UNIRATIONALITY OF CERTAIN UNIVERSAL FAMILIES OF CUBIC FOURFOLDS.

MICHELE BOLOGNESI AND HANINE AWADA

Abstract. The aim of this short note is to define the *universal cubic fourfold* over certain loci of the moduli space. Then, by using some recent results of Farkas-Verra and more classical ones by Mukai, we prove that the universal cubic fourfolds over the Hassett divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{42}$ are unirational. As for the rationality of cubic fourfolds, this relies heavily on the existence of associated K3 surfaces, and the birational geometry of their universal families. Finally, we observe that for explicit infinitely many values of d, the universal cubic fourfold over $\mathcal{C}_d$ can not be unirational.

1. Introduction

Since the celebrated result of Clemens and Griffiths [CG72] on the irrationality of smooth cubic threefolds, at least three generations of algebraic geometers have been working hard on the problem of rationality of cubic fourfolds. The search for the good invariant that would detect (ir)rationality has involved Hodge theoretical [Has16], homological [Kuz10], and even motivic [BP] methods. While the general suspicion is that the generic cubic fourfold should be non rational as well, no explicit example of irrational cubic fourfold is known at the moment. A little more is known, and the expectation are slightly more precise, about rationality. Let $\mathcal{C}$ denote the moduli space of smooth cubic fourfolds and $\mathcal{C}_d \subset \mathcal{C}$ the Hassett divisor of discriminant d, then a well known conjecture due to Harris-Hassett-Kuznetsov (see [Kuz10, Has16, AT14]) states that there should be a countable infinity of Hassett divisors that parametrize rational cubic fourfolds. In particular these divisors should correspond to cubic fourfolds that have an associated K3 surface, in a Hodge theoretical or derived categorical sense (see [Kuz10, Has00]). In the second decade of this century, the search for rational examples of cubic fourfolds has concentrated more on cubics containing some special surfaces [ABBVA14, Nue15, AHTVA19, Awa19]. This has been developed further in the works of Russo-Staglianó, and at the moment the HHK conjecture is known to be true for four low values of d [BD85, BRS19, RS19b, RS19a].

In this note we are also interested in the birational geometry of cubic fourfolds, but in a slightly different direction. Universal families are well known and studied objects in the realm of moduli spaces of curves, where forgetful maps are the algebraic geometer's everyday tools. Also in the context of K3 surfaces several results about these objects have appeared in the literature, also very recently [FV18, FV19, Ma19]. Probably because the birational geometry of a single cubic fourfold is already so difficult to understand, the problem of studying the same problems for universal families of cubics seems to have been overtaken. Thanks to the recent advances in the study of rationality of a single cubic fourfold, it seems however natural to ask what can be said in families, in particular over certain relevant divisors $\mathcal{C}_d$ in the moduli space. In this paper, in particular, we study the universal cubic fourfolds $\mathcal{C}_{14,1} \to \mathcal{C}_{14}$, $\mathcal{C}_{26,1} \to \mathcal{C}_{26}$ and $\mathcal{C}_{42,1} \to \mathcal{C}_{42}$ over the divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{42}$, and prove the following fact.

Theorem 1.1. *The universal cubic fourfolds $\mathcal{C}_{14,1}$, $\mathcal{C}_{26,1}$ and $\mathcal{C}_{42,1}$ are unirational.*

Of course, though the intuitive definition is clear, this requires a few checks that actually the universal family exists, which we make in Sect. 2. In the same Section, we also collect some important properties of cubic fourfolds, the rational scrolls they contain, their associated K3 surfaces and their Fano varieties of lines, that are then necessary in Section 3. Section 3 is rather devoted to the proof of unirationality, which is based on the construction of a couple of rational/unirational moduli quotients involving the families of scrolls inside our cubic fourfolds.

In the second part of Section 3 we underline the special nature of the cases we are considering, by comparing them with preceding results of Nuer [Nue15], Gritsenko-Hulek-Sankaran [GHS13] and Várilly-Alvarado-Tanimoto [TVA19].

In fact, in [GHS13] it is proven that the moduli space $\mathcal{F}_{2m}$ of polarized K3 surfaces of degree $2m$ has positive Kodaira dimension or is even of general type for an infinity of values (see Prop. 3.13 for details).

Furthermore one expects that the gaps in Prop. 3.13 can be filled in by using automorphic form techniques. Várilly-Alvarado and Tanimoto [TVA19] have started pursuing this project obtaining another relevant range of values where the Kodaira dimension of $\mathcal{C}_d$ is positive (see Prop. 3.14).

These results give us *negative* information about the divisors $\mathcal{C}_d$ where the universal cubic $\mathcal{C}_{d,1}$ can not be unirational.

Proposition 1.2. *The universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational if:*

1. $d > 80$, $d \equiv 2\ (mod 6)$, $4 \not| d$ *and such that for any odd prime* p, $p|d$ *implies* $p \equiv 1\ (mod 3)$;
2. $d > 122$;
3. $d = 6n + 2$ *for* $n > 18$ *and* $n \neq 20, 21, 25$;
4. $d = 6n + 2$ *and* $n = 14, 18, 20, 21, 25$;
5. $d = 6n$ *for* $n \geq 34$, $n \neq 19, 21, 24, 25, 26, 28, 29, 30, 31$;
6. $d = 6n$ *for* $n = 17, 23, 27, 33$.

The reader will realize that our proofs are little more than a survey of recent literature on the subject, that - glued together - automatically gives us our results. However, since the object and the question we ask are so natural, it seemed to us worth to put this into a short text.

We wish to thank Francesco Russo and Sandro Verra for lots of discussions on related topics in the last few months. Thanks to Shouhei Ma and Zhiwei Zheng for giving us precise references and information.

2. The existence of the universal cubic fourfold, and some properties of scrolls and associated K3 surfaces.

As it is customary when discussing moduli spaces, one of the first questions one considers is the actual existence of a universal family. In particular this is an important question when the moduli space is obtained as a GIT quotient, which is our case. The invariant theory of cubic fourfolds, with respect to the natural $PGL(6)$-action, is known by the work of Laza [Laz09, Laz10].

Theorem 2.1. [Laz09, Thm. 1.1] *A cubic fourfold $X \subset \mathbb{P}^5$ is not GIT stable if and only if one of the following conditions holds:*

1. *X is singular along a curve C spanning a linear subspace of $\mathbb{P}^5$ of dimension at most 3;*
2. *X contains a singularity that deforms to a singularity of class P_8, X_9 or J_{10}.*

In particular, if X is a cubic fourfold with isolated singularities, then X is stable if and only if X has at worst simple singularities.

The GIT quotient that we are considering is slightly different than the one constructed in [Laz09], since we only consider smooth cubic fourfolds. That is, we consider the complement $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ of the discriminant and take the quotient $\mathcal{C} := U//PGL(6)$. The resulting quotient is a quasi-projective variety of dimension 20 and Thm. 2.1 assures that all point are stable in the GIT sense. We are in particolar concerned by the divisors $\mathcal{C}_{14}, \mathcal{C}_{26}$ and $\mathcal{C}_{42}$ inside $\mathcal{C}$. By the seminal work of Hassett [Has00], and further developments [Nue15, RS19b, RS19a, FV18, FV19], they can be characterized in terms of *special* scrolls - not homologous to linear sections - contained in the generic cubic fourfold in each divisor. Recall in fact that for the general cubic fourfold $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$, generated by a linear section, and the locus where the rank of this group is bigger than one is the union of a countable infinity of divisors in $\mathcal{C}$.

Definition 2.2. We have the following descriptions:

- $\mathcal{C}_{14} := \overline{\{\text{Cubic fourfolds containing a quartic scroll}\}}$
- $\mathcal{C}_{26} := \overline{\{\text{Cubic fourfolds containing a } 3-\text{nodal septic scroll}\}}$
- $\mathcal{C}_{42} := \overline{\{\text{Cubic fourfolds containing a } 8-\text{nodal degree 9 scroll}\}}$.

Moreover, cubic fourfolds X inside these three divisors have an associated K3 surface[1]. As it is well known, this means that the *Kuznetsov component* of the bounded derived category of X is equivalent to the derived category of a K3 surface, or equivalently that the Hodge structure of the nonspecial cohomology of the cubic fourfold is essentially the Hodge structure of the primitive cohomology of a K3 surface. We refrain to give more details about this, since these notions are now well known and the references in the literature are very good [Kuz10, Has00, Has16]. The associated K3 surfaces for cubics from $\mathcal{C}_{14}$ are genus 8 and degree 14, for $\mathcal{C}_{26}$ genus 14 and degree 26, whereas for cubics from $\mathcal{C}_{42}$ they are genus 22 ans degree 42.

Remark 2.3. The case $d = 42$ deserves a few words more. In fact, a priori, a cubic in $\mathcal{C}_{42}$ has *two* associated K3 surfaces, but by considering the degree 9, 8-nodal scrolls scrolls we implicitely choose the genus 22 K3 surface. The explaination is due to [FV19] and roughly speaking is due to the fact that the family of these scrolls contained in a cubic fourfold $X \in \mathcal{C}_{42}$ is morally the genus 22 K3 surface, and not the other - see Prop. 2.7(3). The existence of two associated K3 surfaces should not puzzle us too much, as long as we consider the wealth of K3 surfaces that have

[1]Actually cubics in $\mathcal{C}_{42}$ have two associated K3s but later in the paper it will be clear that we will be concerned only by one type of these.

equivalent (possibly twisted) derived categories. This is why we will not linger any more on this ambiguity.

Remark 2.4. It is impossible to discuss cubic fourfolds inside these divisors without some talk about the recent work of [RS19b, RS19a]. In these two papers a new impressive way of showing rationality of certain cubic fourfolds has been introduced, that is *congruences of rational curves.* The idea is rather simple, but very powerful, and it is a generalization of the argument of [Fan43] and [BRS19] for $\mathcal{C}_{14}$ with quartic scrolls, considered as *one apparent double point* varieties.

Suppose that a cubic fourfold X contains a surface Z such that there exists a 4-dimensional family V of degree d rational curves, intersectig Z in $3d-1$ points. By studying the linear system $|(3d-1)-dZ|$, Russo and Staglianó observe that in the case of our three divisors (and $\mathcal{C}_{38}$ as well), the base V is rational and the rational curves are contracted by the linear system onto V. This induces a natural birational map between X and V, that associates the only point where the generic rational curve intersects X outside of Z to the point of V parametrizing the curve itself. Though the work of Russo and Staglianó is the main inspiration of this circle of ideas, we observe that our results do not make use of theirs.

Let $\mathcal{F}_g$ denote the moduli space of genus g K3 surfaces. For $X \in \mathcal{C}_d$ with discriminant $d = 2(n^2+n+1)$, with $n \geq 2$ (remark that 14, 26 and 42 all verify this equality), Hasset shows [Has00, Sect. 6] that there is an isomorphism

$$F(X) \cong S^{[2]} \tag{2.5}$$

between the Fano variety of lines $F(X)$ and the Hilbert scheme $S^{[2]}$ of (S,H) a polarized K3 surface with $H^2 = d$, the K3 *associated* to X. The moduli spaces of the corrisponding K3 surfaces have dimension 19, the same dimension as divisors in $\mathcal{C}$, and this assignment induces a rational map

$$\mathcal{F}_{\frac{d+2}{2}} \dashrightarrow \mathcal{C}_d \tag{2.6}$$

that is birational for $d \equiv 2\ (mod 6)$ and degree two for $d \equiv 0 (mod 6)$. In particular, for the three divisors we are concerned in, we have rational maps

$$\begin{array}{ccc} \mathcal{C}_{14} & \overset{\sim}{\dashrightarrow} & \mathcal{F}_8; \\ \mathcal{C}_{26} & \overset{\sim}{\dashrightarrow} & \mathcal{F}_{14}; \\ \mathcal{C}_{42} & \overset{\sim}{\dashrightarrow} & \mathcal{F}_{22}; \end{array}$$

where the third map is birational and not degree two because of the observations in Rmk. 2.3.

Another upshot of the isomorphsm (2.5) is that, once we fix a smooth cubic fourfold in $\mathcal{C}_{14}, \mathcal{C}_{26}$ or $\mathcal{C}_{42}$, the family of scrolls contained inside X is precisely parametrized by the associated K3. The construction, roughly speaking, goes as follows. For each $p \in S$, one defines a rational curve

$$\Delta_p := \{y \in S^{[2]} : \{p\} = Supp(y)\}.$$

The image of Δ_p inside $F(X)$ then defines a (possibly singular) scroll $R_p \subset X$. For our range of values of d the precise results are the following.

Proposition 2.7. [BD85, BRS19, FV18, FV19]

(1) *The family of quartic scrolls inside a generic cubic fourfold X in $\mathcal{C}_{14}$ is the genus 8 K3 surface associated to X.*
(2) *The family of septimic 3-nodal scrolls inside a generic cubic fourfold X in $\mathcal{C}_{26}$ is the genus 14 K3 surface associated to X.*
(3) *The family of 8-nodal degree 9 scrolls inside a generic cubic fourfold X in $\mathcal{C}_{42}$ is the genus 22 K3 surface associated to X.*

Remark 2.8. Actually the construction of the scroll R_p for cubics in $\mathcal{C}_{14}$ does not give a quartic scroll, but rather a quintic scroll $V \subset X$. Anyway it is well-known there is always a quartic scroll linked to V. Take one Segre product $\Sigma_{1,2} := \mathbb{P}^1 \times \mathbb{P}^2$ containting V and consider the degree 9 surface $X \cap \Sigma_{1,2} \supset V$. The quartic scroll is the residual surface to V inside $X \cap \Sigma_{1,2}$.

Recall that the generic K3 surface in $\mathcal{F}_8$, $\mathcal{F}_{14}$ and $\mathcal{F}_{22}$ has no automorphism, hence universal families exist:

$$\begin{aligned} f_8 : \mathcal{F}_{8,1} &\to \mathcal{F}_8; \\ f_{14} : \mathcal{F}_{14,1} &\to \mathcal{F}_{14}; \\ f_{22} : \mathcal{F}_{22,1} &\to \mathcal{F}_{22}. \end{aligned}$$

at least over an open subset of each moduli space. This has allowed the study of $\mathcal{F}_{14,1}$ and $\mathcal{F}_{22,1}$ by Faraks-Verra in [FV18, FV19] and of $\mathcal{F}_{8,1}$ by Mukai [Muk88] Their main results are resumed here below.

Proposition 2.9. *The universal K3s $\mathcal{F}_{8,1}$ and $\mathcal{F}_{22,1}$ are unirational. The universal K3 $\mathcal{F}_{14,1}$ is rational.*

Let us now come to the definition of the object that we will study in Sect. 3. Smilarly to the case of curves and K3 surfaces we give the following definition.

Definition 2.10. By *universal cubic fourfold* over a divisor $\mathcal{C}_d$ we mean the moduli space $\mathcal{C}_{d,1}$ of 1-pointed cubic fourfolds.

As it is customary over moduli spaces, the mere existence of a universal cubic fourfold needs a little bit of justification.

Proposition 2.11. *The generic cubic fourfold in any divisor $\mathcal{C}_d$ does not have projective automorphism, hence a universal family of cubic fourfolds exists over an open subset of each divisor.*

Proof. This appears to be an easy corollary of [GAL11, Thm 3.8] where it is proven that the families of cubic fourfolds with a non-trivial projective automorphism have dimension at most 14. Hence clearly also the generic element of any divisor has no automorphism □

Remark 2.12. Given the result of [GAL11], Prop. 2.11 is straightforward and very general. In our particular cases (14, 26, and 42), since the isomorphism (2.5) holds, one could also argue in a more "geometric" way as follows. Any automorphism $\phi \in \mathrm{Aut}(X)$ of a smooth cubic fourfold X induces an automorphism of the Fano variety of lines $F(X)$. Then, via the isomorphism $F(X) \cong S^{[2]}$ with the symmetric square of the associated K3, one can use the fact that the generic K3 has no non-trivial automorphism to conclude that ϕ is the identity. Remark also that, by Prop.

2.7, the fact that the associated K3 has no non-trivial automorphism implies that any $\phi \in \mathrm{Aut}(X)$ should send each scroll onto itself.

3. The unirationality construction

By the results of the preceding section, at least over the a dense subset of $\mathcal{C}_{26}$, $\mathcal{C}_{14}$ and $\mathcal{C}_{42}$, there exists a universal family of cubic fourfolds. For simplicity, and since we are however working in the birational category, we will still denote the three universal families by $\mathcal{C}_{14,1} \to \mathcal{C}_{14}$, $\mathcal{C}_{26,1} \to \mathcal{C}_{26}$ and $\mathcal{C}_{42,1} \to \mathcal{C}_{42}$, without stressing the fact that they may not be defined everywhere.

In this Section we will show that:

Theorem 3.1. *The universal cubic fourfolds $\mathcal{C}_{14,1}, \mathcal{C}_{26,1}$ and $\mathcal{C}_{42,1}$ are unirational.*

3.1. **Unirationality of $\mathcal{C}_{14,1}, \mathcal{C}_{26,1}$ and $\mathcal{C}_{42,1}$.** The strategy will be the same for the three cases, hence we will resume here below shortly the properties, that hold for all the three divisors, that we will need. In order to keep the notation not too tedious, when we will say that a certain property holds for $\mathcal{C}_n$ we will assume $n = 14, 26, 42$.

Let X be a generic smooth cubic fourfold in $\mathcal{C}_n$. We will denote by $K(X)$ the associated K3 surface and by $S(X)$ the family of scrolls (as defined in Def. 2.2) contained in $X \subset \mathbb{P}^5$. For $n = 14$ quartic scrolls, for $n = 26$ septimic 3-nodal scrolls, for $n = 42$ 8-nodal degree 9 scrolls. One can rephrase Prop. 2.7 by saying that $K(X)$ is isomorphic to $S(X)$. More generally, let S_n denote the $PGL(6)$-quotient of the Hilbert scheme of scrolls contained in cubics in $\mathcal{C}_n$, as seen in Def. 2.2. These moduli quotients have been also considered in [FV18, FV19, Lai17]. Taking example from these papers let us give the following definition.

Definition 3.2. Let us denote

$$\Xi_n := \{(X,R) : R \subset X,\ [X] \in \mathcal{C}_n,\ R \in S(X)\}//PGL(6)$$

the "nested" moduli quotient of scrolls inside cubic fourfolds in $\mathcal{C}_n$.

This variety of course comes with two natural projections, as a correspondence between cubic fourfolds and scrolls.

$$\begin{array}{ccc} & \Xi_n & \\ \swarrow & & \searrow \\ \mathcal{C}_n & & S_n \end{array}$$

From [FV18, FV19], and as a straightforward consequence of [Muk88], we have the following result.

Proposition 3.3. *The universal K3 surfaces $\mathcal{F}_{n,1}$ are birational to Ξ_n.*

Let us now consider a slight variation of Ξ_n.

Definition 3.4. We denote by $\hat{\Xi}_n$ the universal cubic over Ξ_n, that is the quotient

$$\hat{\Xi}_n := \{(X,R,p)) : R \subset X,\ [X] \in \mathcal{C}_n,\ R \in S(X),\ p \in X\}//PGL(6).$$

Of course, as a consequence of Prop. 2.11, this is generically a fibration in cubic hypersurfaces over Ξ_n. Finally, we will denote by $\mathbb{P}S_n$ the trivial $\mathbb{P}^5$ bundle over S_n. Then we have a commutative diagram of rational forgetful maps.

(3.5)
$$\begin{array}{ccccc} & & \hat{\Xi}_n & & \\ & \overset{\varphi_X}{\swarrow} & & \overset{\varphi_p}{\searrow} & \\ \mathbb{P}S_n & & & & \Xi_n \\ & \underset{\varphi'_p}{\searrow} & & \underset{\varphi'_X}{\swarrow} & \\ & & S_n & & \end{array} \qquad \begin{array}{ccccc} & & (X,R,p) & & \\ & \swarrow & & \searrow & \\ (R,p) & & & & (X,R) \\ & \searrow & & \swarrow & \\ & & (R) & & \end{array}$$

Theorem 3.6. *The moduli quotients $\hat{\Xi}_n$ are unirational.*

Proof. By Prop. 2.9 and 3.3 we know that the quotients Ξ_n are (at least) unirational, and hence also S_n is unirational. Since $\mathbb{P}S_n$ is the trivial $\mathbb{P}^5$ bundle over S_n we also have the unirationality of S_n. Now consider the relative linear system

$$\sigma_n : \mathbb{P}H^0(\mathbb{P}^5, \mathcal{I}_{R\cup p}(3)) \to (R,p),$$

over S_n, whose fibers are the linear system of cubics in $\mathbb{P}^5$ containing R and p. Over an open subset of S_n (i.e. the locus where $p \notin R$) the rank does not change and it is 26, 11, and 5 respectively for $d = 14, 26$ and 42. This means that $\hat{\Xi}_n$ is dominated by a projective bundle over the unirational variety $\mathbb{P}S_n$. This in turn implies that $\hat{\Xi}_n$ is unirational. □

Proposition 3.7. *The universal cubic fourfold $\mathcal{C}_{n,1} \to \mathcal{C}_n$ is unirational.*

Proof. Recall from Prop. 2.11 that the generic cubic fourfold in the divisors $\mathcal{C}_n$ has no projective automorphism. Call $V \subset \mathcal{C}_n$ the dense locus where cubics have no automorphism. It is straightforward to see that, over V, the universal cubic has a natural quotient structure

$$\mathcal{C}_{n,1} := \{(X,p) : \ X \subset \mathbb{P}^5, \ R_n \subset X, p \in X\}//PGL(6). \tag{3.8}$$

where R_n denotes one of the scrolls defining each divisor $\mathcal{C}_n$ (as seen in Def. 2.2). Hence $\mathcal{C}_{n,1}$ is a natural moduli space for couples (X,p), up to the action of $PGL(6)$.

The upshot is that there exist a natural rational forgetful map

$$\begin{aligned} \varphi_R : \hat{\Xi}_n &\to \mathcal{C}_{n,1}; && (3.9)\\ (X,R,p) &\mapsto (X,p); && (3.10) \end{aligned}$$

that forgets the scroll contained in X. Since by Thm. 3.6, the variety $\hat{\Xi}_n$ is unirational, and it dominates $\mathcal{C}_{n,1}$, also $\mathcal{C}_{n,1}$ is unirational. □

Remark 3.11. The case of $\mathcal{C}_{38}$ seems more complicated to check, since the absence of the isomorphism $F(X) \cong S^{[2]}$ makes it more difficult to describe the families of scrolls in terms of associated K3s.

3.2. **Some results of non-unirationality.** In this last section we collect some recent results about the Kodaira dimension of moduli spaces of K3 surfaces and of Hassett divisors $\mathcal{C}_d$. These allow us to show that for an infinte range of values of d, the universal cubic fourfold over $\mathcal{C}_d$ can not be unirational. The key argument is the following easy Lemma, in which we need to assume that d is even.

Lemma 3.12. *Suppose that the Kodaira dimension of $\mathcal{F}_{\frac{d+2}{2}}$ or of $\mathcal{C}_d$ is positive, then the universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational.*

Proof. The universal cubic $\mathcal{C}_{d,1}$ dominates $\mathcal{C}_d$ via the forgetful map, hence if $\mathcal{C}_{d,1}$ were unirational, then $\mathcal{C}_d$ would also be unirational and have negative Kodaira dimension. If $\mathcal{F}_{\frac{d+2}{2}}$ has positive Kodaira dimension, then via the birational map of (2.6) also $\mathcal{C}_d$ has, and $\mathcal{C}_d$ can not be unirational, as seen before. □

We will apply Lemma 3.12 to the following two Propositions, due to Gritsenko-Hulek-Sankaran and Várilly-Alvarado-Tanimoto. Prop. 3.13 was initially conceived for moduli of K3 surfaces; following [Nue15] we write its "translation" in terms of cubic fourfolds via the birational isomorphism of (2.6).

Proposition 3.13. *Let $d > 80$, $d \equiv 2\ (mod 6)$, $4 \nmid d$ be such that for any odd prime p, $p|d$ implies $p \equiv 1\ (mod 3)$. Then the Kodaira dimension of $\mathcal{C}_d$ is non-negative. If $d > 122$, then $\mathcal{C}_d$ is of general type.*

Proposition 3.14. *The divisor $\mathcal{C}_{6n+2}$ is of general type for $n > 18$ and $n \neq 20, 21, 25$ and has nonnegative Kodaira dimension for $n = 14, 18, 20, 21, 25$. Moreover, $\mathcal{C}_{6n}$ is of general type for $n \geq 34$, $n \neq 19, 21, 24, 25, 26, 28, 29, 30, 31$ and has nonnegative Kodaira dimension for $n = 17, 23, 27, 33$.*

By combining Prop. 3.14 and 3.13 with Lemma 3.12 we obtain the following Proposition.

Proposition 3.15. *Under the hypotheses on d of Prop. 3.14 and 3.13, the universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational.*

The full, lenghty, statement is in Prop. 1.2 in the Introduction.

3.3. **Open questions.** It would be interesting to start a systematic study of the birational geometry of universal cubic fourfolds. Some very natural questions rise, and they are of course related to the classical birational geometry of cubic fourfolds.

Questions:

(1) What is the Kodaira dimension of the universal cubic fourfold over $\mathcal{C}_d$, for different values of d.
(2) In particular, is $\mathcal{C}_{38,1}$ unirational?
(3) Is there a relation between the Kodaira dimension of the universal cubic fourfold over $\mathcal{C}_d$ and that of the universal associated K3?
(4) What is the Kodaira dimension of the universal cubic fourfold over the full moduli space $\mathcal{C}$?

References

[ABBVA14] A. Auel, M. Bernardara, M. Bolognesi, and A. Várilly-Alvarado, *Cubic fourfolds containing a plane and a quintic del Pezzo surface*, Algebr. Geom. **1** (2014), no. 2, 181–193.

[AHTVA19] Nick Addington, Brendan Hassett, Yuri Tschinkel, and Anthony Várilly-Alvarado, *Cubic fourfolds fibered in sextic del pezzo surfaces*, Amer. J. Math. **141** (2019), no. 6, 1479–1500.

[AT14] Nicolas Addington and Richard Thomas, *Hodge theory and derived categories of cubic fourfolds*, Duke Math. J. **163** (2014), no. 10, 1885–1927.

[Awa19] Hanine Awada, *Rational fibered cubic fourfolds with nontrivial brauer classes*, arXiv preprint (2019), 1–20.

[BD85] A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.

[BP] Michele Bolognesi and Claudio Pedrini, *The transcendental motive of a cubic fourfold*, 1–18.

[BRS19] Michele Bolognesi, Francesco Russo, and Giovanni Staglianò, *Some loci of rational cubic fourfolds*, Math. Ann. **373** (2019), no. 1-2, 165–190.

[CG72] C. Herbert Clemens and Phillip A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356. MR 302652

[Fan43] G. Fano, *Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del* 4° *ordine*, Comment. Math. Helv. **15** (1943), no. 1, 71–80.

[FV18] Gavril Farkas and Alessandro Verra, *The universal $K3$ surface of genus 14 via cubic fourfolds*, J. Math. Pures Appl. (9) **111** (2018), 1–20.

[FV19] ———, *The unirationality of the moduli space of k3 surfaces of genus 22*, arXiv preprint (2019), 1–17.

[GAL11] Víctor González-Aguilera and Alvaro Liendo, *Automorphisms of prime order of smooth cubic n-folds*, Arch. Math. (Basel) **97** (2011), no. 1, 25–37.

[GHS13] V. Gritsenko, K. Hulek, and G. K. Sankaran, *Moduli of K3 surfaces and irreducible symplectic manifolds*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 459–526.

[Has00] B. Hassett, *Special cubic fourfolds*, Comp. Math. **120** (2000), no. 1, 1–23.

[Has16] Brendan Hassett, *Cubic fourfolds, K3 surfaces, and rationality *questions*, Rationality problems in algebraic geometry, Lecture Notes in Math., vol. 2172, 2016, pp. 29–66.

[Kuz10] A. Kuznetsov, *Derived categories of cubic fourfolds*, Cohomological and Geometric Approaches to Rationality Problems, Progress in Mathematics, vol. 282, Birkhäuser Boston, 2010, pp. 219–243.

[Lai17] Kuan-Wen Lai, *New cubic fourfolds with odd-degree unirational parametrizations*, Algebra Number Theory **11** (2017), no. 7, 1597–1626.

[Laz09] Radu Laza, *The moduli space of cubic fourfolds*, J. Algebraic Geom. **18** (2009), no. 3, 511–545.

[Laz10] ———, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. (2) **172** (2010), no. 1, 673–711.

[Ma19] Shouhei Ma, *Mukai models and borcherds products*, arXiv preprint (2019), 1–37.

[Muk88] Shigeru Mukai, *Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10*, Algebraic geometry and commutative algebra, Vol. I, Kinokuniya, Tokyo, 1988, pp. 357–377.

[Nue15] H. Nuer, *Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces*, preprint `http://arxiv.org/abs/1503.05256`, 2015.

[RS19a] Francesco Russo and Giovanni Staglianò, *Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds*, Duke Math. J. **168** (2019), no. 5, 849–865.

[RS19b] ———, *Trisecant flops, their associated k3 surfaces and the rationality of some fano fourfolds*, arXiv preprint (2019), 1–30.

[TVA19] Sho Tanimoto and Anthony Várilly-Alvarado, *Kodaira dimension of moduli of special cubic fourfolds*, J. Reine Angew. Math. **752** (2019), 265–300.

Institut Montpellierain Alexander Grothendieck, Université de Montpellier, CNRS Case Courrier 051 - Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
Email address: michele.bolognesi@umontpellier.fr

Institut Montpellierain Alexander Grothendieck, Université de Montpellier, CNRS Case Courrier 051 - Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
Email address: hanine.awada@umontpellier.fr