Applications de la théorie d'intersection des diviseurs de Hassett

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Overview

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- Quadrics on $\mathbb{C}$ are all rational (by projection)

Introduction

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- Dim 3: cubic threefolds are nonrational (Intermediate Jacobian, $\sim$ 1972).
- Dim 4: cubic fourfolds? conjecture: very general cubic fourfolds are nonrational/ only rational examples.

Moduli space

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- $\mathcal{C}$ is a 20-dimensional quasi-projective variety.

Hodge decomposition

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$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X),$$

where $H^{p,q}(X) := H^q(X, \Omega^p)$.

Hodge diamond

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4}
\end{array}
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$$
\begin{array}{ccccccccc}
\color{red}{0} & & \color{red}{1} & & \color{red}{21} & & \color{red}{1} & & \color{red}{0} \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
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Lattice theory for cubic fourfolds

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$$H^4(X,\mathbb{Z}) \cong \ (+1)^{\bigoplus 21} \bigoplus (-1)^{\bigoplus 2},$$

and the primitive cohomology lattice:

$$H^4(X,\mathbb{Z})_{prim} :=< h^2 >^{\perp} \cong \ E_8^{\bigoplus 2} \bigoplus U^{\bigoplus 2} \bigoplus A_2,$$

$h \in H^2(X,\mathbb{Z})$ is a hyperlane section;
E_8 is the positive definite even lattice of rank 8 associated to the corresponding Dynkin diagram;
$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic plane; and $A_2 := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

General stuff

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The integral Hodge conjecture holds, and $\sim_{rat} = \sim_{alg} = \sim_{hom}$ for codimension 2 cycles on cubic 4folds, hence:

- $A(X) = H^2(X, \Omega_X^2) \cap H^4(X, \mathbb{Z})$ the lattice of algebraic 2-cycles on X up to rational equivalence.
- For a very general cubic fourfold X, any algebraic surface $T \subset X$ is homologous to a multiple of h^2, so that $rk(A(X)) = 1$.

Special cubic fourfolds

Definitions (H)

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with d the determinant of the intersection form on K_d;

$$K_d := \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & d_T \\ T & d_T & (T,T) \end{array}$$

where the self-intersection $(T,T) = c_2(\mathcal{N}_{T/X}) =$
$6H^2 + 3H.K_T + K_T^2 - \chi_T$, $H = h|S$, χ_T *top. Euler char*;
$d = 3(T,T) - d_T^2$.

Special cubic fourfolds

- The Noehter-Lefschetz loci $\mathcal{C}_d \subset \mathcal{C}$ parametrizing special cubic fourfolds with a labelling of discriminant d are divisors, i.e. codimension one subvarieties. These divisors $\mathcal{C}_d$ are called Hassett divisors.

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Proposition (H)

$\mathcal{C}_d$ is irreducible and nonempty iff $d > 6$ and $d \equiv 0, 2 \ (mod\ 6)$.

Examples: special cubic fourfolds

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- $\mathcal{C}_{14} \leftrightarrow$ cubic fourfolds containing a quartic scroll S_{14}:

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  It is the same if we pass to codimension 1 sub-Hodge structure of signature (2,19) (up to a Tate twist for $H^2(S)$, that changes weight of the HS and sign of the intersection form).

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For a special cubic fourfold $X \in \mathcal{C}_d$, there exists a polarized K3 surface S of degree d such that there is a Hodge-isometry

$$H^4(X,\mathbb{Z}) \supset K_d^{\perp} \cong H^2_{prim}(S,\mathbb{Z})(-1)$$

iff

d *is not divisible by 4, 9, or any odd prime number* $p \equiv 2\ [3]$ $(**)$.

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- $K_d^{\perp}$ is often called non-special cohomology.

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Work of Kuznetsov, Macrí and others about this.

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For (infinitely many) values of d such that $d = 2(n^2 + n + 1)$ for an integer $n \geq 2$, there is an isomorphism

$$F(X) \cong S^{[2]} \tag{1}$$

between $F(X)$ and the Hilbert scheme of length two subschemes $S^{[2]}$ of the associated K3 surface.

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onto a 8-dimensional HK variety, called the LLSvS 8-fold [LLSvS].

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For infinitely many values of d (see later) there is a birational map

$$L(X) \sim S^{[4]}.$$

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This has a relation with rational normal quartics similar to that of $L(X)$ with twisted cubics.

Recollect some information

Condition on d	Comments for $X \in \mathcal{C}_d$
$d \geq 8$ and $d \equiv 0$ or 2 $(mod\ 6)$	$\mathcal{C}_d$ non empty
$d \not| 4$, 9, or any odd prime number $p \equiv 2$ $(mod\ 3)$	$H^2(S,\mathbb{Z})_{prim}(-1) \simeq K_d^{\perp}$ (Hodge isom.); $Ku(X) \cong \mathrm{D}^b(S)$
$d = \frac{2n^2+2n+2}{a^2}$, for some $n,\ a \in \mathbb{Z}$	$F(X) \sim S^{[2]}$
$d = 2n^2 + 2n + 2$, for some $n,\ a \in \mathbb{Z}$	$F(X) \cong S^{[2]}$
$d = \frac{6n^2+6n+2}{a^2}$, $\quad n, a \in \mathbb{Z}$	$L(X) \sim S^{[4]}$

Intersection of Hassett divisors

Proposition (Yang-Yu)

Any two Hassett divisors intersect i.e. $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \neq \emptyset$ *for any integers* d_1 *and* d_2 *such that the* $\mathcal{C}_{d_i}$ *are not empty.*

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Theorem (ABP)

For $3 \leq n \leq 20$,

$$\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset,$$

for $d_k > 6, d_k \equiv 0, 2[6]$ and $d_3, .., d_n = 6\prod_i p_i^2$ or $6\prod_i p_i^2 + 2$ with p_i a prime number.

Lemma

Let N be a positive definite lattice of rank $r(N) \geq 2$, that admits a saturated embedding

$$h^2 \in N \subset H^4(X, \mathbb{Z}),$$

Let $\mathcal{C}_N \subset \mathcal{C}$ the locus of cubic fourfolds X having algebraic classes with lattice structure $N \subset A(X)$. If $\mathcal{C}_N$ is non-empty, then it has codimension $r(N) - 1$ and there exists $X \in \mathcal{C}_N$ with $A(X) = N$.

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Rephrasing the Theorem: given $d_1, \ldots, d_r$, we have an "algorithm"to construct a lattice $M_{d_1,\ldots,d_r}$ of rank $r+1$ s.t. $\mathcal{C}_{M_{d_1,\ldots,d_r}} \neq \emptyset$ is contained in $\bigcap_{i=1}^r \mathcal{C}_{d_i}$, and is of codimension r in the moduli space.

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- $\mathcal{C}_M$ is non-empty, and of codimension r in $\mathcal{C}$;
- there are rank 2 sublattices K_{d_i} of M;
- this gives conditions on the d_i.

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- $\mathcal{C}_M$ is non-empty, and of codimension r in $\mathcal{C}$;
- there are rank 2 sublattices K_{d_i} of M;
- this gives conditions on the d_i.

The generic cubic in $\mathcal{C}_M$ has $rk(A(X)) = r+1$ and if one of the $\mathcal{C}_{d_i}$ parametrizes cubics X with associated K3 S_X, then (generically) $rk(NS(S_X)) = r$.

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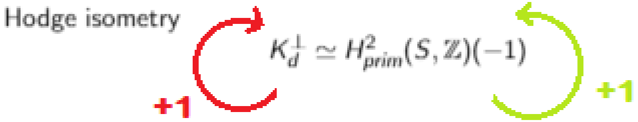

- One can find loci, of dimension $20-n$, parametrizing cubic fourfolds with associated K3 surfaces of Néron-Severi rank n, with $1 \leq n \leq 20$, inside any divisor $\mathcal{C}_d$.

Consequences

Corollary

Call $\mathcal{F} = \mathcal{C}_{M_{d_1,\dots,d_{19}}} \subset \bigcap_{k=1}^{19} \mathcal{C}_{d_k} \neq \emptyset$ *the family constructed this way, and s.t. at least one* $\mathcal{C}_{d_k}$ *has associated K3s. Then cubic fourfolds in* $\mathcal{F}$ *have associated K3 surface* S *s.t.* $rk(NS(S)) \geq 19$. *These are called singular K3 surfaces.*

Definition

The category of Chow motives $\mathcal{M}_{rat}(\mathbb{C})$ consists of triples (X, p, m) with X a projective smooth variety over $\mathbb{C}$, $p \in Corr^0_{rat}(X, X)$ is a projector and m an integer. The morphisms are as follows:

$$Hom_{\mathcal{M}_{rat}(\mathbb{C})}((X, p, m_1), (Y, q, m_2)) = q \circ Corr^{m_2 - m_1}_{rat}(X, Y) \circ p$$

Reminder: $Corr^r_{rat}(X_d, Y) := Z^{d+r}_{rat}(X \times Y, \mathbb{Q})$.

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$$\begin{aligned} h\colon SmProj_{/\mathbb{C}} &\to \mathcal{M}_{rat}(\mathbb{C}) \\ X &\mapsto h(X) = (X, \Delta_X, 0) \\ f : X \to Y &\mapsto h(f) = \Gamma_f^T : h(Y) \to h(X), \end{aligned}$$

with Δ_X the diagonal embedding $X \hookrightarrow X \times X$.

A few important properties

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- $\mathcal{M}_{rat}(\mathbb{C})$ has good properties with respect to Weil cohomological theories (e.g. Betti cohomology). There is a realization functor H^*:

$$\begin{tikzcd} & \mathcal{M}_{rat}(\mathbb{C}) \arrow[dr, "H^*"] & \\ SmProj_{/\mathbb{C}}^{opp} \arrow[ur] \arrow[rr, "cohomology"] & & GrVect_{\mathbb{C}} \end{tikzcd}$$

through which every "good"cohomology theory factors.

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and the dual of $h(X)$ is $h(X) \otimes \mathbb{L}^{\otimes - dim(X)}$.

Chow-Künneth decomposition
We say that the motive of X smooth, projective of dimension d, $h(X) \in \mathcal{M}_{rat}(\mathbb{C})$ has a Chow-Künneth decomposition if there exist orthogonal projectors $\pi_i = \pi_i(X) \in Corr^0_{rat}(X, X)$ for $0 \leq i \leq 2n$, s.t. $\pi_1 + \cdots + \pi_{2d} = \Delta_X$ and there is a direct sum decomposition

$$h(X) = h^0(X) \oplus \cdots \oplus h^{2d}(X),$$

with $h^i(X) = (X, \pi_i, 0)$, such that for any $0 \leq i \leq 2d$, the Betti realization $H^*(h^i(X)) = H^i(X)$.

Finite-dimensional motives

Let Σ_m be the symmetric group of order m. For an object M in $\mathcal{M}_{rat}(\mathbb{C})$, we denote by $\bigwedge^m M$ the $m^{th}-$exterior power of M which is the image of the following projector:

$$\frac{1}{m!}\Sigma_{\sigma\in\Sigma_m} sign(\sigma)[\Gamma_\sigma] : M^{\bigotimes m} \to M^{\bigotimes m},$$

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Conjecture (Kimura and O'Sullivan)

Every Chow motive is finite dimensional.

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- not surprisingly $\mathbb{1}$ and $\mathbb{L}$ are finite dimensional.

Abelian Chow motives

Let $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$ be the full, rigid, tensor subcategory of $\mathcal{M}_{rat}(\mathbb{C})$ generated by the motives of Abelian varieties. All the examples of motives that have been proven to be finite-dimensional belong to the category $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$

Examples

1. projective spaces, Grassmannian varieties, projective homogeneous varieties, toric varieties;
2. smooth projective curves;
3. Kummer K3 surfaces;
4. K3 surfaces with Picard numbers at least 19;
5. Hilbert schemes of points on abelian surfaces;
6. Fermat hypersurfaces ;
7. Cubic 3-folds and their Fano surfaces of lines.

Cubic fourfolds with Chow motives of abelian type

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Proof: The motive of a cubic fourfold has a Chow-Künneth decomposition

$$h(X) = \mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}^{\rho_2(X)} \oplus t(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4,$$

where $\rho_2(X) = rk(CH^2(X))$ and $t(X)$ is the *transcendental motive* of X, i.e. $H^*(t(X)) = H^4_{tr}(X, \mathbb{Q})$, [BP].

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then [BP,Bü] there exists a K3 surface S s.t.

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Proposition (P)

Let S be a smooth complex projective K3 surface with $\rho(S) = 19, 20$. Then the motive $h(S) \in \mathcal{M}_{rat}(\mathbb{C})$ is finite dimensional and of Abelian type.

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Choose $\mathcal{C}_d$ any divisor of special cubic fourfolds. We can choose appropriately 17 divisors $\mathcal{C}_{d_1}, \ldots, \mathcal{C}_{d_{17}}$ such that the family

$$\mathcal{F} = \mathcal{C}_{M_{d,14,d_1,\ldots,d_{17}}} \subset \mathcal{C}_d \cap \mathcal{C}_{14} \cap (\bigcap_{k=1}^{17})\mathcal{C}_{d_k}$$

is non-empty, one-dimensional and contained - by definition - in $\mathcal{C}_{14}$.

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A countable infinity of families like $\mathcal{F}$?

Density of abelian motives in $\mathcal{C}_d$

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Corollary

If $d \not| 4$, 9, or any odd prime $p \equiv 2 \pmod 3$, cubic fourfolds with Abelian motive are dense (in the complex topology) inside the divisors $\mathcal{C}_d$.

An aside on rationality and the condition $t(X) \cong t_2(S)(-1)$

If $X \in \mathcal{C}_d$ and $\exists f, g \in \mathbb{Z}$ $s.t.$ $g|(2n^2+2n+2)$, $n \in \mathbb{N}$ and $d = f^2 g$, then there exists a K3 surface S s.t. $t(X) \cong t_2(S)(-1)$.

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Heuristically: for $d = 8$, $H^4(X, \mathbb{Z}) \supset K_8^\perp$ is an index two sublattice of $H^2_{prim}(S, \mathbb{Z})$ [vG], where S is a degree 2 K3 surface related to the Hilbert scheme of lines in X.

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(H-K condition does not hold for the red values)

Heuristically: for $d = 8$, $H^4(X, \mathbb{Z}) \supset K_8^{\perp}$ is an index two sublattice of $H^2_{prim}(S, \mathbb{Z})$ [vG], where S is a degree 2 K3 surface related to the Hilbert scheme of lines in X. In the category of Chow motives we work with coefficients in $\mathbb{Q}$, hence the index two morally "disappears"

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The condition $t(X) \cong t_2(S)(-1)$ is likely to be necessary for rationality, but not sufficient (motives with $\mathbb{Z}$-coefficients for a proper criterion?).

Consequences: Motives of associated Hyperkähler varieties

Corollary (ABP)

All Hyperkähler 4folds $F(X)$ and Hyperkähler 8folds $L(X)$, $X \in \mathcal{F}$, have finitely generated and Abelian Chow motive.

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Condition on d	Comments for $X \in \mathcal{C}_d$
$d \geq 8$ and $d \equiv 0$ or 2 $(mod\ 6)$	$\mathcal{C}_d$ non empty
$d \not| 4$, 9, or any odd prime number $p \equiv 2$ $(mod\ 3)$	$H^2(S,\mathbb{Z})_{prim}(-1) \simeq K_d^{\perp}$ (Hodge isom.); $Ku(X) \cong \mathrm{D}^b(S)$
$d = \frac{2n^2+2n+2}{a^2}$, for some $n,\ a \in \mathbb{Z}$	$F(X) \sim S^{[2]}$
$d = 2n^2 + 2n + 2$, for some $n,\ a \in \mathbb{Z}$	$F(X) \cong S^{[2]}$
$d = \frac{6n^2+6n+2}{a^2}$, $\quad n, a \in \mathbb{Z}$	$L(X) \sim S^{[4]}$

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Hence we have an isomorphism $F(X) \cong S^{[2]}$ and a birational equivalence $L(X) \simeq S^{[4]}$. By [dCM] and invariance of Chow motives of HK varieties under birational transformations we conclude. ∎

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$$\mathcal{C}_{14} := \overline{\{cubics\ containing\ a\ del\ Pezzo\ quintic\}} \subset \mathcal{C}.$$

Theorem

There exist an infinity of Pfaffian cubic fourfolds $X \in \mathcal{C}_{14}$ such that the motive $h(\mathcal{J}(X))$ of the associated LSV 10-fold is finite dimensional and Abelian. These cubics are dense in $\mathcal{C}_{14}$.

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is birational, and K3 surfaces with $rk(NS(S)) \geq 19$ are dense inside $\mathcal{G}_8$. Hence there are infintely many Pfaffian 4folds with finite dimensional, Abelian motive, since the associated K3 has this property.

Theorem (LSV)

If X is a smooth Pfaffian cubic fourfold, then $\mathcal{J}(X)$ is birational to the moduli space $\mathcal{M}_{2,0,4}(S_X)$ parameterizing rank-2 semi-stable sheaves on S_X with $c_1 = 0$ and $c_2 = 4$.

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Thanks for your patience...

General stuff

Precise definition: X smooth projective variety, an algebraic cycle $Z \in Z^i(X)$ is rationally equivalent to zero if there exist $W \in Z^i(X \times \mathbb{P}^1)$, flat over $\mathbb{P}^1$, $a, b \in \mathbb{P}^1$ s.t. if $W(t) := (pr_X)_*(W \cdot (X \times t))$, we have

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