THE HYPERELLIPTIC THETA MAP AND OSCULATING PROJECTIONS.

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Abstract. Let C be a hyperelliptic curve of genus $g \geq 3$. In this paper we give a new geomtric description of the theta map for moduli spaces of rank 2 semistable vector bundles on C with trivial determinant. In orther to do this, we describe a fibration of (a birational model of) the moduli space, whose fibers are GIT quotients $(\mathbb{P}^1)^{2g}/\!/\operatorname{PGL}(2)$. Then, we identify the restriction of the theta map to these GIT quotients with some explicit degree two osculating projection. As a corollary of this construction, we obtain a birational inclusion of a fibration in Kummer $(g-1)$-varieties over $\mathbb{P}^g$ inside the ramification locus of the theta map.

1. Introduction

Let C be a complex smooth curve of genus $g \geq 3$ and $\mathcal{SU}_C(r)$ the (coarse) moduli space of semistable vector bundles of rank r with trivial determinant on C. It is well known that this moduli space is a normal, projective, unirational variety of dimension $(r^2-1)(g-1)$. The study of the projective geometry of moduli spaces of vector bundles in low rank and genus has produced some beautiful descriptions, frequently mingling constructions issued in the context of classical algebraic geometry and the geometry of Jacobians and theta functions ([26, 24, 12]).

Let $\mathcal{L}$ be the determinant line bundle on $\mathcal{SU}_C(r)$ and $\varphi_{\mathcal{L}} : \mathcal{SU}_C(r) \dashrightarrow |\mathcal{L}|^*$ the map induced by global sections of $\mathcal{L}$. The linear system $|\mathcal{L}|^*$ is isomorphic to the $|r\Theta|$ linear series on the Jacobian variety $\operatorname{Jac}(C)$, by the first declination of *strange duality* [5]. This way, we obtain a (in general) rational map

$$\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|,$$

the clebrated theta map, which is canonically identified to $\varphi_{\mathcal{L}}$ [5].

Let us now fix $r = 2$. In this setting, the map θ is a finite morphism [27]. When $g = 2$, the map θ is an isomorphism onto $\mathbb{P}^3$ [23]. For $g \geq 3$, the map θ is an embedding if C is non-hyperelliptic, and it is a 2:1 map if C is hyperelliptic [12, 3, 11, 28] (see Section 2.1 for more details).

The goal of this paper is to describe the geometry associated to the map θ in the case $r = 2$ when C hyperelliptic. In the non-hyperelliptic case, the papers [8] and [1] outline a connection between the moduli space $\mathcal{SU}_C(2)$ and the moduli space $\mathcal{M}_{0,n}$ of rational curves with n ordered marked points. A generalization of [1] for higher rank vector bundles has been given in [10]. In the present work, we develop once again the link with the moduli space of pointed rational curves (more

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precisely with its GIT compactification $\mathcal{M}_{0,n}^{GIT}$). Thanks to some clever description of the GIT compactification in terms of linear systems on the projective space due to Kumar [19], this also offers a new geometric description of the θ-map if C is hyperelliptic.

Let C be a hyperelliptic curve of genus $g \geq 3$ and D an effective divisor of degree g on C. Let us consider the isomorphism classes of extensions

$$(e) \quad 0 \to \mathcal{O}(-D) \to E_e \to \mathcal{O}(D) \to 0.$$

These are classified by the projective space

$$\mathbb{P}_D^{3g-2} := \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*,$$

where K is the canonical divisor on C. Since the divisor $K + 2D$ is very ample, the linear system $|K + 2D|$ embeds the curve C in $\mathbb{P}_D^{3g-2}$. Let $\mathbb{P}_N^{2g-2}$ be the span in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$ on C defined by the effective divisor $N \in |2D|$ ($\mathbb{P}_N^{2g-2}$ has also a precise description in terms of extensions, see Section 3).

Proposition 1.1. *There exists a fibration* $p_D : \mathcal{SU}_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$ *whose general fiber is birational to* $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$. *Moreover, we have:*

1. *For every generic divisor* $N \in |2D|$, *there exists a* $2g$*-pointed projective space* $\mathbb{P}_N^{2g-2}$ *and a rational dominant map* $h_N : \mathbb{P}_N^{2g-2} \dashrightarrow p_D^{-1}(N)$ *classifying extension classes, such that the fibers of* h_N *are rational normal curves passing by the* $2g$ *marked points.*
2. *The family of rational normal curves defined by* h_N *is the universal family of rational curves over (an open subset of) the generic fiber* $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.

Our aim is to describe the map θ restricted to the generic fibers of the fibration p_D. To this end, the following construction is crucial:

Let $p, i(p)$ be two involution-conjugate points in C; and consider the line $l \subset \mathbb{P}_D^{3g-2}$ secant to C and passing through p and $i(p)$. We show that this line intersects the subspace $\mathbb{P}_N^{2g-2}$ in a point. Moreover, the locus $\Gamma \subset \mathbb{P}_N^{2g-2}$ of these intersections as we vary $p \in C$ is a rational normal curve passing by the points $p_1, \ldots, p_{2g}$. It follows from Proposition 1.1, the map h_N contracts the curve Γ onto a point $P \in p_D^{-1}(N) \cong \mathcal{M}_{0,2g}$.

In [19], Kumar defines the linear system Ω of $(g-1)$-forms on $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ at $2g-1$ general points. He shows that Γ induces a birational map $i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \mathcal{M}_{0,2g}^{\mathrm{GIT}}$ onto the GIT compactification of the moduli space $\mathcal{M}_{0,2g}$. The partial linear system $\Lambda \subset \Omega$ of forms vanishing with multiplicity $g-2$ at an additional general point $e \in \mathbb{P}^{2g-3}$ induces a projection $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow |\Lambda|^*$. More precisely, κ is a 2-to-1 osculating projection centered on the point $w = i_\Omega(e)$. We describe birationally the restrictions of θ to the fibers $p_D^{-1}(N)$ using Kumar's map:

Theorem A. *The map* θ *restricted to the fibers* $p_D^{-1}(N)$ *is the osculating projection* κ *centered at the point* $P = h_N(\Gamma)$, *up to composition with a birational map.*

Furthermore, the image of κ is a connected component of the moduli space $\mathcal{SU}_{C_w}(2)^{inv}$ of hyperelliptic invariant semistable vector bundles with trivial determinant on C_w, where C_w is the hyperelliptic 2-to-1 cover of $\mathbb{P}^1$ ramifying over the

$2g$ points defined by w. He also proves that the ramification locus of the map κ is the Kummer variety $\mathrm{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)^{inv}$. These results, combined with Theorem A, allow us to give a quite accurate description of the ramification locus of the map θ:

Theorem B. *The ramification locus of the map θ has an irreducible component birational to a fibration in Kummer varieties of dimension $g-1$ over $|2D| \cong \mathbb{P}^g$.*

In low genus we are able to give a more precise description of the Theta map and its interplay with maps classifying extensions. Let $f_D : \mathbb{P}^{3g-2}_D \dashrightarrow \mathcal{SU}_C(2)$ denote the natural map sending an extension class onto its rank 2 vector bundle and let us define φ_D as $\theta \circ f_D$.

Theorem C. *Let C be a hyperelliptic curve of genus 3. Then, for generic N, the restriction of φ_D to the subspace $\mathbb{P}^{2g-2}_N$ is exactly the composition $\kappa \circ h_N$. If $g = 4$ or 5, then φ_D is defined by a (possibly equal) linear subsystem of the one defining $\kappa \circ h_N$, and set-theoretically the base loci of the two linear systems coincide.*

Notation. $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ will denote the n-dimensional complex projective space of dim 1 subspaces. Throughout this paper, a form F of degree r on $\mathbb{P}^n$ will denote element of the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \mathrm{Sym}^r(\mathbb{C}^{n+1})^*$. If we fix a basis $x_0, \ldots, x_n$ of $(\mathbb{C}^{n+1})^*$, F is simply a homogeneous polynomial of degree r on $x_0, \ldots, x_n$. Most of the maps in this paper will be rational maps, hence we will often offend good taste by just dropping the adjective *rational* . We apologize for that.

2. Moduli of vector bundles

We briefly recall here some results about moduli of vector bundles. For a more detailed reference, see [4].

2.1. **Moduli of vector bundles and the map θ.** Let C be a smooth genus g algebraic curve with $g \geq 2$. Let us denote by $\mathrm{Pic}^d(C)$ the Picard variety of degree d line bundles on C. The Jacobian of C is $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$. The canonical divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$ is defined set-theoretically as

$$\Theta := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, L) \neq 0\}.$$

Let $\mathcal{SU}_C(2)$ be the moduli space of semistable rank 2 vector bundles on C with trivial determinant. This variety parametrizes S-equivalence classes of such vector bundles.

The Picard group $\mathrm{Pic}(\mathcal{SU}_C(2))$ is isomorphic to $\mathbb{Z}$, and it is generated by the *determinant line bundle* $\mathcal{L}$ [14]. For every $E \in \mathcal{SU}_C(2)$, let us define the *theta divisor*

$$\theta(E) := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.$$

In the rank 2 case, $\theta(E)$ is a divisor in the linear system $|2\Theta| \cong \mathbb{P}^{2^g-1}$ and $|2\Theta|$ is isomorphic to the linear system $|\mathcal{L}|^*$ [5]. It is well known that we can identify the map $\mathcal{SU}_C(2) \to |\mathcal{L}|^*$ with the Theta map

$$\theta : \mathcal{SU}_C(2) \to |2\Theta|; \tag{1}$$

$$E \mapsto \theta(E). \tag{2}$$

In rank 2, the map θ is a finite morphism. If C is not hyperelliptic, θ is known to be an embedding [11, 28]. This is also the case in genus 2, where θ is an isomorphism onto $\mathbb{P}^3$ [23]. If C is hyperelliptic of genus $g \geq 3$, we have that θ factors through the involution

$$E \mapsto i^*E$$

induced by the hyperelliptic involution i, embedding the quotient $\mathcal{SU}_C(2)/i^*$ into $|2\Theta|$ [12, 3]. An interesting explicit description of the image of the hyperelliptic theta map is given in [12].

2.2. **The classifying maps.** Let D be a general degree g effective divisor on C. Let us consider isomorphism classes of extensions

$$(e) \quad 0 \to \mathcal{O}(-D) \to E_e \to \mathcal{O}(D) \to 0.$$

These extensions are classified by the $(3g-2)$-dimensional projective space

$$\mathbb{P}_D^{3g-2} := \mathbb{P}\operatorname{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*,$$

where K is the canonical divisor of C. The divisor $K + 2D$ is very ample and embeds C as a degree $4g-2$ curve in $\mathbb{P}_D^{3g-2}$. Let us define the rational surjective classifying map

$$f_D : \mathbb{P}_D^{3g-2} \dashrightarrow \mathcal{SU}_C(2)$$

which sends the extension class (e) to the vector bundle E_e. The composed map

$$\varphi_D := \theta \circ f_D : \mathbb{P}_D^{3g-2} \dashrightarrow |2\Theta|$$

can be described in terms of polynomial maps. From [6, Thm. 2] we have an isomorphism

$$H^0(\mathcal{SU}_C(2), \mathcal{L}) \cong H^0(\mathbb{P}_D^{3g-2}, \mathcal{I}_C^{g-1}(g)), \tag{3}$$

where $\mathcal{I}_C$ is the ideal sheaf of C. In particular we have

Theorem 2.1. *The map φ_D is given by the linear system $|\mathcal{I}_C^{g-1}(g)|$ of forms of degree g vanishing with multiplicity at least $g-1$ on C.*

Let us denote by $\operatorname{Sec}^n(C)$ the variety of $(n+1)$-secant n-planes on C. We have that the singular locus of $\operatorname{Sec}^{n+1}(C)$ is the secant variety $\operatorname{Sec}^n(C)$ for every n. The linear system $|\mathcal{I}_C^{g-1}(g)|$ is characterized as follows:

Proposition 2.2 ([1, Lemma 2.5])**.** *The linear system $|\mathcal{I}_C^{g-1}(g)|$ and $|\mathcal{I}_{\operatorname{Sec}^{g-2}(C)}(g)|$ on $\mathbb{P}_D^{3g-2}$ are the same.*

Proof. We reproduce here the proof for the reader's convenience. The elements of both linear systems can be seen as symmetric g-linear forms on the vector space

$H^0(C, K+2D)^*$. Let F, G be such forms. Then, F belongs to $|\mathcal{I}_C^{g-1}(g)|$ (resp. G belongs to $|\mathcal{I}_{\mathrm{Sec}^{g-2}(C)}(g)|$) if and only if

$$F(p_1,\dots,p_g)=0 \qquad \text{for all } p_k\in C \text{ such that } p_i=p_j \text{ for some } 1\le i,j\le g$$

$$G(p,\dots,p)=0 \qquad \text{for any linear combination } p=\sum_{k=1}^{g-1}\lambda_i p_i, \text{ where } p_i\in C.$$

One can show that these conditions are equivalent by exhibiting appropriate choices of λ_i. □

2.3. **The exceptional fibers of the classifying map f_D.** Since $\dim \mathcal{SU}_C(2) = 3g-3$, the generic fiber of f_D has dimension one. The set of stable bundles for which $\dim(f_D^{-1}(E)) > 1$ is a proper subset of $\mathcal{SU}_C(2)$. For simplicity, let us define the "Serre dual" divisor

$$B := K - D$$

with $\deg(B) = g-2$. As in the previous paragraphs, the isomorphism classes of extensions

$$0 \to \mathcal{O}(-B) \to E \to \mathcal{O}(B) \to 0$$

are classified by the projective space

$$\mathbb{P}_B^{3g-6} := \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K+2B|^*,$$

which is endowed with the rational classifying map $f_B : \mathbb{P}_B^{3g-6} \dashrightarrow \mathcal{SU}_C(2)$ defined in the same way as f_D.

Proposition 2.3. *Let $E \in \mathcal{SU}_C(2)$ be a stable bundle. Then*

$$\dim(f_D^{-1}(E)) \geq 2 \qquad \textit{if and only if} \qquad E \in \overline{f_B(\mathbb{P}_B^{3g-6})}.$$

Proof. Let E be a stable bundle. Then, by Riemann-Roch and Serre duality theorems, the dimension of $f_D^{-1}(E)$ is given by

$$\begin{aligned} h^0(C, E\otimes\mathcal{O}(D)) &= h^0(C, E\otimes\mathcal{O}(B)) + 2g - 2(g-1) \\ &= h^0(C, E\otimes\mathcal{O}(B)) + 1 \end{aligned}$$

Thus, $\dim(f_D^{-1}(E)) > 2$ if and only if there exists a non-zero sheaf morphism $\mathcal{O}(-B) \to E$. This is equivalent to $E \in \overline{f_B(\mathbb{P}_B^{3g-6})}$. □

If $g > 2$, the divisor $|K+2B|$ embeds C as a degree $4g-6$ curve in $\mathbb{P}_B^{3g-6}$ (recall that $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K+2B|^*$). Again by Theorem 2.1, the map φ_B is given by the linear system $|\mathcal{I}_C^{g-3}(g-2)|$. Moreover, by [25, Theorem 4.1] this linear system has projective dimension $\left(\sum_{i=0}^{g-2}\binom{g}{i}\right) - 1$.

Let us denote by $\mathbb{P}_c$ the linear span of $\theta(\overline{f_B(\mathbb{P}_B^{3g-6})})$ in $|2\Theta|$. Since the map θ is finite, $\mathbb{P}_c$ has projective dimension $\left[\sum_{i=0}^{g-2}\binom{g}{i}\right] - 1$, and Proposition 2.3 also applies to φ_D: the fibers of φ_D with dimension ≥ 2 are those over $\mathbb{P}_c$.

3. A linear projection in $|2\Theta|$

The goal of this Section is to describe the map $\mathcal{SU}_C(2) \to \mathbb{P}^g$ whose fibers will be birational - and in some cases even biregular - to the GIT compactification of the moduli space of $2g$-pointed rational curves. In order to do this, we describe the projection with center $\mathbb{P}_c$, seen as a linear subspace of $|2\Theta|$. The fibers of this map are what we are looking for, and so far we are not resricting to the case where C is hyperelliptic.

Let $p_{\mathbb{P}_c}$ be the linear projection in $|2\Theta|$ with center $\mathbb{P}_c$. Recall that $\dim \mathbb{P}_c = \left[\sum_{i=0}^{g-2}\binom{g}{i}\right] - 1$. A straightforward calculation shows that the supplementary linear subspaces of $\mathbb{P}_c$ in $|2\Theta|$ are of projective dimension g. Thus, the image of $p_{\mathbb{P}_c}$ is a g-dimensional projective space. Let us set

$$\widehat{\mathcal{SU}}_C(2) := \mathcal{SU}_C(2) \setminus (\mathrm{Kum}(C) \cup \overline{\varphi_D(\mathbb{P}^{3g-6}_B)}).$$

This is the open subset of $\mathcal{SU}_C(2)$ we will be mostly concerned by. Recall that the space $H^0(C, E \otimes \mathcal{O}(D))$ has dimension 2 for $E \in \widehat{\mathcal{SU}}_C(2)$. Consequently, we can pick two sections s_1 and s_2 that constitute a basis for this space.

Theorem 3.1. *The image of the projection $p_{\mathbb{P}_c}$ can be identified with the linear system $|2D|$ on C, in a way such that the restriction of the projection $p_{\mathbb{P}_c}$ to $\theta(\widehat{\mathcal{SU}}_C(2))$ coincides with the map*

$$\theta(\widehat{\mathcal{SU}}_C(2)) \to |2D|$$
$$\theta(E) \mapsto \mathrm{Zeroes}(s_1 \wedge s_2)$$

Proof. This result was proved in [1] for C non hyperelliptic, but the proof extends harmlessly to the hyperelliptic case. We will mention explicitly where the proof for C hyperelliptic differs. The Picard variety $\mathrm{Pic}^{g-1}(C)$ contains a model $\widetilde{C}$ of C, made up by line bundles of type $\mathcal{O}(B + p)$, with $p \in C$. The span of $\widetilde{C}$ inside $|2\Theta|^*$ corresponds to the complete linear system $|2D|^*$. Moreover, the linear span of $\widetilde{C}$ is the annihilator of $\mathbb{P}_c$. In particular, the projection $p_{\mathbb{P}_c|\theta(\widehat{\mathcal{SU}}_C(2))}$ determines a hyperplane in the annihilator of $\mathbb{P}_c$, which is a point in $|2D|$. This projection can be identified with the map

$$p_{\mathbb{P}_c}|_{\theta(\widehat{\mathcal{SU}}_C(2))} : \theta(\widehat{\mathcal{SU}}_C(2)) \to |2D|,$$
$$\theta(E) \mapsto \Delta(E),$$

where $\Delta(E)$ is the divisor defined by

$$\Delta(E) := \{p \in C \mid h^0(C, E \otimes \mathcal{O}(B + p)) \neq 0\}. \tag{4}$$

Equivalently, we have that $\Delta(E) = \theta(E) \cap \widetilde{C}$. Now, in order to adapt to the hyperelliptic case, it is enough to observe that since $\theta(E) = \theta(i^*E)$, we directly obtain that $\Delta(E) = \Delta(i^*E)$. Finally, an easy Riemann-Roch argument shows that that $\Delta(E)$ is the divisor of zeroes of $s_1 \wedge s_2$. □

Recall that the linear system $|K + 2D|$ embeds the curve C in the projective space $\mathbb{P}^{3g-2}_D$. Let $N \in |2D|$ be a generic effective reduced divisor and consider the linear span $\langle N\rangle \subset \mathbb{P}^{3g-2}_D$. The annihilator of $\langle N\rangle$ is the vector space $H^0(C, 2D + K - N)$, which has dimension g. In particular, the linear span $\langle N\rangle$ has dimension $(3g-2) - g = 2g-2$. Let us write

$$\mathbb{P}^{2g-2}_N := \langle N\rangle \subset \mathbb{P}^{3g-2}_D.$$

We will study the classifying map φ_D in relation with a fibration $\mathcal{SU}_C(2) \to \mathbb{P}^g$ by considering the restrictions of φ_D to $\mathbb{P}^{2g-2}_N$, as N varies in the linear system $|2D|$. The spaces $\mathbb{P}^{2g-2}_N$ have a very explicit description in terms of extension classes (see [21]).

Notation. For simplicity, let us write $\varphi_{D,N}$ for the restricted map $\varphi_D|_{\mathbb{P}^{2g-2}_N}$.

Proposition 3.2. *Let N in $|2D|$ be a general divisor on $C \subset \mathbb{P}^{3g-2}_D$. Then, the image of*

$$\varphi_{D,N} : \mathbb{P}^{2g-2}_N \dashrightarrow \theta(\mathcal{SU}_C(2))$$

is the closure in $\theta(\mathcal{SU}_C(2))$ of the fiber over $N \in |2D|$ of the projection $p_{\mathbb{P}_c}$.

Proof. Let $(e) \in \mathbb{P}^{3g-2}_D$ be an extension

$$(e) \quad 0 \to \mathcal{O}(-D) \xrightarrow{i_e} E_e \xrightarrow{\pi_e} \mathcal{O}(D) \to 0.$$

By [21, Proposition 1.1], we have that $e \in \mathbb{P}^{2g-2}_N$ if and only if there exists a section

$$\alpha \in H^0(C, \mathrm{Hom}(\mathcal{O}(-D), E))$$

such that $\mathrm{Zeroes}(\pi_e \circ \alpha) = N$. This means that α and i_e are two independent sections of $E_e \otimes \mathcal{O}(D)$ with $\mathrm{Zeroes}(\alpha \wedge i_e) = N$. Consequently, $\theta(E_e) = \varphi_{D,N}(e)$ is projected by $p_{\mathbb{P}_c}$ on $N \in |2D|$ by Theorem 3.1. Hence, the image of $\varphi_{D,N}$ is contained in $\overline{p^{-1}_{\mathbb{P}_c}(N)}$.

Conversely, by the proof of Theorem 3.1, we have that for every bundle $E \in \widehat{\mathcal{SU}}_C(2)$, $\theta(E)$ is projected by $p_{\mathbb{P}_c}$ to a divisor $\Delta(E) \in |2D|$. The argument used above implies that the fiber $\varphi^{-1}_D(\theta(E)) = f^{-1}_D(E)$ of such bundle is contained in $\mathbb{P}^{2g-2}_{\Delta(E)}$. Consequently, the fiber of a general divisor $N \in |2D|$ by $p_{\mathbb{P}_c}$ is contained in the image of $\varphi_{D,N}$. □

4. The modular fibration $\mathcal{SU}_C(2) \to \mathbb{P}^g$

Let C be a smooth genus $g \geq 3$ curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$. Consider the span $\mathbb{P}^{2g-2}_N$ in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$. In this Section, building on [9] and [1], we will give more information about restricted map

$$\varphi_{D,N} = \varphi_D|_{\mathbb{P}^{2g-2}_N} : \mathbb{P}^{2g-2}_N \dashrightarrow \mathcal{SU}_C(2).$$

In particular, we will explain the interplay between these maps, rational normal curves in $\mathbb{P}^{2g-2}_N$ and moduli spaces of rational pointed curves.

4.1. **Linear systems in $\mathbb{P}^{2g-2}_N$ contracting rational normal curves.** Recall that the secant variety $\mathrm{Sec}^{g-2}(C)$ is the base locus for φ_D (see Theorem 2.1 and Proposition 2.2). Hence we will distinguish the following two secant varieties in $\mathbb{P}^{2g-2}_N$:

$$\mathrm{Sec}^N := \mathrm{Sec}^{g-2}(C) \cap \mathbb{P}^{2g-2}_N \ ,$$
$$\mathrm{Sec}^{g-2}(N) := \bigcup_{\substack{M \subset N \\ \# M = g-1}} \mathrm{span}\{M\} \ .$$

Note that, since the points of N are already in $\mathbb{P}^{2g-2}_N$, we have the inclusion $\mathrm{Sec}^{g-2}(N) \subset \mathrm{Sec}^N$. We will also need to consider the linear systems on $\mathbb{P}^{2g-2}_N$

$$\mathcal{I}_{\mathrm{Sec}^N}(g) \ , \quad \text{and} \quad \mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)$$

of forms of degree g vanishing on the corresponding secant varieties. The previous inclusion of secant varieties implies that $\mathcal{I}_{\mathrm{Sec}^N}(g)$ is in general a linear subsystem of $\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)$.

Lemma 4.1. *The restricted map $\varphi_{D,N}$ is given by a linear subsystem $\mathcal{R}$ of $|\mathcal{I}_{\mathrm{Sec}^N}(g)|$.*

Proof. This is a direct consequence of Theorem 2.1 and Proposition 2.2. □

4.2. **Moduli spaces of pointed rational curves.** In this Section we will outline the relation between the restricted map $\varphi_{D,N}$ and the moduli spaces of pointed rational curves.

4.2.1. *Two compactifications of $\mathcal{M}_{0,n}$.* The moduli space $\mathcal{M}_{0,n}$ of ordered configurations of n disctinct points on the projective line is not compact. We will consider two compactifications of $\mathcal{M}_{0,n}$. The first one is the GIT quotient

$$\mathcal{M}^{\mathrm{GIT}}_{0,n} := (\mathbb{P}^1)^n // \mathrm{PGL}(2,\mathbb{C})$$

of $(\mathbb{P}^1)^n$ by the diagonal action of $G = \mathrm{PGL}(2,\mathbb{C})$ for the natural G-linearization of the line bundle $L = \boxtimes_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(1)$ (see [13]). The quotient $\mathcal{M}^{\mathrm{GIT}}_{0,n}$ is naturally embedded in the projective space $\mathbb{P}(H^0((\mathbb{P}^1)^n, L)^G)$ of invariant sections.

The second one is the Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$ [18]. The points of $\overline{\mathcal{M}}_{0,n}$ represent isomorphism classes of stable curves. More details on these constructions can be found in [16] and [18].

Both $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$ and $\overline{\mathcal{M}}_{0,n}$ contain $\mathcal{M}_{0,n}$ as an open set. However, the Mumford-Knudsen compactification is finer on the boundary: there exists a contraction dominant morphism

$$c_n : \overline{\mathcal{M}}_{0,n} \to \mathcal{M}^{\mathrm{GIT}}_{0,n}$$

contracting some components of the boundary of $\overline{\mathcal{M}}_{0,n}$, that restricts to the identity on $\mathcal{M}_{0,n}$ [9]. Moreover, we will denote by

$$\lambda_k : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1},$$

for $k = 1, \ldots, n$, the forgetful morphism that forgets the labelling of the k-th point.

4.2.2. *A variety of rational normal curves.* Let $e_1, \dots e_n \in \mathbb{P}^{n-2}$ be n points in general position. Let $\mathcal{H}$ be the Hilbert scheme of subschemes of $\mathbb{P}^{n-2}$. Let $V_0(e_1, \dots, e_n) \subset \mathcal{H}$ be the subvariety of rational normal curves in $\mathbb{P}^{n-2}$ passing through the points $e_1, \dots, e_n$, and let $V(e_1, \dots, e_n)$ be the closure of $V_0(e_1, \dots, e_n)$ inside the Hilbert scheme of subschemes of $\mathbb{P}^{n-2}$. The boundary $V(e_1, \dots, e_n) \setminus V_0(e_1, \dots, e_n)$ consists on reducible rational normal curves, i.e. reducible non-degenerate curves of degree n such that each component is a rational normal curve in its projective span.

There exists an isomorphism $V_0(e_1, \dots, e_n) \cong \mathcal{M}_{0,n}$ (see [13]) associating to a rational normal curve passing by $e_1, \dots, e_n$ the corresponding ordered configuration of n points in $\mathbb{P}^1$. This can be extended [16] to an isomorphism between $V(e_1, \dots, e_n)$ and $\overline{\mathcal{M}}_{0,n}$.

4.2.3. *The blow-up construction.* The following construction is due to Kapranov [15]: let $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$ be $(n-1)$ points in general position. Consider the following sequence of blow-ups:

(1) Blow-up the points $e_1, \dots, e_{n-1}$.
(2) Blow-up the proper transforms of lines spanned by pairs of points from $\{e_1, \dots, e_{n-1}\}$.
(3) Blow-up the proper transforms of planes spanned by triples of points from $\{e_1, \dots, e_{n-1}\}$.

$\vdots$

$(n-4)$. Blow-up the proper transforms of linear subspaces spanned by $(n-4)$-ples of points from $\{e_1, \dots, e_{n-1}\}$.

Let $\mathrm{Bl}(\mathbb{P}^{n-3})$ be the $(n-3)$-variety obtained in this way, and $b : \mathrm{Bl}(\mathbb{P}^{n-3}) \to \mathbb{P}^{n-3}$ the composition of this sequence of blow-ups. We will call this map the *Kapranov blow-up map centered in the points* $e_1, \dots, e_{n-1}$.

Theorem 4.2 (Kapranov [15])**.** *Let $n \geq 4$. Then, the moduli space $\overline{\mathcal{M}}_{0,n}$ is isomorphic to* $\mathrm{Bl}(\mathbb{P}^{n-3})$.

Moreover, the images by b of the fibres of the map λ_k over the points in the open set $\mathcal{M}_{0,n-1} \subset \overline{\mathcal{M}}_{0,n-1}$ are the rational normal curves in $\mathbb{P}^{n-3}$ passing through the $n-1$ points $e_1, \dots, e_{n-1}$ (see [17, Prop. 3.1]).

4.2.4. *The Cremona inversion.* Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Without loss of generality, we may assume $e_k = [0 : \dots : 1 : \dots : 0]$ for $k = 1, \dots, n-2$; and $e_{n-1} = [1 : \dots : 1]$. The *Cremona inversion with respect to* e_{n-1} is the birational map

$$\begin{aligned} \mathrm{Cr}_{n-1} : \mathbb{P}^{n-3} &\dashrightarrow \mathbb{P}^{n-3} \\ [x_0 : \dots : x_{n-3}] &\mapsto [1/x_0 : \dots : 1/x_{n-3}] \,. \end{aligned}$$

On $\mathbb{P}^2$ the Cremona inversion is given by the linear system of quadrics passing through e_1, e_2 and e_3, on $\mathbb{P}^3$ by the cubics that vanish on the six secant lines of e_1, e_2, e_3 and e_4, and so on. The Cremona inversion has the following property: any non-degenerate rational normal curve passing through the points $e_1, \dots, e_{n-1}$ is transformed into a line passing by the point $\mathrm{Cr}_{n-1}(e_{n-1})$. Let $\tau_{n-1} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-4}$ be the linear projection with center $\mathrm{Cr}_{n-1}(e_{n-1})$. From the previous property, we obtain that the composition $\tau_{n-1} \circ \mathrm{Cr}_{n-1}$ contracts non-degenerate rational normal curves passing through $e_1, \dots, e_{n-1}$.

Let $k \in \{1, \dots, n-1\}$. It is straightforward to see that one can let e_k play the role of e_{n-1} in the definition of Cr_{n-1}, and define similarly the Cremona inversion Cr_k. Let $\tau_k : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-4}$ be the linear projection with center $\mathrm{Cr}_k(e_k)$.

Lemma 4.3. *Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Then, the composition $\tau_k \circ \mathrm{Cr}_k$ contracts the non-degenerate rational normal curves passing through $e_1, \dots, e_n$.*

Let us denote H_t, for $t \neq k$, the hyperplane in $\mathbb{P}^{n-3}$ spanned the points e_i, with $i \neq k, t$. There are $n-2$ such hyperplanes and each one is contracted to a point by Cr_k.

Proposition 4.4 ([16])**.** *Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Let $k \in \{1, \dots, n-1\}$. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n} & \xrightarrow{\lambda_k} & \overline{\mathcal{M}}_{0,n-1} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b_k} \\ \mathbb{P}^{n-3} & \overset{\tau_k \circ \mathrm{Cr}_k}{\dashrightarrow} & \mathbb{P}^{n-4} \end{array}$$

Here, the map b_k is the Kapranov blow-up map centered in the images of the hyperplanes H_t, for $1 \leq t \leq n-1$ and $t \neq k$, by $\tau_k \circ \mathrm{Cr}_k$.

Remark. We observe here a little subtlety. We only get here $n-1$ forgetful maps through Cremona transformations, because we are tacitly assuming that Kapranov's blow-up construction of $\overline{\mathcal{M}}_{0,n}$ labels with integers from 1 to $n-1$ the points $e_1, \dots, e_{n-1}$ of the projective base of $\mathbb{P}^{n-3}$, and labels as n the last point (which is free to move inside the $\mathbb{P}^{n-3}$ birational model of $\overline{\mathcal{M}}_{0,n}$). This is due to this small asymmetric aspect of Kapranov's construction, but it is clear that one could assume that the last, free, point is labeled with any $k \in \{1, \dots, n-1\}$, and obtain other forgetful maps.

4.2.5. *Rationalizations of $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.* Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ in general position. Let Ω be the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$.

Theorem 4.5 ([19])**.** *Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ be in general position. Then, the rational map*

$$i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \Omega^*$$

induced by the linear system Ω maps $\mathbb{P}^{2g-3}$ birationally onto $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.

We also observe that the contraction map c_{2n} can also be described in terms of Kumar's linear system Ω:

Lemma 4.6 ([9])**.** *Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ in general position. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2g} & & \\ {\scriptstyle b}\downarrow & \searrow^{c_{2n}} & \\ \mathbb{P}^{2g-3} & \overset{i_\Omega}{\dashrightarrow} & \mathcal{M}_{0,2g}^{\mathrm{GIT}} \end{array}$$

Here, the map b is the Kapranov blow-up map centered in $p_1, \dots, p_{2g-1}$.

Let $e_0 \in \mathbb{P}^{2g-3}$ such that $w = i_\Omega(e_0)$ lies in the open set $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. The point w can represent a hyperelliptic genus $(g-1)$ curve C_w (namely the double cover of $\mathbb{P}^1$ ramifying in the $2g$ marked points) together with an ordering of the Weierstrass points. Let $\mathcal{SU}_{C_w}(2)^{inv}$ be the moduli space of rank 2 semistable vector bundles with trivial determinant over the curve C_w, that are invariant w.r.t. the hyperelliptic involution.

Consider the partial linear system Λ of Ω consisting of the $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in all the points $e_0, e_1, \ldots, e_{2g-1}$. Let $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow \Lambda^*$ be the rational projection induced by the linear system Λ.

Theorem 4.7 ([19])**.** *Let $g \geq 3$, and let $e_1, \ldots, e_{2g-1} \in \mathbb{P}^{2g-3}$ be $2g-1$ points in general position. Let $e_0 \in \mathbb{P}^{2g-3}$ such that $w = i_\Omega(e_0)$ lies in the open set $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. Then, the map κ induced by the linear system Λ is of degree 2 onto a connected component of the moduli space $\mathcal{SU}_{C_w}(2)^{inv}$. Furthermore, the map κ ramifies along the Kummer variety $\mathrm{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)^{inv}$.*

$$\begin{array}{ccc} \mathbb{P}^{2g-3} & \overset{i_\Omega}{\dashrightarrow} & \mathcal{M}_{0,2g}^{\mathrm{GIT}} \\ & {\scriptstyle i_\Lambda}\searrow & \downarrow{\scriptstyle \kappa} \\ & & \mathcal{SU}_{C_w}(2)^{inv} \end{array}$$

4.3. **Forgetful linear systems and $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.** Let C be a smooth genus $g \geq 3$ curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general reduced divisor in the linear system $|2D|$. Consider the span $\mathbb{P}_N^{2g-2}$ in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

We will now apply the discussion of Section 4.2 to the general points $p_1, \ldots, p_{2g}$ in the projective space $\mathbb{P}_N^{2g-2}$, taking $n = 2g+1$. For every $k = 1, \ldots, 2g$, we can compose Prop. 4.4 and Lemma 4.6 and get a commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{0,2g+1} & \xrightarrow{\lambda_k} & \overline{\mathcal{M}}_{0,2g} & & \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b_k} & \overset{c_{2n}}{\searrow} & \\ \mathbb{P}_N^{2g-2} & \overset{\tau_k \circ \mathrm{Cr}_k}{\dashrightarrow} & \mathbb{P}^{2g-3} & \overset{i_\Omega}{\dashrightarrow} & \mathcal{M}_{0,2g}^{\mathrm{GIT}} \end{array} \tag{5}$$

where Ω is the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ at the $2g-1$ points $\tau_k \circ \mathrm{Cr}_k(H_i)$, with $i \neq k$ and $1 \leq i \leq 2g$. Let us define the rational map

$$h_N : \mathbb{P}_N^{2g-2} \dashrightarrow |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)| * .$$

Proposition 4.8 ([9])**.** *Let $N = p_1 + \cdots + p_{2g}$ be a general reduced divisor in the linear system $|2D|$. Then, the map h_N coincides with the composition $i_\Omega \circ \tau_k \circ \mathrm{Cr}_k$ for every $k = 1, \ldots, 2g$. In particular, the composition $i_\Omega \circ \tau_k \circ \mathrm{Cr}_k$ does not depend on k.*

This is due to the fact that the linear system $|\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$ is invariant w.r.t the action of the symmetric group Σ_{2g} that operates on $\mathbb{P}_N^{2g-2}$ by linear automorphisms. Let us put together the results of Lemma 4.3, Theorem 4.5 and Proposition 4.8 in the following Proposition:

Proposition 4.9. *The image of h_N is isomorphic to the GIT moduli space $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ of ordered configurations of $2g$ points in $\mathbb{P}^1$. The map h_N is dominant and its general fiber is of dimension 1. More precisely, h_N contracts every rational normal curve Z passing through the $2g$ points N to a point z in $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$. This point represents an ordered configuration of the $2g$ points N on the rational curve Z.*

This is why these maps where dubbed "forgetful linear systems". In fact the rational normal curves passing through the $2g$ points make up the universal curve over an open subset of $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.

Since $\mathcal{R}$ is a linear subsystem of $|\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$ by Lemma 4.1, we have that $\varphi_{D,N}$ factors through h_N:

$$\begin{array}{ccc} \mathbb{P}_N^{2g-2} & \overset{h_N}{\dashrightarrow} & \mathcal{M}_{0,2g}^{\mathrm{GIT}} \\ & \searrow^{\varphi_{D,N}} & \downarrow \\ & & \theta(\mathcal{SU}_C(2)) \end{array} \tag{6}$$

4.3.1. *A comparison of base loci.* For future use, we need to compare the locus $\mathrm{Sec}^{g-2}(N)$ and the more intricate locus Sec^N obtained by intersecting the base locus of φ_D with $\mathbb{P}_N^{2g-2}$. This section is devoted to this comparison.

By definition, the points in Sec^N are given by the intersections $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$, where L_{g-1} is an effective divisor of degree $g-1$ and $\langle L_{g-1}\rangle$ is its linear span in $\mathbb{P}_D^{3g-2}$. If L_{g-1} is contained in N, it is clear that $\langle L_{g-1}\rangle \subset \mathrm{Sec}^{g-2}(N) \subset \mathbb{P}_N^{2g-2}$.

Lemma 4.10. *Let L_{g-1} be an effective divisor on C of degree $g-1$, not contained in N. Then,*

$$\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2} \neq \phi \qquad \textit{if and only if} \qquad \dim |L_{g-1}| \geq 1.$$

Moreover, if the intersection is non-empty, we have that

$$\dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1.$$

Proof. First, let us suppose that L_{g-1} and N have no points in common. The vector space $V := H^0(C, 2D + K - L_{g-1})$ is the annihilator of the span $\langle L_{g-1}\rangle$ in $\mathbb{P}_D^{3g-2}$. By the Riemann-Roch theorem, we see that V has dimension $2g$, hence

$$\dim\langle L_{g-1}\rangle = (3g-2) - 2g = g - 2.$$

Let d be the dimension of the span $\langle L_{g-1}, N\rangle$ of the points of L_{g-1} and N. Since the dimension of $\mathbb{P}_N^{2g-2} = \langle N\rangle$ is $2g-2$, we have that $d \leq (g-2)+(2g-2)+1 = 3g-3$, where the equality holds iff $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$ is empty.

In particular, this intersection is non-empty iff $d \leq 3g-4$. Since $\dim |K+2D|^* = \dim \mathbb{P}_D^{3g-2} = 3g-2$, this is equivalent to the annihilator space

$$W := H^0(C, 2D + K - L_{g-1} - N) = H^0(C, K - L_{g-1})$$

being of dimension ≥ 2. By Riemann-Roch and Serre duality, we obtain that this condition is equivalent to $\dim |L_{g-1}| \geq 1$.

More precisely, let us suppose that $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$ is non-empty and let $e := \dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2})$. Then, we have that

$$d = 3g - 3 - (e+1),$$

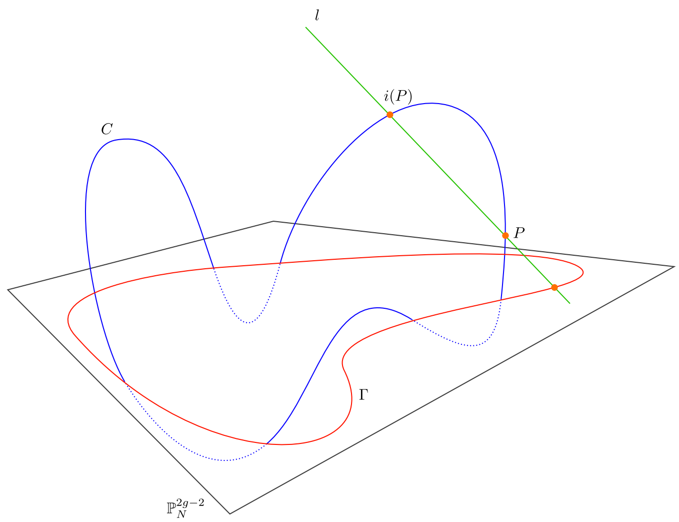

FIGURE 1. The situation in genus 3. The curves Γ and C intersect along the divisor D, of degree 6. The secant lines l cutting out the hyperelliptic pencil define the curve Γ.

and the annihilator space W is of dimension $2 + e$. Again by a Riemann-Roch computation, we conclude that $e = \dim |L_{g-1}| - 1$.

Finally, if L_{g-1} and N have some points in common, we have to count them only once when defining the vector space W to avoid requiring higher vanishing multiplicity to the sections.

□

From this Lemma, we conclude that $\mathrm{Sec}^{g-2}(N)$ is a proper subset of Sec^N if and only if there exists a divisor L_{g-1} not contained in N with $\dim |L_{g-1}| \geq 1$. By the Existence Theorem of Brill-Noether theory (see [2, Theorem 1.1, page 206]) this is possible only if $g \geq 4$ in the non-hyperelliptic case, whereas such a linear system may exist also when $g = 3$ when C is hyperelliptic. We will discuss the first low genera cases in Section 7.

5. The hyperelliptic case

From now on, C will be a *hyperelliptic* curve of genus $g \geq 3$.

As we have seen in Lemma 4.1, the base locus of the map $\varphi_{D,N}$ contains Sec^N. We have seen that the secant variety $\mathrm{Sec}^{g-2}(N)$ is contained in Sec^N and that this inclusion is strict for $g \geq 4$ in the non-hyperelliptic case.

5.1. **A rational normal curve coming from involution invariant secant lines.** In the hyperelliptic case, we have an additional base locus for every $g \geq 3$, which appears due to the hyperelliptic nature of the curve. This locus arises as follows: for each pair $\{p, i(p)\}$ of involution-conjugate points in C, consider the secant line l in $\mathbb{P}_D^{3g-2}$ passing through the points p and $i(p)$. Let Q_p be the intersection of the line l with $\mathbb{P}_N^{2g-2}$. Let us define $\Gamma \subset \mathbb{P}_N^{2g-2}$ as the locus of intersection points Q_p when p moves inside C.

Lemma 5.1. *The locus $\Gamma \subset \mathbb{P}^{2g-2}_N$ is a non-degenerate rational normal curve in $\mathbb{P}^{2g-2}_N$. Moreover, Γ passes through the $2g$ points $N \subset C$.*

Proof. Let us start by showing that the intersection Q_p is non-empty for every line $l = \overline{p, i(p)}$, with $p \in C$. Since $\dim |p + i(p)| = \dim |h| = 1$, the intersection $l \cap \mathbb{P}^{2g-2}_N$ is non-empty by Lemma 4.10.

Let us show that this intersection is a point, i.e. that the line l is not contained in $\mathbb{P}^{2g-2}_N$. Recall that $\mathbb{P}^{3g-2}_D = |2D + K|^*$. If the points p and $i(p)$ are both not contained in the divisor N, the vector space

$$V := H^0(C, 2D + K - N - (p + i(p)) = H^0(C, 2D + K - N - h)$$

is exactly the annihilator of the span $\langle l, \mathbb{P}^{2g-2}_N \rangle$ in $\mathbb{P}^{3g-2}_D$. In particular, the codimension of $\langle l, \mathbb{P}^{2g-2}_N \rangle$ in $\mathbb{P}^{3g-2}_D$ is the dimension of V. By Riemann-Roch and Serre duality, we get that $\dim V = g - 2$, thus $\dim \langle l, \mathbb{P}^{2g-2}_N \rangle = 3g - 2 - (g - 2) = 2g$. This means that the intersection $l \cap \mathbb{P}^{2g-2}_N$ is a point.

For the case $p \in N$ and $i(p) \notin N$, let us remark that the the annihilator of the span $\langle l, \mathbb{P}^{2g-2}_N \rangle$ is now the vector space $H^0(C, 2D + K - N - i(p))$. Since

$$h^0(C, 2D + K - N - i(p)) < h^0(C, 2D + K - N),$$

we conclude that the line l is not contained in $\mathbb{P}^{2g-2}_N$. The case $\{p, i(p)\} \subset N$ is excluded by our genericity hypotheses on N. Hence we deduce that the locus Γ is a curve in $\mathbb{P}^{2g-2}_N$.

Let q be a point of N. Then, q is a point of $\mathbb{P}^{2g-2}_N$. Consequently, the line passing through q and $i(q)$ intersects the plane $\mathbb{P}^{2g-2}_N$ at q. Thus, we have that Γ passes through the points of N. Moreover, it is clear that N is the only intersection of Γ and C, i.e. $\Gamma \cap C = N$.

Let us prove now that Γ is a rational normal curve. Since Γ is defined by the hyperelliptic pencil, it is straightforward to see that Γ is rational. Moreover, since the divisor D is general, the span of any subset of $2g - 1$ points of D is $\mathbb{P}^{2g-2}_N$. Thus, it suffices to show that the degree of $\Gamma \subset \mathbb{P}^{2g-2}_N$ is precisely $2g - 2$.

Let us set $N = q_1 + \cdots + q_{2g}$ with $q_1, \dots, q_{2g} \in C$. By the previous paragraph, Γ passes through these $2g$ points. Let us consider a hyperplane H of $\mathbb{P}^{2g-2}_N$ spanned by $2g - 2$ points of N. Without loss of generality, we can suppose that these points are the first $2g - 2$ points $q_1, \dots, q_{2g-2}$. To show that the degree of Γ is $2g - 2$, we have to show that the intersection of Γ with H consists exactly only of these points.

Let l be the secant line $\overline{q, i(q)}$, $q \in C$. The intersection $l \cap H$ is empty if and only if the linear span $\langle l, H \rangle$ of l and H in $\mathbb{P}^{3g-2}_D$ is of maximal dimension $2g - 1$, i.e. of codimension $g - 1$ in $\mathbb{P}^{3g-2}_D$. Consider the divisors

$$D_H = q_1 + \cdots + q_{2g-2} \qquad \text{and} \qquad D_l = q + i(q) \ .$$

As before, if $\{q, i(q)\} \cap \{q_1, \dots, q_{2g-2}\}$ is empty, the vector space $W = H^0(C, 2D + K - D_H - D_l)$ is the annihilator of the span $\langle l, H \rangle$ in $\mathbb{P}^{3g-2}_D$. In particular, the codimension of $\langle l, H \rangle$ in $\mathbb{P}^{3g-2}_D$ is given by the dimension of W. Again by Riemann-Roch and Serre duality theorems, we can check that

$$\dim W = h^0(C, -2D + D_H + D_l) + g - 1.$$

Thus, the codimension of $\langle l, H\rangle$ in $\mathbb{P}^{3g-2}_D$ is greater than $g-1$ if and only if $h^0(C, -2D + D_H + D_l) > 0$. Since $\deg(-2D + D_H + D_l) = 0$, this is equivalent to $-2D + D_H + D_l \sim 0$. Since $N = q_1 + \cdots + q_{2g} \sim 2D$, we have that

$$\begin{aligned} -2D + D_H + D_l \sim 0 &\iff q + i(q) \sim q_{2g-1} + q_{2g} \\ &\iff h \sim q_{2g-1} + q_{2g} \\ &\iff i(q_{2g-1}) = q_{2g}. \end{aligned}$$

By our genericity hypothesis on N, the last condition is not satisfied. Consequently, we conclude that the line l intersects the hyperplane H iff $\{q, i(q)\} \cap \{q_1, \ldots, q_{2g-2}\}$ is non-empty, i.e. iff q or $i(q)$ is one of the q_k for $k = 1, \ldots, 2g-2$. In particular,

$$\Gamma \cap H = \{q_1, \ldots, q_{2g-2}\}$$

as we wanted to show. □

Hence, the curve Γ is contracted by the map h_N to a point $w \in \mathcal{M}^{\mathrm{GIT}}_{0,2g}$ by Proposition 4.9. The point w represents a hyperelliptic curve C_w of genus $g-1$ together with an ordering of the Weierstrass points that correspond to the points of N on the rational curve Γ.

5.2. **The restriction of the theta map to $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$.** Let us set once again $N = p_1 + \cdots + p_{2g}$, a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}^{2g-2}_N$ in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

In this Section, we describe birationally the restrictions of θ to the fibers $p_D^{-1}(N)$ by means of the maps presented in Section 4.

5.2.1. *The global factorization.* Recall that the base locus of the map φ_D is the secant variety $\mathrm{Sec}^{g-2}(C)$ by Proposition 2.2. As in [6], one can construct a resolution $\widetilde{\varphi_D}$ of the map φ_D via sequence of blow-ups

$$\begin{array}{ccc} \widetilde{\mathbb{P}^{3g-2}_D} & & \\ \downarrow \mathrm{Bl}_{g-1} & \searrow \widetilde{\varphi_D} & \\ \vdots & & \\ \downarrow \mathrm{Bl}_1 & & \\ \mathbb{P}^{3g-2}_D & \overset{\varphi_D}{\dashrightarrow} & |2\Theta| \end{array}$$

along the secant varieties

$$C = \mathrm{Sec}^0(C) \subset \mathrm{Sec}^1(C) \subset \cdots \subset \mathrm{Sec}^{g-1}(C) \subset \mathbb{P}^{3g-2}_D.$$

This chain of morphisms is defined inductively as follows: the center of the first blow-up Bl_1 is the curve $C = \mathrm{Sec}^0(C)$. For $k = 2, \ldots, g-1$, the center of the blow-up Bl_k is the strict transform of the secant variety $\mathrm{Sec}^{k-1}(C)$ under the blow-up Bl_{k-1}.

The map φ_D is, by definition, the composition of the classifying map f_D and the degree 2 map θ. Thus, the map f_D lifts to a map $\widetilde{f_D}$ which makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{\mathbb{P}^{3g-2}_D} & \xrightarrow{\widetilde{f_D}} & \mathcal{SU}_C(2) \\ & \searrow^{\widetilde{\varphi_D}} & \downarrow \theta \\ & & |2\Theta| \end{array} \tag{7}$$

5.2.2. *Osculating projections.* We recall here a generalization of linear projections that will allow us to describe the map p in higher genus. For a more complete reference, see for example [22]. Let $X \subset \mathbb{P}^N$ be an integral projective variety of dimension n, and $p \in X$ a smooth point. Let

$$\begin{aligned} \phi : \mathcal{U} \subset \mathbb{C}^n &\longrightarrow \mathbb{C}^N \\ (t_1, \dots, t_n) &\longmapsto \phi(t_1, \dots, t_n) \end{aligned}$$

be a local parametrization of X in a neighborhood of $p = \phi(0) \in X$. For $m \geq 0$, let O^m_p be the affine subspace of $\mathbb{C}^N$ passing through $p \in X$ and generated by the vectors $\phi_I(0)$, where ϕ_I is a partial derivative of ϕ of order $\leq m$.

By definition, the *m-osculating space* $T^m_p X$ of X at p is the projective closure in $\mathbb{P}^N$ of O^m_p. The *m-osculating projection*

$$\Pi^m_p : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_m}$$

is the corresponding linear projection with center T^m_p.

5.2.3. *Osculating projections of $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$.* In this section we show how the map $\varphi_{D,N}$ induces an osculating projection on $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$.

Lemma 5.2. *Let Q be a r-form in $\mathbb{P}^n$ vanishing at the points P_1 and P_2 with multiplicity l_1 and l_2 respectively. Then, Q vanishes on the line passing through P_1 and P_2 with multiplicity at least $l_1 + l_2 - r$.*

Proof. See, for example, [20, page 2]. □

Let us now consider the linear system $|\mathcal{I}_{\mathrm{Sec}^N}(g)|$ on $\varphi_{D,N}$ (see Section 4). The forms in $|\mathcal{I}_{\mathrm{Sec}^N}(g)|$ vanish with multiplicity $g-1$ along the points of C (see Lemma 2.2). By Lemma 5.2, these forms vanish then with multiplicity $(g-1)+(g-1)-g = g-2$ along the secant lines l cutting out the hyperelliptic pencil. Thus, these forms vanish with multiplicity $g-2$ on the curve Γ.

Let us also consider the linear system $|\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$. Let $\mathcal{I}(\Gamma) \subset |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$ be the partial linear system of forms vanishing (with multiplicity 1) along $\mathrm{Sec}^{g-2}(N)$, and with multiplicity $g-2$ along Γ. By our previous observation and Lemma 4.1, we have the following inclusions of linear systems:

$$\mathcal{R} \subset |\mathcal{I}_{\mathrm{Sec}^N}(g)| \subset \mathcal{I}(\Gamma) \subset |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|.$$

These inclusions yield a factorization

$$\begin{array}{ccccc} \mathbb{P}^{2g-2}_N & \overset{h_N}{\dashrightarrow} & \mathcal{M}^{\mathrm{GIT}}_{0,2g} \subset |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|^* & \overset{\pi_N}{\dashrightarrow} & |\mathcal{I}(\Gamma)|^* \\ & \searrow_{\varphi_{D,N}} & & & \downarrow l_N \\ & & & & \theta(\mathcal{SU}_C(2)) \end{array} \tag{8}$$

The first map h_N is the one defined in Section 4.3, its image is the GIT quotient $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$. According to Proposition 4.9, this map contracts the curve Γ to a point $h_N(\Gamma)$.

Proposition 5.3. *The map π_N is the $(g-3)$-osculating projection Π_P^{g-3} with center the point $w = h_N(\Gamma)$.*

Proof. From the geometric description of the linear systems $\mathcal{I}(\Gamma)$ and $|\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$ (Prop. 4.8 and 4.9), the base locus of the map π_N is the point $w = h_N(\Gamma)$, with multiplicity $g-2$. In particular, since the forms in $\mathcal{I}(\Gamma)$ vanish with multiplicity $g-2$ along Γ, the order the projection π_N is $g-3$. □

According to Proposition 4.9, the map h_N contracts the curve Γ to a point w in $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ representing an ordered configuration of the $2g$ marked points N. This point in turn corresponds to a hyperelliptic genus $(g-1)$ curve C_w together with an ordering of the Weierstrass points. Now recall from Sec. 4.3 that the bottom composed map of Diag. 5 is the map h_N. The rational normal curve $\Gamma \subset \mathbb{P}_N^{2g-2}$ is contracted to a point $e_0 \in \mathbb{P}^{2g-3}$ s.t. $w = i_\Omega(e_0)$. Recall, once again from Sec. 4.3 that $\mathbb{P}^{2g-3}$ also contains the $2g-1$ points $\tau_k \circ \mathrm{Cr}_k(H_i)$, imagese of the hyperplanes $H_i \subset \mathbb{P}_N^{2g-2}$, with $i \neq k$ and $1 \leq i \leq 2g$. Let us lable them $e_1, \dots, e_{2g-1}$. Let now Λ be the partial linear system of Ω consisting of the $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in all the points $e_0, e_1, \dots, e_{2g-1}$. This linear sub-system induces an osculating linear projection $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow \Lambda^*$, as seen in Thm. 4.7.

Theorem 5.4. *The map π_N coincides with the map κ. In particular, the map π_N is of degree 2.*

Proof. Consider the GIT quotient $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ embedded in $|\Omega|^*$ as we have seen in Thm. 4.5. The osculating projection π_N is given by the linear system $|\mathcal{O}_{\mathcal{M}_{0,2g}^{\mathrm{GIT}}}(1) - (g-2)w|$ of hyperplanes vanishing in w with multiplicity $g-2$. By definition of Ω, this linear system pulls back via i_Ω to the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in $e_1, \dots, e_{2g-1}$, and also with multiplicity $g-2$ in e_0, which is precisely Λ. Hence, the map π_N is the map induced by the same linear system as κ (see Thm 4.7). □

We will show in the next Section that the map l_N from Diag 8 is actually birational, and that the map π_N coincides with the restriction of the map θ.

5.3. **The hyperelliptic theta map and Rational involutions on $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ and $\mathcal{SU}_C(2)$.** The resolution $\widetilde{\varphi_D}$ of φ_D factors through the degree 2 map θ as shown in Diagram 7. In the preceding Section we have shown that, when we restrict $\varphi_{D,N}$ to $\mathbb{P}_N^{2g-2}$, it factors through the degree 2 map π_N. Now we link these two factorizations. The identification of maps in the following claim must be intended as rational maps, since for example π_N is not everywhere defined.

Theorem 5.5. *Let $N \in |2D|$ be a general effective divisor. Then, the restricted map $\theta|_{f_D(\mathbb{P}_N^{2g-2})}$ is the map π_N up to composition with a birational map.*

Proof. Let us place ourselves on the open set $\widehat{\mathcal{SU}}_C(2) \subset \mathcal{SU}_C(2)$ of general stable bundles. First we remarl that the factorization $\widetilde{\varphi_D} = \theta \circ \widetilde{f_D}$ of Diagram 7 is the Stein factorization of the map $\widetilde{\varphi_D}$ along $\widetilde{\mathbb{P}_D^{3g-2}}$. Indeed, the map θ is of degree 2 as explained in Section 1. Moreover, the preimage of a general stable bundle E by the map f_D is the $\mathbb{P}^1$ arising as the projectivisation of the space of extensions of the form

$$0 \to \mathcal{O}(-D)) \to E \to \mathcal{O}(D) \to 0.$$

In particular, the fibers of $\widetilde{f_D}$ over $\widehat{\mathcal{SU}}_C(2)$ are connected.

The restriction of φ_D to $\mathbb{P}^{2g-2}_N$ factors through the maps h_N and π_N (see Diagrams 8), followed by the map l_N. According to Proposition 4.9, the fibers of h_N are rational normal curves, thus connected. Moreover, the map π_N is degree 2 by Theorem 5.4. By unicity of the Stein factorization, we have our result.

Comparing with the factorization $\widetilde{\varphi_D} = \theta \circ \widetilde{f_D}$, we see that l_N cannot have relative dimension > 0. Hence, l_N is a finite map. Since the degree of the map θ in the Stein factorization is 2, which is equal to the degree of π_N, we have that l_N cannot have degree > 1. In particular, we have that the map l_N is a birational map. □

From this description, the arguments of Section 5.2.3 and Thm. 4.7 we obtain the following

Theorem 5.6. *The restriction of θ to the general fiber of $p_{\mathbb{P}_c}$ ramifies on the Kummer variety of dimension $g-1$, obtained from the Jacobian of the hyperelliptic curve that is the double cover of $\mathbb{P}^1$ ramified along the $2g$ points represented by $P = h_N(\Gamma)$.*

Corollary 5.7. *One of the irreducible components of the ramification locus of the theta map is birational to a fibration in Kummer $(g-1)$-folds over $\mathbb{P}^g$.*

Results from [29, App. E] imply that the ramification locus is in fact non-irreducible.

6. The case $g = 3$

Let us now illustrate the geometric situation by explaining in detail the first case in low genus. We will often tacitly assume that when we say *map* we mean a rational map. Let C be a hyperelliptic curve of genus 3. In this setting, we have that the map θ factors through the involution i^*, and embeds the quotient $\mathcal{SU}_C(2)/\langle i^* \rangle$ in $\mathbb{P}^7 = |2\Theta|$ as a quadric hypersurface (see [3] and [12]). Let D be a general effective divisor of degree 3. The projective space $\mathbb{P}^7_D$, as defined in Section 1, parametrizes the extension classes in $\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D))$. The classifying map φ_D is given in this case by the complete linear system $|\mathcal{I}^2_C(3)|$ of cubics vanishing on C with multiplicity 2. The forms from this linear system vanish along the secant lines of C, and in particular along the secant lines passing through involution-conjugate points. These form a pencil parametrized by the linear system $|h|$.

The image of the projection of $\theta(\mathcal{SU}_C(2))$ with center $\mathbb{P}_c = \mathbb{P}^3 \subset |2\Theta|$ is also a $\mathbb{P}^3$, that is identified with $|2D|$ by Theorem 3.1. Let $N \in |2D|$ be a general reduced divisor. By Proposition 3.2, the closure of the fiber $p^{-1}_{\mathbb{P}_c}(N)$ is the image via φ_D of the $\mathbb{P}^4_N$ spanned by the six points of N.

6.1. The restriction to $\mathbb{P}^4_N$.

The base locus of the restricted map $\varphi_{D,N} = \varphi_D|_{\mathbb{P}^4_N}$ contains $\mathrm{Sec}^N = \mathrm{Sec}^1(C) \cap \mathbb{P}^4_N$ by Lemma 4.1. The secant variety $\mathrm{Sec}^1(N) \subset \mathrm{Sec}^N$ is the union of the 15 lines passing through pairs of the 6 points of N. According to Lemma 4.10, the further base locus $\mathrm{Sec}^N \setminus \mathrm{Sec}^1(N)$ is given by the intersections of $\mathbb{P}^4_N$ with the lines spanned by degree 2 divisors L_2 on C not contained in N satisfying $\dim |L_2| \geq 1$. By Brill-Noether theory, there is only one linear system of such divisors on a genus 3 curve, namely the hyperelliptic linear system $|h|$ (see, for example, [2], Chapter V). We will review these ideas in Section 7. This linear system defines, by the intersections with $\mathbb{P}^{2g-2}_N$ of the lines spanned by the hyperelliptic pencil, the curve Γ that we introduced in Section 5. Hence, we have that $\mathrm{Sec}^N = \{15 \text{ lines}\} \cup \Gamma$, and the restricted map $\varphi_{D,N}$ factors as

$$\begin{array}{ccc} \mathbb{P}^4_N & \overset{h_N}{\dashrightarrow} & \mathcal{M}_{0,6}^{\mathrm{GIT}} \subset \mathbb{P}^4 \\ & \underset{\varphi_{D,N}}{\searrow} & \downarrow p \\ & & \mathbb{P}^3 \end{array}$$

where h_N is the map defined by the complete linear system $|\mathcal{I}_{\mathrm{Sec}^1(N)}(3)|$ of cubics vanishing along the 15 lines defined by the points of N, and p is the projection with center the image via h_N of the rational normal curve Γ.

The image of $\varphi_{D,N}$ is a $\mathbb{P}^3$. Indeed, this image cannot have higher dimension, since the map factors through the projection from a point of $\mathcal{M}_{0,6}^{\mathrm{GIT}} \subset \mathbb{P}^4$. Also, it cannot have dimension strictly smaller than 3 since otherwise the relative dimension of $\varphi_{D,N}$ would be bigger than 1, or equivalently the global map φ_D would not surjet onto $\mathcal{SU}_C(2)$. Hence, in this case the map $\varphi_{D,N}$ is defined exactly by the system of cubics in $\mathbb{P}^4_N$ vanishing on Sec^N.

According to Proposition 4.9, the image of h_N is the GIT moduli space $\mathcal{M}_{0,6}^{\mathrm{GIT}}$ if N is general and reduced. It is a classical result that this GIT quotient is embedded in $\mathbb{P}^4$ as the Segre cubic S_3 (see for instance [13]). This 3-fold arises by considering the linear system of quadrics in $\mathbb{P}^3$ that pass through five points in general position, thus it is isomorphic to the blow-up of $\mathbb{P}^3$ at these points, followed by the blow-down of all lines joining any two points. The composition of this map with the projection off a smooth point of S_3 gives a $2:1$ rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ whose ramification locus is a Weddle surface ([19, 7]). The curve $\Gamma \subset \mathbb{P}^4_N$ is a rational normal curve by Lemma 5.1, hence Γ is contracted to a point P by h_N again by Proposition 4.9.

By [6] and Lemma 4.1, the linear system $|\mathcal{O}_{S_3}(1)|$ of hyperplanes in S_3 is pulled back by h_N to $|\mathcal{I}_{\mathrm{Sec}^1(N)}(3)|$ on $\mathbb{P}^4_N$. The linear system $|\mathcal{O}_{S_3}(1) - P|$ of hyperplanes in S_3 passing through P is pulled back to the complete linear system $|\mathcal{I}_{\mathrm{Sec}^N}(3)|$ defining $\varphi_{D,N}$. Hence, the map p is the linear projection with center P. Since S_3 is a cubic, the projection p is a degree 2 map. We will see in the next Section that this will be also the case for higher genus. The following proposition resumes what we have seen so far in this Section.

Proposition 6.1. *Let C be a hyperelliptic curve of genus 3. Then, for generic N, the restriction of φ_D to the subspace $\mathbb{P}^{2g-2}_N$ is exactly the composition $\kappa \circ h_N$.*

The point P in $\mathcal{M}_{0,6}^{\mathrm{GIT}}$ represents a rational curve with 6 marked points. Let C' be the hyperelliptic genus 2 curve constructed as the double cover of this rational curve ramified in these 6 points. According to Theorem 4.2 of [19], the Kummer variety $\mathrm{Kum}(C')$ is contained in the image of p, and it is precisely the ramification locus of π. Recall that, when $g = 3$, the linear system $|2D|$ is a $\mathbb{P}^3$. By Proposition 3.2, the image of $\mathbb{P}^4_N$ by φ_D is the closure of the fiber $p_{\mathbb{P}_c}^{-1}(N)$. For each point N in $|2D|$, this image is $\mathbb{P}^3 = |\mathcal{I}^2_{\mathrm{Sec}^N}(3)|^*$, which is the image of the Segre variety

$\mathcal{M}_{0,6}^{\mathrm{GIT}}$ under the projection with center P. Thus, the image of the global map φ_D is birational to a $\mathbb{P}^3$-bundle over $|2D| = \mathbb{P}^3$. Of course this is also the case since the image of the theta map is a quadric hypersurface in $\mathbb{P}^7$ [12].

7. Explicit descriptions in low genera

In this Section we will go through an explicit description of the classifying maps and how they factor through forgetful linear systems and osculating projections, for low values of the genus $g(C)$ of the hyperelliptic curve. In these cases the map remain fairly simple. These computations seem completely out of reach without the help of a computer for higher genus.

Recall from Section 4 that the intersection $\mathrm{Sec}^N = \mathrm{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2}$ arises naturally as part of the base locus of the restricted map $\varphi_{D,N}$. The subvarieties $\mathrm{Sec}^{g-2}(N)$ and Γ of Sec^N yield the factorization of $\varphi_{D,N}$ through the maps h_N and π_N of Proposition 5.3. Let us now describe the set

$$\mathrm{Sec}^{N\prime} = \mathrm{Sec}^N \setminus \{\Gamma \cup \mathrm{Sec}^{g-2}(N)\}.$$

This set is empty for $g = 3$, and the map $\varphi_{D,N}$ is exactly the composition of h_N and π_N, as described in Section 6. In higher genus, the existence of non-empty additional base locus Sec^N corresponds to the fact that the map $\varphi_{D,N}$ may not be exactly the composition of the maps h_N and π_N. In other words, the map l_N from Diag. 8 may not be non-trivial in higher genus.

This supplementary base locus is given by the intersections of $\mathbb{P}_N^{2g-2}$ with $(g-2)$-dimensional $(g-1)$-secant planes of C in $\mathbb{P}_D^{3g-2}$, which are not already supported on $\mathrm{Sec}^{g-2}(N)$ and Γ. According to Lemma 4.10, these intersections are given by effective divisors L_{g-1} on C of degree $g-1$, not contained in $\mathbb{P}_N^{2g-2}$, and satisfying $\dim |L_{g-1}| \geq 1$. Also by Lemma 4.10, we obtain $\dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1$.

We will now give account of the situation in low genera.

Case $g = 4$. In this case, the divisor N is of degree 8 and the map

$$\varphi_D|_N : \mathbb{P}_N^6 \subset \mathbb{P}_D^{10} \dashrightarrow |2\Theta| = \mathbb{P}^{15}$$

is given by the linear system $|\mathcal{I}_C^3(4)|$. This map factors through the map π_N which coincides with the 1-osculating projection Π_w^1, where $w = h_N(\Gamma)$.

We are looking for degree 3 divisors L_3 with $\dim |L_3| \geq 1$. These satisfy all $\dim |L_3| = 1$ and are of the form

$$L_3 = h + q \qquad \text{for } q \in C,$$

where h is the hyperelliptic divisor. Let p be a point of C. Then $L_3 = p + i(p) + q$. Since $\dim |L_3| = 1$, the secant plane $\mathbb{P}_{L_3}^2$ in $\mathbb{P}_D^{10}$ spanned by p, $i(p)$ and q intersects $\mathbb{P}_N^6$ in a point. But this point necessarily lies on Γ, since the line passing through p and $i(p)$ is already contained in this plane. Hence, we do not obtain any additional locus.

Case $g = 5$. In this case, the divisors L_4 of degree 4 are all of the form

$$L_4 = h + q + r \qquad \text{for } q, r \in C,$$

and satisfy $\dim|L_4| = 1$. Thus, the corresponding secant $\mathbb{P}^3_{L_4}$ spanned by p, $i(p)$, q and r intersects $\mathbb{P}^8_N$ in a point. As before, this point lies on Γ, thus we do not obtain any additional locus. The upshot is the following

Proposition 7.1. *Let C be a hyperelliptic curve of genus 4 or 5, then φ_D is defined by a (possibly equal) linear subsystem of the one defining $\kappa \circ h_N$, and set-theoretically the base loci of the two linear systems coincide.*

Case $g = 6$. Here we have, as in the genus 5 case, the divisors of the form

$$L_3 = h + q \qquad \text{for } q \in C,$$

which do not give rise to any additional base locus. But there is a new family of divisors

$$L_5 = 2h + r \qquad \text{for } r \in C.$$

These divisors satisfy $\dim|L_5| = 2$. In particular, the intersection of the $\mathbb{P}^4_{L_5}$ spanned by p, $i(p)$, q, $i(q)$ and r, for $p, q \in C$, with $\mathbb{P}^{10}_N$ is a line m in $\mathbb{P}^{10}_N$. The line l_1 (resp. l_2) spanned by p and $i(p)$ (resp. q, $i(q)$) intersects Γ in a point $\widetilde{p}$ (resp. $\widetilde{q}$). In particular, the line m is secant to Γ and passes through $\widetilde{p}$ and $\widetilde{q}$. Since every point of Γ comes as an intersection of a secant line in C with $\mathbb{P}^{10}_N$, we obtain the following description of the base locus of $\varphi_{D,N}$:

Proposition 7.2. *Let C be a curve of genus $g = 6$. Then, the base locus of the restricted map $\varphi_{D,N}$ contains the ruled 3-fold* $\mathrm{Sec}^1(\Gamma)$.

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