac{\phi^2}{r} \right\|^2_{\Omega_{r_0}} \leq C\epsilon^4 \leq C.$$

For the other part we get from Lemma 5.4 i)

$$\left\| \frac{\phi^2}{r} \right\|^2_{\Omega \backslash \Omega_{r_0}} \leq \frac{1}{{r_0}^2} \|1\|^2_{\Omega \backslash \Omega_{r_0}} \leq C.$$

Combining the above estimates we finally get

$$\|\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}\| \leq C.$$

We summarize:

$$|\mathbf{l}(\boldsymbol{\phi})| \leq \left(\frac{C_1}{Re} + C_2 \right) \|\boldsymbol{\phi}_H\|$$

with some constant C_2. □

Lemma 5.6. *The bilinear form* $\mathbf{a}$ *satisfies the coercivity condition*

$$\mathbf{a}(\mathbf{w}, \mathbf{w}) \geq \alpha \|\nabla \mathbf{w}\|^2 - \beta \|\mathbf{w}\|^2$$

for $\mathbf{w} \in \mathbf{V}$*, where* α *and* β *are positive constants depending on* ϵ *and* Re*.*

Proof. We first prove the assertion for $\mathbf{w} \in \mathcal{V}$, for general $\mathbf{w}$ it follows from a density argument. We compute

$$\mathbf{a}(\mathbf{w}, \mathbf{w}) = \frac{1}{Re} \|\nabla_H \mathbf{w}_H\|^2 + \frac{1}{Re\,\epsilon^2} \|\partial_3 \mathbf{w}_H\|^2 + \frac{\epsilon^2}{Re} \|\nabla_H w_3\|^2 + \frac{1}{Re} \|\partial_3 w_3\|^2 - (\mathbf{w} \cdot \nabla \mathbf{w}_H, \tilde{\mathbf{u}}).$$

The terms $(\tilde{\mathbf{u}} \cdot \nabla \mathbf{w}_H, \mathbf{w}_H)$, $\epsilon^2(\tilde{\mathbf{u}} \cdot \nabla w_3, w_3)$ vanish because $\tilde{\mathbf{u}}$ is divergence-free and w has compact support. Therefore we only need to estimate

$$(\mathbf{w} \cdot \nabla \mathbf{w}_H, \tilde{\mathbf{u}}) \leq \|\tilde{\mathbf{u}}\|_\infty \|\nabla \mathbf{w}_H\| \cdot \|\mathbf{w}\| \leq \frac{c}{2} \|\nabla \mathbf{w}_H\|^2 + \frac{1}{2c} \|\mathbf{w}\|^2.$$

Choosing $c = \min \left(\frac{1}{Re}, \frac{1}{Re\,\epsilon^2} \right)$ yields the desired result. □

Proceeding in the usual way to derive a weak formulation for the Navier–Stokes equations from (2.3a)–(2.3e) and taking into account the above considerations for the nonlinear terms we arrive at the following notion for a weak formulation.

Definition 5.7. Let $\mathbf{u}_0 \in \mathbf{H}$ be a divergence-free initial function. We call $\mathbf{u}$ a weak solution of problem (2.3a)–(2.3e) iff

i) $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}} \in L^\infty_{loc}(]0,\infty[;\mathbf{H}) \cap L^2_{loc}(]0,\infty[;\mathbf{V})$,

ii) for all $\boldsymbol{\phi} \in \mathbf{V}$ the following identity holds in the distributional sense:

$$\langle \partial_t \mathbf{v}, \boldsymbol{\phi}\rangle + \mathbf{a}(\mathbf{v}, \boldsymbol{\phi}) + \mathbf{b}(\mathbf{v}, \mathbf{v}, \boldsymbol{\phi}) = \mathbf{l}(\boldsymbol{\phi}),$$

iii) $\mathbf{v}$ satisfies the energy inequality

$$|||\mathbf{v}(t)|||^2 \le 2\int_0^t \Big\{\mathbf{a}(\mathbf{v}, \mathbf{v}) + \mathbf{l}(\mathbf{v})\Big\} + |||\mathbf{u}_0 - \tilde{\mathbf{u}}|||^2 \quad \text{for a.e.}\, t > 0,$$

iv) the initial condition is fulfilled in the weak L^2-sense:

$$\mathbf{u}(t,\cdot) \rightharpoonup \mathbf{u}_0 \quad \text{for } t \to 0 \quad \text{weakly in } L^2(\Omega).$$

Remark 5.8. We *require* that weak solutions $\mathbf{u}$ of the problem are such that $\mathbf{u} - \tilde{\mathbf{u}}$ are in $L^2_{loc}(]0,\infty[;\mathbf{V})$ and satisfy the energy inequality. Note that it is still an open problem whether *every* distributional solution of Navier–Stokes equations with regular boundary data fulfills the energy inequality.

Theorem 5.9. *Let* $\mathbf{u}_0 \in \mathbf{H}$. *Then there is at least one weak solution of problem* (2.3a)–(2.3e).

Proof. Owing to Lemma 5.5 and Lemma 5.6, the existence of such a weak solution $\mathbf{u}$ can be established by a Galerkin procedure in the usual way, see for instance [12]. □

6. Asymptotic stability

We can only expect the primary flow to be a good approximation for $\mathbf{u}$, if the flow is somehow laminar. This means we need an assumption on the smallness of Re. The analysis below shows that it is sufficient to assume $Re\,\epsilon \le C$. Note that $Re\,\epsilon$ can be interpreted as a Reynolds number UL/ν with typical velocity $U = R\omega$ and typical length equal to the gap width $L = R\epsilon$. Our numerical results confirm the necessity of a condition on Re. However, computationally the condition seems to be a little bit less restrictive, namely to be of the kind $Re\,\epsilon^2 \le C$, see Section 7.

We now state our analytical results on the stability up to the boundary of the corrected primary flow, under the above mentioned smallness assumption.

Theorem 6.1. *Let* $\mathbf{u}_0 \in \mathbf{H}$ *be given,* $0 < \epsilon \le 1/2$ *and* $1 \le Re \le \frac{1}{\epsilon}\sqrt{4/3}$. *Then the following estimate holds for any weak solution* $\mathbf{u}$ *and for a.e.* $t > 0$*:*

$$|||\mathbf{u}(t) - \tilde{\mathbf{u}}||| \le |||\mathbf{u}_0 - \tilde{\mathbf{u}}||| \exp\left(-\frac{t}{2\,Re\,\epsilon^2}\right) + C\,Re\,\epsilon^2.$$

For the shear strain we have the bound

$$\frac{1}{t}\int_0^t \|\partial_3(\mathbf{u}_H(s)-\tilde{\mathbf{u}})\|^2 ds \le \frac{2\,Re\,\epsilon^2}{t}|||\mathbf{u}_0-\tilde{\mathbf{u}}|||^2 + C\,Re^2\epsilon^4$$

for all $t>0$. *In the above estimates the constants* C *are independent of* ϵ *and* Re.

Remark 6.2. If we scale back to the original domain Ω_ϵ, and if we still denote the original scale velocity by $\mathbf{u}$, we obtain the relative error

$$\frac{\|\mathbf{u}(t)-\tilde{\mathbf{u}}\|_{L^2(\Omega_\epsilon)}}{\|\tilde{\mathbf{u}}\|_{L^2(\Omega_\epsilon)}} \le \frac{\|\mathbf{u}_0-\tilde{\mathbf{u}}\|_{L^2(\Omega_\epsilon)}}{\|\tilde{\mathbf{u}}\|_{L^2(\Omega_\epsilon)}} \exp\left(-\frac{t}{2\,Re\,\epsilon^2}\right) + C\,Re\,\epsilon^2.$$

The error consists of two parts, the first one decreasing exponentially with a relaxation time proportional to $Re\,\epsilon^2$, the second part does not vanish because $\tilde{\mathbf{u}}$ is not a solution of Navier–Stokes equations.

Now, due to the specific shape of the boundary corrector, we are able to obtain the same estimates for the primary flow in a vicinity of the cone apex.

Theorem 6.3. *Let* $\mathbf{u}_0 \in \mathbf{H}$ *be given,* $0<\epsilon\le 1/2$, $0<r_0<1$ *and* $1\le Re\le \frac{1}{\epsilon}\sqrt{\frac{4}{3}}$. *Then the following estimate holds for any weak solution* $\mathbf{u}$ *and for a.e.* $t>0$*:*

$$|||\mathbf{u}(t)-\bar{\mathbf{u}}|||_{\Omega_{r_0}} \le |||\mathbf{u}_0-\tilde{\mathbf{u}}|||\exp\left(-\frac{t}{2\,Re\,\epsilon^2}\right) + C\,Re\,\epsilon^2.$$

For the shear strain we have the bound

$$\frac{1}{t}\int_0^t \|\partial_3\mathbf{u}_H(s)-\mathbf{e}_\theta\|^2_{\Omega_{r_0}} ds \le \frac{2\,Re\,\epsilon^2}{t}|||\mathbf{u}_0-\tilde{\mathbf{u}}|||^2 + C\,Re^2\epsilon^4$$

for all $t>0$. *In the above estimates the constants* $C=C(r_0)$ *are independent of* ϵ *and* Re.

Proof. The statement is a direct consequence of Theorem 6.1, the triangle inequality and Lemma 5.4 iii), iv) respectively. Note that we have made the assumption $Re\ge 1$, which is the "bad" case of large Reynolds numbers. □

This last result explains why the primary flow is a good approximation of the true solution *in the vicinity of the cone apex*, when $Re\,\epsilon^2$ is small enough, according to physical experiments [9] and it explains also why the CPA device can be used as a viscometer.

Proof of Theorem 6.1. The basic idea of the proof is to take advantage of the very high vertical diffusion terms which predominates due to the factor $\frac{1}{Re\,\epsilon^2}\gg 1$. We start with the energy inequality iii) of Definition 5.7: For a.e. $t > 0$

$$|||\mathbf{v}(t)|||^2 + \frac{1}{Re}\int_0^t \Big\{|||\nabla_H \mathbf{v}|||^2 + \frac{1}{\epsilon^2}|||\partial_3 \mathbf{v}|||^2\Big\} \le \int_0^t \Big\{\mathbf{l}(\mathbf{v}) + (\mathbf{v}\cdot\nabla\mathbf{v}_H, \tilde{\mathbf{u}})\Big\} + |||\mathbf{u}_0 - \tilde{\mathbf{u}}|||^2. \tag{6.1}$$

The terms $(\tilde{\mathbf{u}}\cdot\nabla\mathbf{v}_H, \mathbf{v}_H)$, $\epsilon^2(\tilde{\mathbf{u}}\cdot\nabla v_3, v_3)$, $(\mathbf{v}\cdot\nabla\mathbf{v}_H, \mathbf{v}_H)$ and $\epsilon^2(\mathbf{v}\cdot\nabla v_3, v_3)$ vanish because both $\mathbf{v}$ and $\tilde{\mathbf{u}}$ are divergence-free and $\mathbf{v}$ is zero on the boundary.

The right-hand side of (6.1) was already estimated in the proof of Lemma 5.5. Thus we conclude the following energy estimate in the distributional sense:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||\mathbf{v}|||^2 &+ \frac{1}{Re}(\|\nabla_H \mathbf{v}_H\|^2 + \epsilon^2\|\nabla_H v_3\|^2) + \frac{1}{Re\,\epsilon^2}(\|\partial_3\mathbf{v}_H\|^2 + \epsilon^2\|\partial_3 v_3\|^2)\\ &\le C_3\|\mathbf{v}_H\| + \|\mathbf{v}_H\|\cdot\|\nabla_H\mathbf{v}_H\| + \|v_3\|\cdot\|\partial_3\mathbf{v}_H\|\end{aligned} \tag{6.2}$$

with $C_3 = \frac{C1}{Re} + C_2$, the constants C_1, C_2 as in Lemma 5.5.

The last two terms on the right-hand side of (6.2) can be estimated by Young's inequality, the vertical Poincaré inequality which for this particular domain reads $\|\mathbf{w}\| \le 1/\sqrt{2}\|\partial_3\mathbf{w}\|$ for $\mathbf{w}\in\mathbf{V}$, and the condition $Re\,\epsilon \le \sqrt{4/3}$:

$$\begin{aligned}\|\mathbf{v}_H\|\cdot\|\nabla_H\mathbf{v}_H\| &\le \frac{1}{Re}\|\nabla_H\mathbf{v}_H\|^2 + \frac{Re}{4}\|\mathbf{v}_H\|^2\\ &\le \frac{1}{Re}\|\nabla_H\mathbf{v}_H\|^2 + \frac{Re}{8}\|\partial_3\mathbf{v}_H\|^2\\ &\le \frac{1}{Re}\|\nabla_H\mathbf{v}_H\|^2 + \frac{1}{6Re\,\epsilon^2}\|\partial_3\mathbf{v}_H\|^2\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\|v_3\|\cdot\|\partial_3\mathbf{v}_H\| &\le \frac{1}{3Re\,\epsilon^2}\|\partial_3\mathbf{v}_H\|^2 + \frac{3Re\,\epsilon^2}{4}\|v_3\|^2\\ &\le \frac{1}{3Re\,\epsilon^2}\|\partial_3\mathbf{v}_H\|^2 + \frac{3Re\,\epsilon^2}{8}\|\partial_3 v_3\|^2\\ &\le \frac{1}{3Re\,\epsilon^2}\|\partial_3\mathbf{v}_H\|^2 + \frac{1}{2Re}\|\partial_3 v_3\|^2.\end{aligned} \tag{6.4}$$

Absorbing the quadratic terms (6.3), (6.4) on the left-hand side of (6.2) and then throwing away the terms with horizontal gradients we arrive at

$$\begin{aligned}\frac{d}{dt}|||\mathbf{v}|||^2 &+ \frac{1}{Re\,\epsilon^2}\big(\|\partial_3\mathbf{v}_H\|^2 + \epsilon^2\|\partial_3 v_3\|^2\big)\\ &\le 2\,C_3\|\mathbf{v}_H\| \le \frac{1}{Re\,\epsilon^2}\|\mathbf{v}_H\|^2 + C_3^2 Re\,\epsilon^2.\end{aligned} \tag{6.5}$$

Using again the vertical Poincaré inequality in (6.5) yields

$$\frac{d}{dt}|||\mathbf{v}|||^2 + \frac{1}{Re\,\epsilon^2}|||\mathbf{v}|||^2 \le C_3^2 Re\,\epsilon^2, \tag{6.6}$$

With the help of Gronwall's Lemma we conclude

$$|||\mathbf{v}(t)|||^2 \le |||\mathbf{u}_0 - \tilde{\mathbf{u}}|||^2 \exp\left(\frac{-t}{Re\,\epsilon^2}\right) + C_3^2 Re^2\,\epsilon^4.$$

The estimate for the shear stress is now a direct consequence of the above bounds: From (6.5) we can derive in the same way

$$\frac{d}{dt}|||\mathbf{v}|||^2 + \frac{1}{2\,Re\,\epsilon^2}|||\partial_3 \mathbf{v}|||^2 \le C_3^2 Re\,\epsilon^2.$$

Integrating this relation with respect to time from 0 to t and using the estimate for $|||\mathbf{v}(t)|||$ we get the stated result. This completes the proof of Theorem 6.1. □

7. Numerical simulations

In this section we present some numerical results to study the behavior of the solution and compare with the estimates of the analysis.

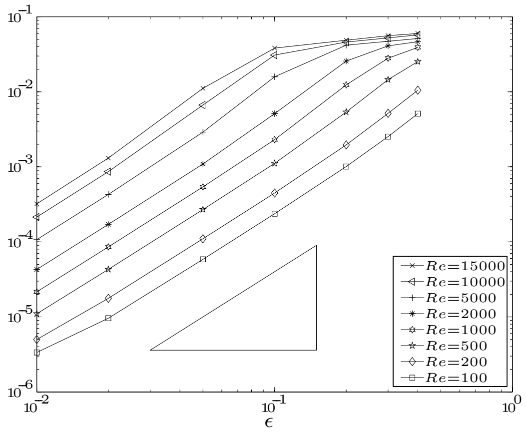

FIG. 7.1. $|||\mathbf{u} - \bar{\mathbf{u}}|||_{\Omega_{r_0}}$ versus ϵ, $r_0 = 0.5$; the triangle has slope 2.

Since for small Reynolds numbers there is a unique stationary limit which in turn is axisymmetric it is convenient to numerically solve for axisymmetric solutions of problem (2.3a)–(2.3e). To this end we use the axisymmetric version [13] of the method and the implementation described in [1, 2]. The code is based on a finite element discretization by the *Taylor–Hood* element in space and by a variant of the fractional step θ-scheme, an operator splitting for time discretization. The axisymmetric code solves for the unknowns $(u_s(r,x_3), u_\beta(r,x_3), u_3(r,x_3))$ and $p(r,x_3)$, where $u_s := \frac{u_r}{r}$, $u_\beta := \frac{u_\theta}{r}$. This scaling allows for a proper variational formulation of the axisymmetric Navier–Stokes equation in appropriately r-weighted Sobolev spaces, see also [8].

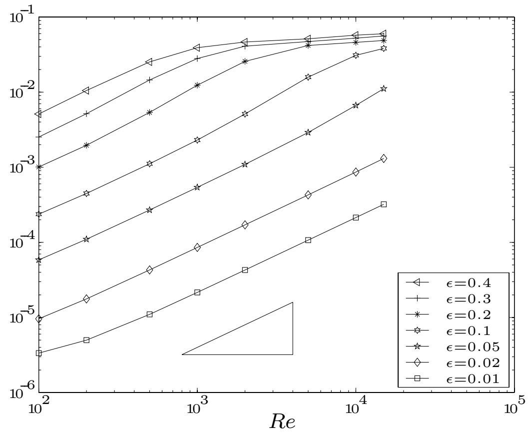

FIG. 7.2. $|||\mathbf{u}-\bar{\mathbf{u}}|||_{\Omega_{r_0}}$ versus Re, $r_0 = 0.5$; the triangle has slope 1.

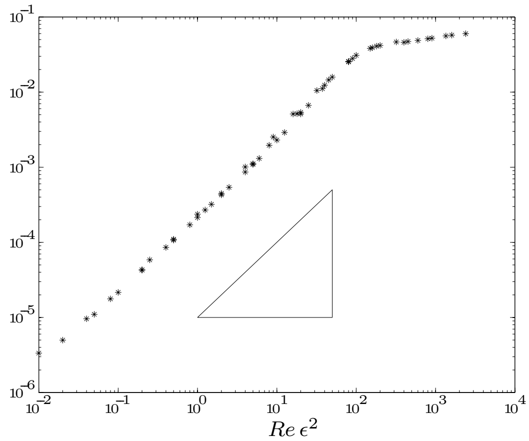

FIG. 7.3. $|||\mathbf{u}-\bar{\mathbf{u}}|||_{\Omega_{r_0}}$ versus $Re\,\epsilon^2$, $r_0 = 0.5$; the triangle has slope 1.

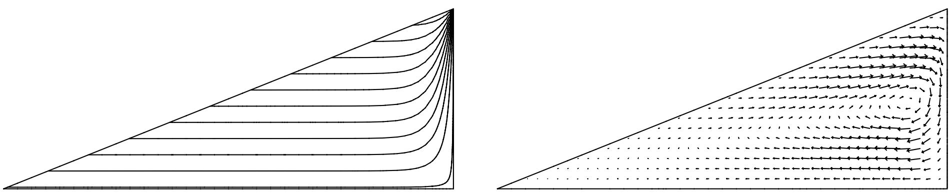

FIG. 7.4. Level lines of u_θ (left) and meridional flow $(u_r, \epsilon\, u_3)$ (right); $\epsilon = 0.1$, $Re = 100$, $\|(u_r, \epsilon\, u_3)\|_\infty = 0.589\, e\text{-}2$ (scale of x_2-axis exaggerated).

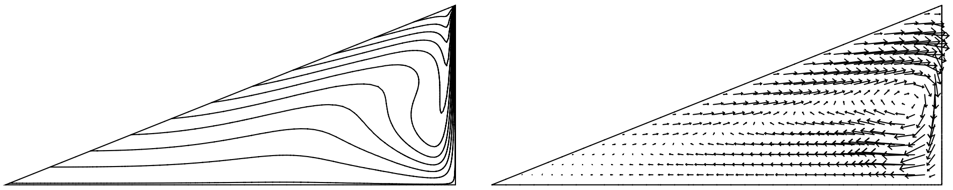

FIG. 7.5. Level lines of u_θ (left) and meridional flow $(u_r, \epsilon\, u_3)$ (right); $\epsilon = 0.1$, $Re = 5000$, $\|(u_r, \epsilon\, u_3)\|_\infty = 0.133$ (scale of x_2-axis exaggerated).

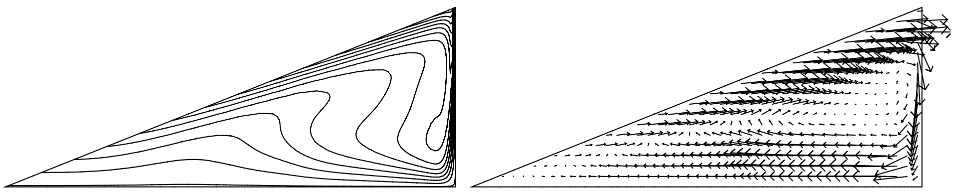

FIG. 7.6. Level lines of u_θ (left) and meridional flow $(u_r, \epsilon\, u_3)$ (right); $\epsilon = 0.1$, $Re = 15000$, $\|(u_r, \epsilon\, u_3)\|_\infty = 0.126$ (scale of x_2-axis exaggerated).

The numerical examples were carried out on a sequence of successively refined meshes to make sure that the discretization error is sufficiently small.

We performed simulations for all

$$(Re, \epsilon) \in \{100, 200, 500, 1000, 5000, 10000, 15000\} \times \{0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.4\}.$$

The time evolution was run until a steady state was reached. Figures 7.1–7.3 show the difference of the stationary limit for $\mathbf{u}$ to the primary flow $\bar{\mathbf{u}}$ with respect to ϵ, Re and $Re\, \epsilon^2$, respectively. The numerical results confirm the estimate of Theorem 6.3 very well. From Figure 7.3 it can be seen that the behavior is determined by the single parameter $Re\, \epsilon^2$. Figures 7.4–7.6 show the stationary solution on an axisymmetric cut of Ω for fixed $\epsilon = 0.1$ and different values of Re. Clearly the influence of increasing Re on the secondary flow can be seen.

A closer quantitative analysis of the data in Figures 7.1–7.3 suggests that the "threshold", for which the decay of the error in terms of $Re\, \epsilon^2$ is linear, is given by a condition like $Re\, \epsilon^2 \leq C$, $C \approx 150$. This is somewhat in contrast to the assumption made for the analysis, namely that $Re\, \epsilon$ has to be small, which is a little bit more restrictive.

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Pascal Azerad
Département de Mathématiques
Université de Montpellier II
F-34095 Montpellier cedex 5
France
e-mail: azerad@math.univ-montp2.fr

Eberhard Bänsch
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39
D-10117 Berlin
Germany
and
Freie Universität
Berlin
Germany
e-mail: baensch@wias-berlin.de

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