Solutions of the Hamilton-Jacobi-Bellman Equation and Invariance Domains

Continuous and discontinuous value functions

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Framework

$$(\Sigma): \quad \dot{x}(t) = f(x(t), u(t)), \quad x(t) \in \mathrm{X} \subset \mathbb{R}^n, \quad u(t) \in \mathrm{U} \subset \mathbb{R}^p$$
$$\text{where } u(\cdot) \in \mathcal{U}_{[t_0, T]} := \{u(\cdot) \text{ meas. } [t_0, T] \mapsto \mathrm{U}\}$$

Hypotheses H1.

1. U compact
2. f continuous in (x, u)
3. f Lipschitz in x, uniformly in u i.e. $\forall x \in \mathrm{X}$, $\forall R > 0$, $\exists L > 0$

$$||f(x, u) - f(y, u)|| \leq L||x - y||, \ \forall y \in \mathbb{B}(x, R) \cap \mathrm{X}, \ \forall u \in \mathrm{U}$$

4. f linear growth i.e. $\exists C > 0$ s.t.

$$||f(x, u)|| \leq C(1 + ||x||), \ \forall (x, u) \in \mathrm{X} \times \mathrm{U}$$

▶ For any $(t_0, x_0) \in (-\infty, T] \times \mathrm{X}$ and $u(\cdot) \in \mathcal{U}_{[t_0, T]}$, there exists an **unique** $x(\cdot)$ abs. cont. solution of (Σ) for a.e. $t \in [t_0, T]$.

Set of trajectories

$$\mathcal{S}_{[t_0,T]}(x_0) = \{x(\cdot) \in AC([t_0,T],\mathrm{X}) \text{ s.t. } (\Sigma) \text{ a.e. } t \in [t_0,T]\}_{u(\cdot \in \mathcal{U}_{[t_0,T]}}$$

Hypotheses H2. For any $x \in \mathrm{X}$,

$$f(x,\mathrm{U}) = \bigcup_{u \in \mathrm{U}} f(x,u) \text{ is convex.}$$

Proposition 1 . Under H1, H2, for any $(t_0,x_0) \in (-\infty,T] \times \mathrm{X}$, $\mathcal{S}_{[t_0,T]}(x_0)$ is compact (for the $||\cdot||_\infty$ norm).

Then for any $\phi : \mathrm{X} \mapsto \mathbb{R} \cup \{+\infty\}$ lower semi-continuous,

$$V(t_0,x_0) = \inf_{u(\cdot)\in\mathcal{U}_{[t_0,T]}} \phi(x_{t_0,x_0,u(\cdot)}(T)) = \min_{x(\cdot)\in\mathcal{S}_{[t_0,T]}(x_0)} \phi(x(T))$$

Proposition 2. Under H1, H2, the map

$$\begin{array}{ccc} u & \longmapsto & x_{t_0,x_0,u(\cdot)} \\ L^1([t_0,T],\mathrm{U}) & & C^0([t_0,T],\mathrm{X}) \end{array} \text{ is continuous}$$

The piecewise C^1 case

Generic transversality of trajectories

Definition. Let $x(\cdot)$ be an A.C. sol. in $\mathbb{R}^n$ of $\dot{x} = g(t,x)$ and $\mathcal{M}$ a C^1 manifold of $\mathbb{R}^{n+1}$: $\mathcal{M} = \{(t,x) \in \mathbb{R}^{n+1}\,,\ \mu(t,x) = 0\}$.
We say that $x(\cdot)$ *intersects* $\mathcal{M}$ *transversally* at $(t, x(t)) \in \mathcal{M}$ if

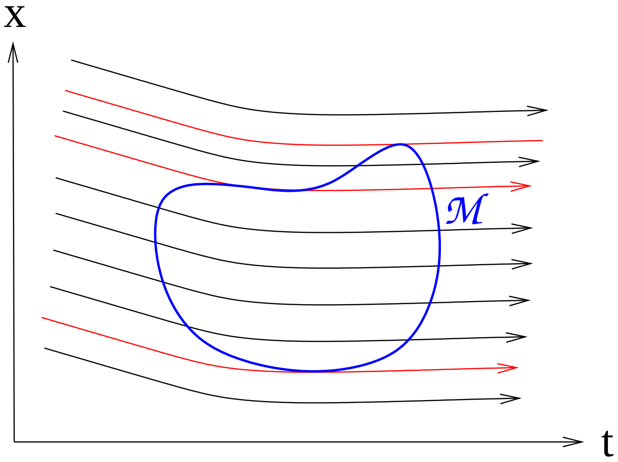
$$\mu_t(t, x(t)) + \mu_x(t, x(t))\, g(t, x(t)) \neq 0$$

Proposition. Assume g is C^1. Let $\mathcal{O}$ be an open set of $\mathbb{R}^n$, $[t_0, t_1]$ s.t. for any $x_0 \in \mathcal{O}$, $x_{(t_0,x_0)}(\cdot)$ is defined on $[t_0, t_1]$, and

$$\mathcal{N} = \{x_0 \in \mathcal{O}\,, \exists t \in [t_0, t_1] \text{ s.t. } (t, x_{(t_0,x_0)}(t)) \text{ intersects } \mathcal{M} \text{ non transversally}\}$$

Then, $\mathcal{N}$ is of null measure.

Generic transversality of trajectories



Sufficient conditions for the Mayer problem

Proposition. If there exists $W \in \mathcal{C}^0([\bar{t}, T] \times \mathrm{X}, \mathbb{R})$ s.t.

1. $W(t,x) \geq V(t,x)$ for any $(t,x) \in [\bar{t}, T] \times \mathrm{X}$
2. $W(T,x) = \phi(x)$ for any $x \in \mathrm{X}$
3. There exists a finite number of C^1 manifolds $\mathcal{M}_i$ s.t.

$$(t,x) \notin \bigcup_i \mathcal{M}_i \Rightarrow W \text{ is } C^1 \text{ and } W_t(t,x) + \bar{H}(x, \nabla W(t,x)) = 0$$

Then, $W = V$ on $[\bar{t}, T] \times \mathrm{X}$.

Sketch of proof

1. Let (t_0, x_0) and $u^\star(\cdot) \in \mathcal{U}_{[t_0,T]}$ be such that

$$V(t_0, x_0) \leq W(t_0, x_0) - \epsilon \quad \text{and} \quad \phi(x^\star(T)) < V(t_0, x_0) + \epsilon/2$$

W cont. $\Rightarrow \exists \eta > 0,\ ||\xi - x_0|| < \eta \Rightarrow W(t_0, \xi) > V(t_0, x_0) + \epsilon/2$

2. Take a sequence of **piecewise constant** $v_n(\cdot)$ such that $\lim_{n\to\infty} \int_{t_0}^{T} ||v_n(t) - u^\star(T-t)||\, dt = 0$ and consider

$$\dot{y}_n = -f(y_n, v_n(t)), \quad y_n(0) = x^\star(T)$$

By continuity w.r.t. to the control, $x_n(\cdot) = y_n(T - \cdot) \xrightarrow{\text{unif}} x^\star(\cdot)$.

Sketch of proof

3. $x_n(\cdot)$ constant on $[t_j, t_{j+1}) \Rightarrow \exists \xi_k \to x_n(t_j)$ s.t.

$$\dot{\tilde{x}}_k = f(\tilde{x}_k, u_n(t_j)),\ \tilde{x}_k(t_j) = \xi_k \Rightarrow \tilde{x}_k(t) \notin \bigcup_i \mathcal{M}_i\,,\ \text{a.e. } t \in [t_j, t_{j+1}]\,.$$

4. $W(\cdot)$ being solution of the the H.J.B. equation outside $\bigcup_i \mathcal{M}_i$,

$$W(t_{j+1}, \tilde{x}_k(t_{j+1})) \geq W(t_j, \tilde{x}_k(t_j)) \overset{k\to+\infty}{\Longrightarrow} W(t_{j+1}, x_n(t_{j+1})) \geq W(t_j, x_n(t_j))$$

$$\overset{\sum_j}{\Longrightarrow} W(T, x_n(T)) \geq W(t_0, x_n(t_0))$$

5. But $W(T, x_n(T)) = \phi(x^\star(T))$ and for n large enough, one has $||x_n(t_0) - x^\star(t_0)|| < \eta$

$$\Rightarrow\ W(t_0, x_n(t_0)) < V(t_0, x_0) + \epsilon/2 : \textit{contradiction...}$$

Sufficient conditions for the minimal time problem

Proposition. If there exists $W \in \mathcal{C}(0, \mathbb{R})$, where $\mathcal{O}$ is an open set that contains $\mathcal{T}$, s.t.

1. $W(x) \geq V(x)$ for any $x \in \mathcal{O}$
2. $W(x) = 0$ for any $x \in \mathcal{T}$
3. There exists a finite number of C^1 manifolds $\mathcal{M}_i$ s.t.

$$x \in \mathcal{O} \setminus \bigcup_i \mathcal{M}_i \Rightarrow W \text{ is } C^1 \text{ and } 1 + H(x, \nabla W(x)) = 0$$

4. For any $x \in \partial\mathcal{O}$, one has

$$W(x) = \sup_{\xi \in \mathcal{O}} W(\xi)$$

Then, $W = V$ on $\mathcal{O}$.

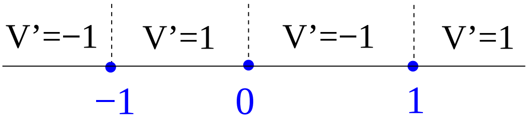
Come back to Example 3

$$\dot{x} = u \in [-1,1]\,, \quad x(t_0) = x_0 \in \mathbb{R} \quad \to \inf_{u(\cdot)} \{t_f | x(t_f) \in \{-1,1\}\}$$

$V(x) = ||x|-1|$ is the cost function for the control

$$u^\star = \left| \begin{array}{rl} 1, & x_0 \in]-\infty,-1[\bigcup[0,1[\\ -1, & x_0 \in]-1,0]\bigcup]1,+\infty[\end{array} \right.$$

and fulfills $1 - |V'(x)| = 0$ outside $\{-1,0,1\}$.



Remark. Other functions

$$W(x) = \int_{-1}^{x} w(\xi) d\xi$$

where $w(\cdot)$ is piecewise constant in $L_1(\mathbb{R},\{-1,1\})$ with $W(1)=0$ are not the cost of admissible trajectories.

The servo-mecanism problem without the PMP

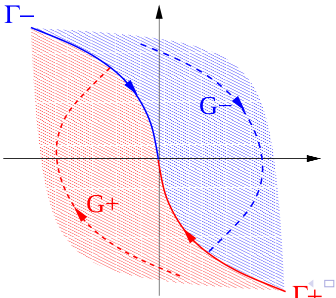
$$\dot{x} = \begin{bmatrix} x_2 \\ u(t) \end{bmatrix}, \quad u(t) \in \mathrm{U} = [-1, 1], \quad \mathcal{T} = \{(0,0)\}$$

Particular solutions that arrives at $\mathcal{T}$ with no commutation :

$$\Gamma_+ = \left\{ x \,|\, x_1 = \frac{1}{2} x_2^2 \,,\, x_2 < 0 \right\},\ \Gamma_- = \left\{ x \,|\, x_1 = -\frac{1}{2} x_2^2 \,,\, x_2 > 0 \right\}$$

Domains from which the constants controls $u = \pm 1$ reach $\Gamma_\mp$:

$$G_+ = \left\{ x \,|\, x_1 < -\mathrm{sgn}(x_2) \frac{1}{2} x_2^2 \right\},\ G_- = \left\{ x \,|\, x_1 > -\mathrm{sgn}(x_2) \frac{1}{2} x_2^2 \right\}$$



The servo-mecanism problem without the PMP

$$\psi^\star(x) = \left| \begin{array}{ll} +1 & \text{si } x \in \Gamma_+ \cup G_+ \\ -1 & \text{si } x \in \Gamma_- \cup G_- \end{array} \right. \quad \text{gives}$$

$$W(x) = \left| \begin{array}{ll} W_+(x) = 2\sqrt{\frac{1}{2}x_2^2 - x_1} - x_2 & x \in \Gamma_+ \cup G_+ \\ \\ W_-(x) = 2\sqrt{\frac{1}{2}x_2^2 + x_1} + x_2 & x \in \Gamma_- \cup G_- \end{array} \right.$$

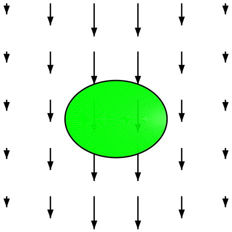
which is C^1 outside $\Gamma_+ \cup \Gamma_-$ with

$$1 + \partial_{x_1} W(x) x_2 - |\partial_{x_2} W(x)| = 0, \quad x \notin \Gamma_+ \cup \Gamma_-$$

Regularity of the value function

The swimmer problem

A swimmer wants to reach as fast as possible an **island**, where $\vec{v}_c(\cdot)$ is the river stream vector :



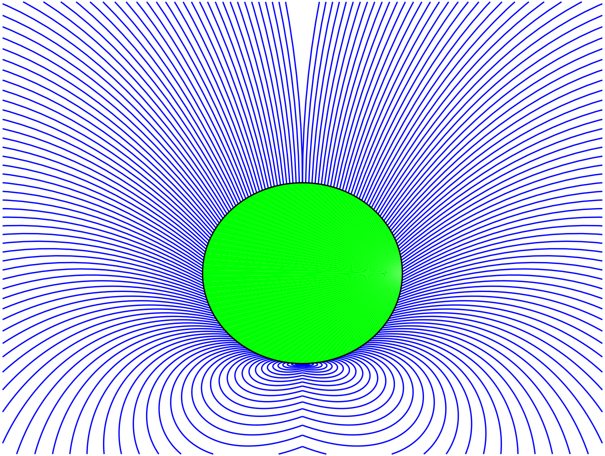
Swimmer dynamics :

$$\left\{ \begin{array}{lcl} \dot{x} & = & v_s \cos(u) \\ \dot{y} & = & v_s \sin(u) - v_c(x) \end{array} \right.$$

where v_s is the swimmer speed and $u \in [0, 2\pi)$ is the control.

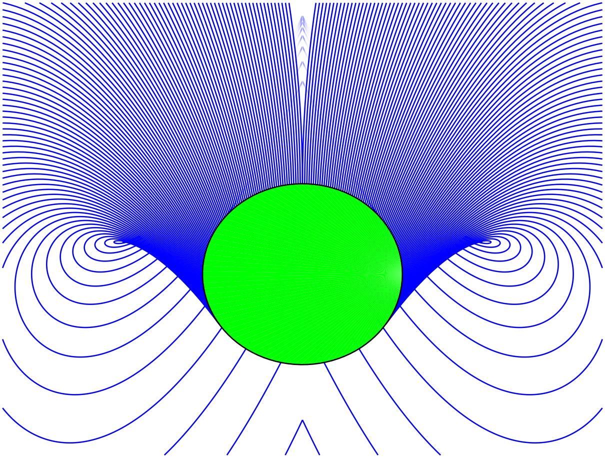
Optimal trajectories

$$X = [-2, 2] \times \mathbb{R} \,, \quad v_c(x) = 2 - |x| \,, \quad v_s > 2$$



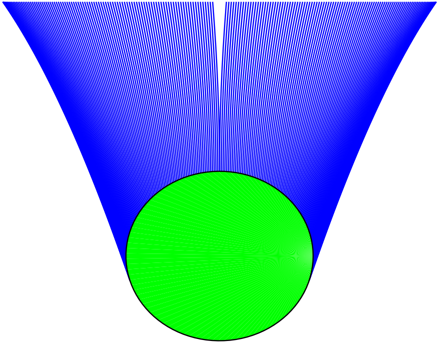
Optimal trajectories

$$X = [-2, 2] \times \mathbb{R} \,, \quad {\color{blue} v_c(x) = 2 - |x|} \,, \quad {\color{red} v_s < 2}$$



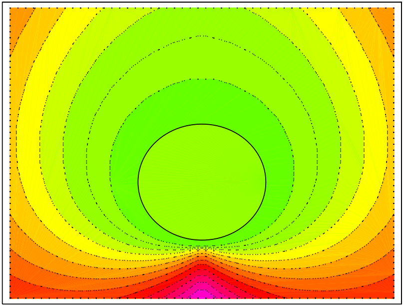
Optimal trajectories

$$X = [-2, 2] \times \mathbb{R}\ , \quad v_c(x) = 3 - |x|\ , \quad v_s < 1$$

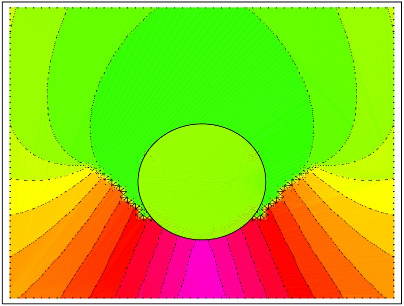


Value functions

$v_s > 2$:



$v_s < 2$:



The differential inclusion framework

$$\dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [t_0, T], \quad x(t_0) = x_0$$

A1. F is a *Marchaud* map :

1. For any $x \in \mathrm{X}$, $F(x)$ is a non-empty convex compact set
2. F is upper semi-continuous
3. F is linear growth i.e. there exists $C > 0$ s.t.

$$||v|| \leq C(1 + ||x||), \quad \forall v \in F(x),\ \forall x \in \mathrm{X}$$

A2. F is *Lipschitz* i.e. for any $x \in \mathrm{X}$ et $R > 0$, there exists $L > 0$

$$y \in \mathbb{B}(x, R) \cap \mathrm{X} \Rightarrow F(y) \subseteq F(x) + \mathbb{B}(0, L||x - y||)\ .$$

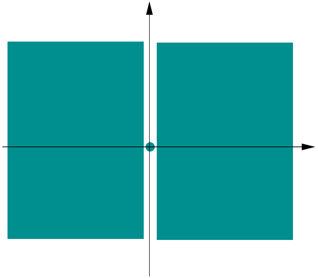
Semi-continuity for set-valued maps

- **F upper semi-continuous :** for any $x \in X$ and $\epsilon > 0$ there exits $\eta > 0$ s.t.

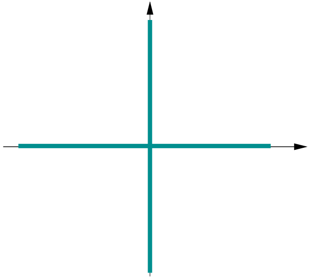
$$||y - x|| \leq \eta \Rightarrow F(y) \subset F(x) + \mathbb{B}(0, \epsilon)$$

- **F lower semi-continuous :** for any $x \in X$,

$$\forall x_n \to x, \; \exists y_n \in F(x_n) \text{ s.t. } y_n \to F(x)$$



l.s.c. but not u.s.c.



u.s.c. but not l.s.c.

Tools

Filippov lemma. Under H1, H2, the set of a.c. solutions of

$$\dot{x} \underset{\text{a.e.}}{\in} F(x) := \bigcup_{u \in \mathrm{U}} f(x, u), \quad x(t_0) = x_0$$

is exactly $\mathcal{S}_{[t_0,T]}(x_0)$.

Theorem of compactness of approximate trajectories.
Under A1 only, let $x_n(\cdot)$ be a sequence of a.c. functions s.t.

$$\dot{x}_n(t) \in F(x_n(t) + \xi_n(t)) + \mathbb{B}(0, r_n(t)) \text{ a.e. } t \in [t_0, T]$$

with $x_n(t_0) = x_0^n \to x_0$, $\xi_n(\cdot)$ and $r_n(\cdot)$ are measurable and converge to 0 in L^2.
Then, there exists a sub-sequence that converges *uniformly* to $x(\cdot)$, and whose derivatives converge *weakly* to $\dot{x}(\cdot)$, where $x(\cdot)$ belongs to $\mathcal{S}_{[t_0,T]}(x_0)$.

Mayer problem

Under Hypotheses H1, H2,

Proposition 1.

$\phi_T : \mathrm{X} \mapsto \mathbb{R} \cup \{+\infty\}$ s.c.i. $\Rightarrow V : (-\infty, T] \times \mathrm{X} \mapsto \mathbb{R} \cup \{+\infty\}$ s.c.i.

Proposition 2.

$$\phi_T : \mathrm{X} \mapsto \mathbb{R} \text{ Lipschitz } \Rightarrow V : (-\infty, T] \times \mathrm{X} \mapsto \mathbb{R} \text{ Lipschitz}$$

Reachable sets

Let $\mathcal{T} \subset \mathrm{X}$ be a closed set and V be the minimal time function

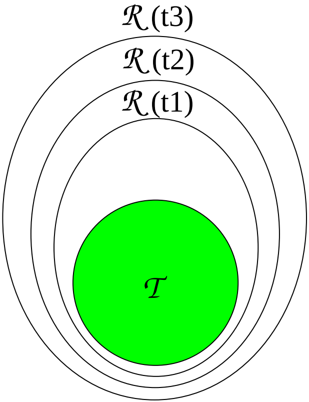
Definitions.

- reachable set at time t : $\mathcal{R}(t) := \{x \in \mathrm{X}\,,\ V(x) < t\}$
- reachable set : $\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t)$

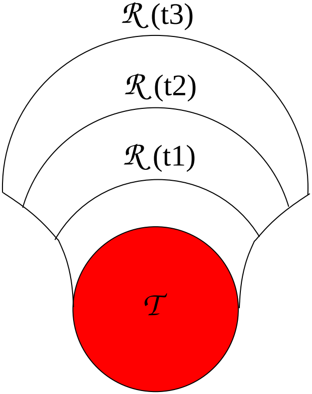
Definition. F (or (f, U)) is small-time controllable (STC) on $\mathcal{T}$ if

$$\mathcal{C} \subseteq \mathrm{int}\mathcal{R}(t)\,, \quad \forall t > 0\ .$$

The small time local controllability of the target



small−time controllable



non small−time controllable

The small time local controllability

A3. $\mathcal{T}$ is closed and $\partial\mathcal{T}$ is compact.

Lemma. Under A1, A2, A3, the following assertions are equivalent

1. $(f, \mathcal{U})$ is STC on $\mathcal{T}$
2. V is continuous at $\partial\mathcal{T}$
3. there exists $\delta > 0$ and a map $\omega : [0, \delta] \mapsto \mathbb{R}_+$ s.t. $\lim_{d \to 0} \omega(d) = 0$ and $V(x) \leq \omega(\mathrm{d}_{\mathcal{T}}(x))$ for any $x \in \mathbb{B}(\mathcal{T}, \delta)$.

Proposition. Under A1, A2, A3, if $(f, \mathcal{U})$ is STC on $\mathcal{T}$, then

1. $\mathcal{R}$ is open
2. V is continuous on $\mathcal{R}$
3. one has $\lim_{x \in \mathcal{R} \to \partial\mathcal{R}} V(x) = +\infty$

The Petrov condition

A3. $\mathcal{T}$ is closed with non empty interior. $\partial\mathcal{T}$ is a C^2 compact manifold.

Definitions. The oriented distance function to $\mathcal{T}$ is

$$\mathrm{d}^o_{\mathcal{T}}(x) = \left| \begin{array}{rl} \mathrm{d}_{\mathcal{T}}(x) & \text{si } x \notin \mathcal{T} \\ -\mathrm{d}_{\partial\mathcal{T}}(x) & \text{si } x \in \mathcal{T} \end{array} \right.$$

and the exterior normal is $n_{\mathcal{T}}(x) = \nabla \mathrm{d}^o_{\mathcal{T}}(x)\,,\ x \in \partial\mathcal{T}$

Proposition Under A1, A2, A4 and the Petrov condition

$$\overline{H}(x, n_{\mathcal{T}}(x)) < 0\,, \quad \forall x \in \partial\mathcal{T}$$

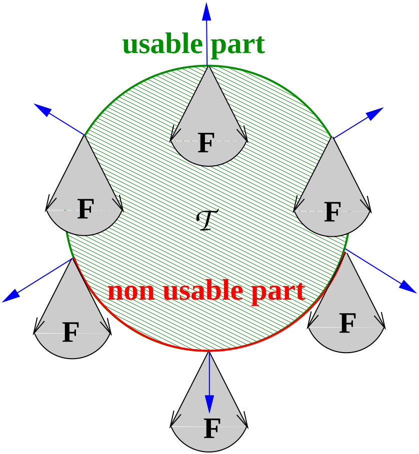
the value function of the minimum time problem is continuous on $\mathcal{R}$ and Lipschitz in a neighborhood of $\mathcal{T}$

The Petrov condition

Definition The **usable part** of the boundary of the target is

$$\sqcap(\mathcal{T}) := \{x \in \partial\mathcal{T}(x) \,|\, \exists v \in F(X),\ v.n_{\mathcal{T}}(x) < 0\}$$

Petrov condition : $\sqcap(\mathcal{T}) = \partial\mathcal{T}$



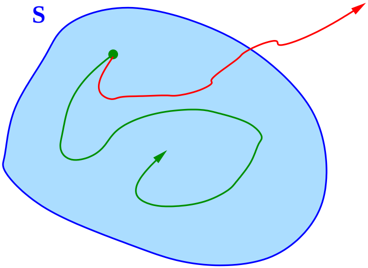
Invariant domains

Invariant domains

Consider $F : \mathrm{X} \rightrightarrows \mathbb{R}^n$ and S closed subset of X.

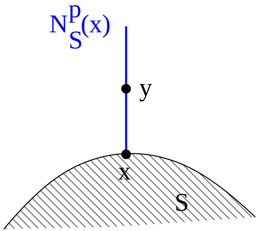
Definitions.

- (S, F) is weakly invariant (or viable) if for any $x_0 \in S$, there exists $x(\cdot) \in \mathcal{S}_{[t_0,+\infty)}(x_0)$ such that $x(t) \in S$ for any $t > t_0$.
- (S, F) is (strongly) invariant if for any $x_0 \in S$, any $x(\cdot) \in \mathcal{S}_{[t_0,+\infty)}(x_0)$ such that $x(t) \in S$ for any $t > t_0$.



S is weakly but not strongly invariant

Notions of proximal analysis



Let S be a closed subset of $\mathbb{R}^n$.

Proximal normal cone at S in x :

$$N_S^p(x) := \{t(y-x) \,|\, x \in \mathrm{Proj}_S(y)\,,\ y \notin S\,,\ t \geq 0\}$$

Bouligand tangent cone at $x \in S$:

$$T_S^B(x) := \left\{ \lim_{n\to+\infty} \frac{x_n - x}{t_n}\,,\ x_n \in S\,,\ x_n \to x\,,\ t_n > 0\,,\ t_n \to 0 \right\}\ .$$

Proposition. $\forall v \in T_S^B(x),\ \forall \xi \in N_S^p(x),\ v.\xi \leq 0$

Proximal sub-differentials

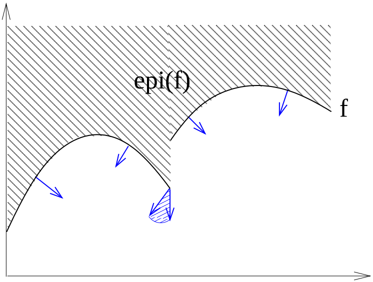
Let $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ s.c.i.

Epigraph of f : $\text{epi}(f) := \{(x, z) \in \mathbb{R}^n \times \bar{\mathbb{R}} \,|\, z \geq f(x)\}$

Property. f s.c.i $\Leftrightarrow$ epi(f) closed

Proximal sub-diffential :

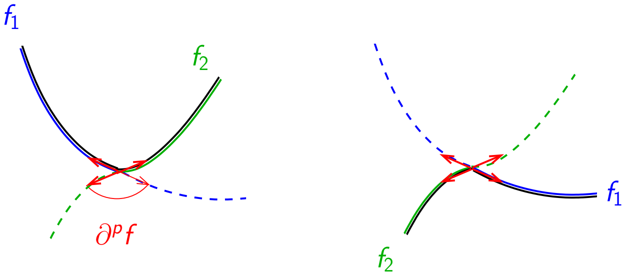
$$\partial^p f(x) = \{\xi \in \mathbb{R}^n \,|\, (\xi, -1) \in N^p_{\text{epi}(f)}(x, f(x))\}$$



Proximal sub-differentials

- $\partial^p f(x) \neq \emptyset \;\Rightarrow\; \partial^p f(x)$ convex
- f C^2 at $x \Rightarrow \partial^p f(x) = \{\nabla f(x)\}$
- $\mathrm{epi}(f_1 \vee f_2) = \mathrm{epi}(f_1) \cap \mathrm{epi}(f_2)$, $\mathrm{epi}(f_1 \wedge f_2) = \mathrm{epi}(f_1) \cup \mathrm{epi}(f_2)$

$$\left\{ \begin{array}{l} f_1, f_2 \; C^2 \text{ at } \bar{x} \\ f_1(\bar{x}) = f_2(\bar{x}) \\ \nabla f_1(\bar{x}) \neq \nabla f_2(\bar{x}) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \partial^p (f_1 \vee f_2)(\bar{x}) = \overline{\mathrm{co}}\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\} \\ \partial^p (f_1 \wedge f_2)(\bar{x}) = \emptyset \end{array} \right.$$



$f = f_1 \vee f_2$ $\qquad$ $f = f_1 \wedge f_2$

Properties

Lemma. Let $x \in S$ and $\xi \in \mathbb{R}^n$

$$\exists \sigma, \delta > 0 \text{ s.t. } \xi.(y-x) \leq \sigma ||y-x||^2 \,,\ \forall y \in S \cap \mathbb{B}(x,\delta) \Rightarrow \xi \in N_S^p(x)$$

Remark. If f is C^2, one has

$$\exists (\sigma, \eta) \text{ t.q. } f(y) \geq f(x) + \partial_x f(x)(y-x) - \sigma ||y-x||^2 \,,\ \forall y \in \mathbb{B}(x,\eta)$$

Proposition.

1. $\xi \in \partial^p f(x)$
2. $\exists (\sigma, \eta)$ t.q. $f(y) \geq f(x) + \xi.(y-x) - \sigma ||y-x||^2 \,,\ \forall y \in \mathbb{B}(x,\eta)$

are equivalent.

Characterization of invariant domains

Theorem 1. Assume A1 (only),

1. (S, F) weakly invariant
2. $\forall x \in S,\ \forall \xi \in N_S^p(x),\ \exists u \in \mathrm{U}$ s.t. $f(x,u).\xi \leq 0$

are equivalent.

Theorem 2. Assume A1 and A2,

1. (S, F) strongly invariant
2. $\forall x \in S,\ \forall \xi \in N_S^p(x),\ \forall u \in \mathrm{U},\ f(x,u).\xi \leq 0$

are equivalent.

Sketch of proof (weak invariance)

Take $x(\cdot) \subset S$. F u.s.c. $\Rightarrow \forall \epsilon > 0$, $\exists \bar{t} > t_0$ s.t.

$$\frac{x(t) - x_0}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^{t} \dot{x}(\tau) d\tau \in F(x_0) + \mathbb{B}(0, \epsilon), \quad \forall t \in]t_0, \bar{t}]$$

$$\Rightarrow v = \lim_{t \to t_0} \frac{x(t) - x_0}{t - t_0} \in T_S^B(x_0) \cap F(x_0) \Rightarrow v.\xi \leq 0, \ \forall \xi \in N_S^p(x_0)$$

Conversely, let $x \in \mathrm{X}$ and take $s(x) \in \mathrm{Proj}_S(x)$

$x - s(x) \in N_S^p(s(x)) \Rightarrow \exists u^\star(x) \in \mathrm{U}$ s.t. $f(s(x), u^\star(x)).(x - s(x)) \leq 0$

Consider $\pi = \{t_0, \cdots, t_N\}$ avec $t_i < t_{i+1}$ and define

$$\dot{x}_\pi(t) = f(s(x_i), u^\star(x_i)), \ x_\pi(t_i) = x_i, \ t \in [t_i, t_{i+1}[\ .$$

Gronwall Lemma $\Rightarrow \sup_{t \in [t_0, T]} ||\dot{x}_\pi(t)|| < M$

Sketch of proof (weak invariance)

$$\mathrm{d}_S(x_1) \leq M(t_1 - t_0)$$

$$\begin{aligned}
\mathrm{d}_S(x_2)^2 &\leq ||x_2 - s(x_1)||^2 \\
&= ||x_2 - x_1||^2 + ||x_1 - s(x_1)||^2 + 2(x_2 - x_1).(x_1 - s(x_1)) \\
&\leq M^2(t_2 - t_1)^2 + d_S(x_1)^2 + 2(t_2 - t_1) f(s(x_1), u^\star(x_1)).(x_1 - s(x_1)) \\
&\leq M^2(t_2 - t_1)^2 + M^2(t_1 - t_0)^2 \\
&\dots \\
\mathrm{d}_S(x_k)^2 &\leq M^2(T - t_0)\mathrm{diam}(\pi)
\end{aligned}$$

$$\mathrm{diam}(\pi_j) \to 0 \overset{\text{Ascoli Th.}}{\Longrightarrow} x_{\pi_j}(\cdot) \to x^\star(\cdot) \text{ with } x^\star(t) \in S, \forall s \in [0, T]$$

$$\dot{x}_{\pi_j}(t) = f(x_i, u^\star(x_i)) \in F(x_{\pi_j}(t) + \epsilon_j(t))$$

Th. of compactness of approximate trajectories $\Rightarrow x^\star(\cdot) \in \mathcal{S}_{[t_0, T]}$

Sketch of proof (strong invariance)

Take $x_0 \in S$ and $v_0 \in F(x_0)$.

Let $\tilde{f}(x) \in \text{Proj}_{F(x)}(v_0)$

One can show F Lipschitz $\Rightarrow$ $\tilde{f}$ continuous

For $\tilde{F}(x) = \{\tilde{f}(x)\}$, $(S, \tilde{F})$ is strongly and weakly invariant

$$\Rightarrow v_0.\xi \leq 0,\ \forall \xi \in N_S^p(x_0)$$

Sketch of proof (strong invariance)

Conversely, take $\bar{x}(\cdot) \in \mathcal{S}_{[t_0,+\infty)}(x_0)$

F linear growth $\Rightarrow ||\bar{x}(\cdot)|| < M$
F Lipschitz $\Rightarrow \exists L$

$$\tilde{F}(t,x) = \left| \begin{array}{l} \{v \in F(x) \text{ s.t.} ||v - \dot{\bar{x}}(t)|| \leq L||x - \bar{x}(t)||\}, \ t \in [t_0, T] \\ F(x), \ t > T \end{array} \right.$$

Clearly, $(\tilde{F}, S)$ is weakly invariant $\Rightarrow \exists x(\cdot) \in \mathcal{S}_{[t_0,+\infty)}(x_0)$ in S

$$||\dot{x}(t) - \dot{\bar{x}}(t)|| \leq L||x(t) - \bar{x}(t)||, \ \forall t \in [t_0, T] \overset{Gronwall}{\Longrightarrow} \bar{x}(\cdot) \equiv x(\cdot)$$

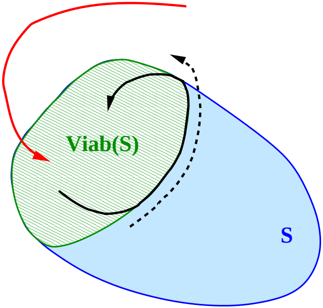
Viability kernels

Let S be a closed non-empty subset of X.

Definition. The Viability kernel of S under F, denoted $\text{Viab}_F(S)$, is the largest subset K of S such that (K, F) is viable (or weakly invariant).

Properties. Assume $\text{Viab}_F(S)$ is non-empty.

- $\partial\text{Viab}_F(S) \setminus \partial S$ is a locus of trajectories of $\dot{x} \subset F(x)$
- A trajectory can enter $\text{Viab}_F(S)$ only on $\partial\text{Viab}_F(S) \cap \partial S$

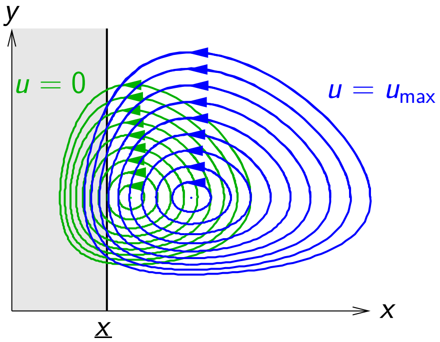


An example

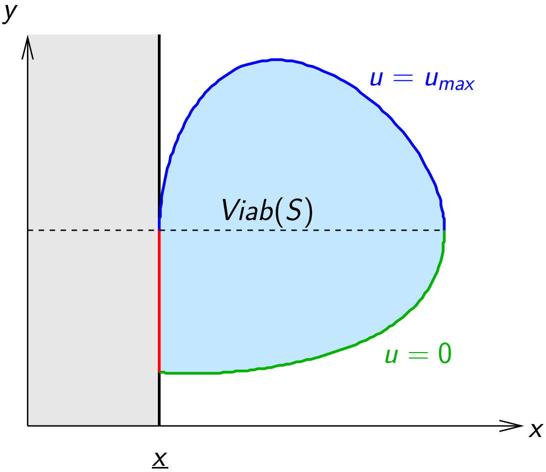
$$\left\{\begin{array}{lcll} \dot{x} & = & rx - xy & \textit{prey} \\ \dot{y} & = & -my + xy {\color{red}- uy} & \textit{predator} \end{array}\right. \quad \in [0, u_{\max}], \qquad S = \{x \geq \underline{x}\}$$

Consider $V_u(x,y) = x - (m+u)\ln(x) + y - r\ln(y)$

$$u \text{ cst } \Rightarrow \frac{d}{dt} V_u(x(t), y(t)) = 0$$



An example



Generalized solutions of the Hamilton-Jacobi-Bellman equation

Proximal solutions for the Mayer problem

Theorem. Assume A1, A2 and let ϕ_T be s.c.i. Then V is the unique s.c.i. function that satisfies the proximal H.J.B. eq.

$$\theta + \bar{H}(x, \lambda) = 0, \quad \forall \begin{bmatrix} \theta \\ \lambda \end{bmatrix} \in \partial^p V(t, x), \ \forall (t, x) \in D_V$$

with the boundary condition

$$\liminf_{t' \to T, x' \to x} V(t', x') = \phi_T(x), \quad \forall x \in \mathrm{X}$$

Sketch of proof

$$\text{Define } G : X = \begin{bmatrix} \tau \\ \xi \\ \zeta \end{bmatrix} \in \mathbb{R} \times \mathrm{X} \times \mathbb{R} \mapsto \begin{bmatrix} \{1\} \\ F(\xi) \\ \{0\} \end{bmatrix} \subset \mathbb{R}^{n+2}$$

Take $(\bar{t}, \bar{x}, \bar{z})$ s.t. $\bar{z} \geq V(\bar{t}, \bar{x})$

$$\dot{X} \in -G(X), \; X(0) = \begin{bmatrix} \bar{t} \\ \bar{x} \\ \bar{z} \end{bmatrix} \stackrel{D.P.}{\Longrightarrow} V(s, \tilde{x}(s)) \leq \bar{z}, \; \forall s \leq \bar{t}$$

i.e. $(\mathrm{epi}(V), -G)$ is strongly invariant

$$\Rightarrow \quad -\theta - \lambda.v \leq 0, \quad \forall \begin{bmatrix} \theta \\ \lambda \end{bmatrix} \in \partial^p V(t, x), \quad \forall v \in F(x)$$

$$\text{where } -\theta - \lambda.v \leq 0, \; \forall v \in F(x) \iff \theta + \bar{H}(x, \lambda) \geq 0$$

Sketch of proof

Conversely, let $x^\star(\cdot)$ be an optimal trajectory.

$$t \mapsto V(t, x^\star(t)) \text{ constant } \Rightarrow (\text{epi}(V), G) \text{ weakly invariant}$$
$$\Rightarrow \theta + \bar{H}(x, \lambda) \leq 0, \ \forall \begin{bmatrix} \theta \\ \lambda \end{bmatrix} \in \partial^p V(t, x)$$

Terminal condition : Take $x \in \mathrm{X}$.

$$(\text{epi}(V), -G) \text{ strongly invariant } \Rightarrow \exists \tilde{x}(\cdot) \in \mathcal{S}_{[t_0, T]}(x_0) \text{ s.t. } \tilde{x}(T) = x$$

$$t \mapsto V(t, \tilde{x}(t)) \text{ non decreas. } \Rightarrow \liminf_{t \to T} V(t, \tilde{x}(t)) \leq \underbrace{V(T, \tilde{x}(T))}_{\phi_T(x)}$$

$$\text{and } V \text{ s.c.i. } \Rightarrow \liminf_{t' \to T, x' \to x} V(t', x') \geq V(T, x) = \phi_T(x)$$

Sketch of proof (uniqueness)

Let W be a proximal solution of the H.J.B. eq.

$$(\text{epi}(W), G) \text{ weakly inv. } \Rightarrow W(t_0, x_0) \geq \phi_T(x(T)) \geq V(t_0, x_0)$$

Take $x^\star(\cdot) \in \mathcal{S}_{[t_0,T]}(x_0)$ s.t. $\phi(x^\star(T)) = V(t_0, x_0)$.

$$X^\star(t) = \begin{bmatrix} T - t \\ x^\star(T - t) \\ V(t_0, x_0) \end{bmatrix} \text{ is sol. of } \dot{X} = -G(X)$$

$$(\text{epi}(W), -G) \text{ strongly inv. } \Rightarrow W(t_0, \underbrace{x^\star(t_0)}_{x_0}) \leq V(t_0, x_0)$$

Example (Linear-quadratic with input constraint)

$$\begin{cases} \dot{x} = au + b \quad (a > b > 0) \\ u \in [-1, 1] \end{cases} \qquad \phi_T(x) = -x^2 \text{ with } T = 1$$

HJB : $V_t(t,x) - a|V_x(t,x)| + bV_x(t,x) = 0,\ t < 1$ with $V(1,x) = -x^2$

- $V_x > 0 \quad \Rightarrow \quad V_t(t,x) - (a-b)V_x(t,x) = 0$: **transport eq.** (E^+)

  $\Rightarrow \quad V^+(t,x) = -(x - (a-b)(1-t))^2$ is sol. of (E^+)

  But $V_x^+ > 0$ on $D^+ = \{(t,x) \text{ s.t. } x < (a-b)(1-t)\}$

- $V_x < 0 \quad \Rightarrow \quad V_t(t,x) + (a+b)V_x(t,x) = 0$ **transport eq.** (E^-)

  $\Rightarrow \quad V^-(t,x) = -(x + (a+b)(1-t))^2$ is sol. of (E^-)

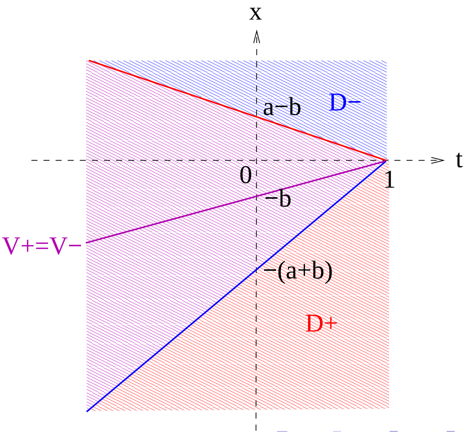
  But $V_x^- < 0$ on $D^- = \{(t,x) \text{ s.t. } x > -(a+b)(1-t)\}$

Example (Linear-quadratic with input constraint)

$$V(t,x) = \left| \begin{array}{ll} V^-(t,x) & (t,x) \in D^- \setminus D^+ \\ \min(V^-(t,x), V^+(t,x)) & (t,x) \in D^- \cap D^+ \\ V^+(t,x) & (t,x) \in D^+ \setminus D^- \end{array} \right.$$

$$V^+(\bar{t},\bar{x}) = V(\bar{t},\bar{x})$$

$$\Rightarrow \partial^p V(\bar{t},\bar{x}) = \emptyset$$



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- $V_x > 0 \quad \Rightarrow \quad V_t(t,x) - (a-b)V_x(t,x) = 0$: **transport eq.** (E^+)

  $\Rightarrow \quad V^+(t,x) = (x - (a-b)(1-t))^2$ is sol. of (E^+)

  But $V_x^+ > 0$ on $D^+ = \{(t,x) \text{ s.t. } x > (a-b)(1-t)\}$

- $V_x < 0 \quad \Rightarrow \quad V_t(t,x) + (a+b)V_x(t,x) = 0$ **transport eq.** (E^-)

  $\Rightarrow \quad V^-(t,x) = (x + (a+b)(1-t))^2$ is sol. of (E^-)

  But $V_x^- < 0$ on $D^- = \{(t,x) \text{ s.t. } x < -(a+b)(1-t)\}$

Example (Linear-quadratic with input constraint)

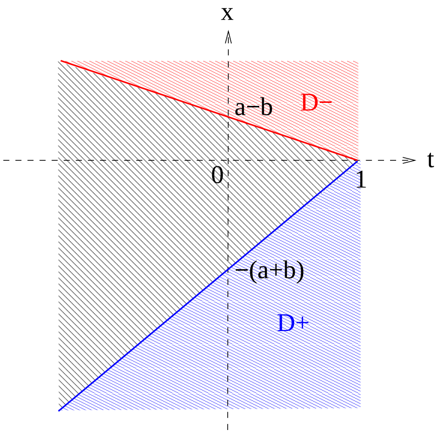
$$V(t,x) = \left| \begin{array}{ll} V^-(t,x) & (t,x) \in D^- \\ V^+(t,x) & (t,x) \in D^+ \\ 0 & (t,x) \notin D^- \cup D^+ \end{array} \right.$$

$$V^+(\bar{t},\bar{x}) = 0 \Rightarrow$$

$$\partial^p V(\bar{t},\bar{x}) = [0,1] \left[\begin{array}{c} \partial_t V^+(\bar{t},\bar{x}) \\ \partial_x V^+(\bar{t},\bar{x}) \end{array} \right]$$

$$V^-(\bar{t},\bar{x}) = 0 \Rightarrow$$

$$\partial^p V(\bar{t},\bar{x}) = [0,1] \left[\begin{array}{c} \partial_t V^-(\bar{t},\bar{x}) \\ \partial_x V^-(\bar{t},\bar{x}) \end{array} \right]$$



Proximal solutions for the minimal time problem

Theorem. Assume A1, A2 and $\mathcal{T}$ is a closed subset of X. Then V is the unique s.c.i. function bounded from below that satisfies the proximal H.J.B. eq.

$$1 + \bar{H}(x, \lambda) = 0, \quad \forall \lambda \in \partial^p V(x), \quad \forall x \in D_V \cap (X \setminus \mathcal{T})$$

with the boundary condition

$$\begin{cases} V(x) = 0, \ \forall x \in \mathcal{T} \\ 1 + \bar{H}(x, \lambda) \geq 0, \ \forall \lambda \in \partial^p V(x), \ \forall x \in \partial \mathcal{T} \end{cases}$$

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$$\begin{cases} V(x) = 0, \ \forall x \in \mathcal{T} \\ 1 + \bar{H}(x, \lambda) \geq 0, \ \forall \lambda \in \partial^p V(x), \ \forall x \in \partial \mathcal{T} \end{cases}$$

Proximal solutions for infinite horizon problems

$$V(t_0,x_0) = \inf_{u(\cdot)} \lim_{T\to+\infty} \int_{t_0}^{T} e^{-\delta\tau} l(x(\tau),u(\tau))d\tau \qquad (\delta > 0)$$

Theorem. Let $\bar{l}(x) = \max_{u\in \mathrm{U}} l(x,u)$. Assume that f and l satisfies H1 and for any $x \in \mathrm{X}$

$$\bigcup_{\substack{u\in \mathrm{U} \\ v\in[-1,1]}} \begin{bmatrix} f(x,u) \\ vl(x,u) + (1-v)\bar{l}(x) \end{bmatrix} \text{ is convex}$$

Then the value function is $V(t,x) = e^{-\delta t}W(x)$ where W is the unique s.c.i. function bounded from below that satisfies

$$\bar{H}(x,\lambda) - \delta W(x) = 0, \quad \forall\lambda \in \partial^p W(x), \quad \forall x \in D_W$$

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