Exercises of Algebraic Geometry I

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Introduction

These notes collect a series of solved exercises for the course of *Algebraic Geometry*.Most of them from the book *Algebraic Geometry* by *R. Hartshorne* [**Har**]. Many others from the notes by *Ph. Ellia* [**PhE**]. I am very thankful to Alex Massarenti for providing most of the resolutions presented in here.

CHAPTER 1

Affine varieties

Exercise 1. [**Har**, Exercise 1.1]

(a) The coordinate ring of the curve $C = \{y - x^2 = 0\} \subset \mathbb{A}^2$ is given by

$$A(C) = k[x,y]/(y - x^2) \cong k[x, x^2] \cong k[x].$$

(b) $A(Z) = k[x,y]/(xy - 1)$ is isomorphic to the localization of $k[x]$ at x. Let $f : A(Z) \to k[x]$ be a morphism of k-algebras. Since $x \in A(Z)$ is invertible $f(x) \in k$. Therefore, f can not be an isomorphism.

Exercise 2. [**Har**, Exercise 1.3] Consider $Y = \{x^2 - yz = xz - x = 0\} \subset \mathbb{A}^3$. Then

$$Y = \{x^2 - y = z - 1 = 0\} \cup \{x = y = 0\} \cup \{x = z = 0\},$$

and Y is the union of two lines and a plane irreducible curve of degree two. In order to show that each component is irreducible, one shows that $k[x,y,z]/equations$ is an integral domain, i.e. has no zero divisor.

In particular, the coordinate ring of each irreducible component is isomorphic to $k[t]$.

Exercise 3. [**Har**, Exercise 1.5] Let B be a finitely generated k-algebra. Then we may write $B = k[x_1, ..., x_n]/I$ for some ideal $I = (f_1, ..., f_r)$ in $k[x_1, ..., x_n]$. Let $X = \{f_1 = ... = f_r = 0\} \subseteq \mathbb{A}^n$. Let $f \in I(X)$ then, by the Nullstellensatz we have $f^k \in I$ for some $k > 0$. Now, B does not have nilpotents, so $f \in I$. Clearly $I \subseteq I(X)$. This yields $I = I(X)$ and $B \cong A(X)$.
Conversely, assume to have $B = A(X)$ for some algebraic set $X \subset \mathbb{A}^n$. Let $I(X)$ be the ideal of X. Then $B \cong k[x_1, ..., x_n]/I(X)$ is a finitely generated k-algebra. Let $f \in B$ be a nilpotent element. Then $f^k = 0$ for some k, that is $f^k \in I$. Since I is radical we get $f \in I$, that is $f = 0$ in B.

Exercise 4. [**Har**, Exercise 1.8] Let $Y \subset \mathbb{A}^n$ be an affine variety of dimension r. Let $H \subset \mathbb{A}^n$ be an hypersurface such that Y is not contained in H and $Y \cap H \neq \emptyset$. Since Y is not contained in H we have $I(H) \nsubseteq I(Y)$. Let f be the polynomial defining H. Then, the irreducible components of $Y \cap H$ corresponds to the minimal prime ideals of $A(Y)$ containing f. Note that $Y \nsubseteq H$ implies that f is not a zero-divisor in $A(Y)$. By the Hauptidealsatz any minimal prime ideal containing f has height one. Finally, by [**Har**, Theorem 1.8A] we get that the any irreducible component of $Y \cap H$ has dimension $\dim(Y) - 1$.

Exercise 5. [**Har**, Exercise 1.9] Let $\mathfrak{a} \subseteq k[x_1, ..., x_n]$ be an ideal that can be generated by r elements $f_1, ..., f_r$. Note that $\{f_i = 0\}$ defines an hypersurface for any $i = 1, ..., r$. We apply r times Exercise 1.8 and we distinguish two cases:

- at any step the variety $H_k = \{f_1 = ...f_k = 0\}$ is not contained in the hypersurface $\{f_{k+1} = 0\}$. Then at each step the dimension of the intersection drops by one. We get that the dimension of each irreducible component of Y is $n - r$,
- if H_k is contained in $\{f_{k+1} = 0\}$ for some k, then the intersection with $\{f_{k+1} = 0\}$ will not drop the dimension. Then each irreducible component of Y has dimension greater than $n - r$.

In any case we have that the dimension of each irreducible component of Y is greater or equal than $n - r$.

Exercise 6. [**Har**, Exercise 1.11] The curve Y is the image of the morphism

$$\begin{array}{rcc} \phi : \mathbb{A}^1 & \longrightarrow & \mathbb{A}^3 \\ t & \longmapsto & (t^3, t^4, t^5) \end{array}$$

Note that since $\mathbb{A}^1$ is irreducible Y is irreducible as well. Therefore $I = I(Y)$ is prime. Furthermore $\dim(Y) = \dim(A(Y)) = 1$ and by [**Har**, Theorem 1.8A] we get height$(I(Y)) = 2$. Note that the three polynomials $z^2 - x^2y$, $xz - y^2$ and $yz - x^3$ are in $I(Y)$ and they are independent.
Let $J = (z^2 - x^2y, xz - y^2, yz - x^3) \subseteq I(Y)$. By [**Ku**, Page 138] we have that $I(Y) = J$ and that we need three elements to generate $I(Y)$.

Exercise 7. [**Har**, Exercise 1.12] Consider the polynomial

$$f = (x^2 - 1 + iy)(x^2 - 1 - iy) = x^4 - 2x^2 + y^2 + 1.$$

Since $\mathbb{R}[x, y] \subset \mathbb{C}[x, y]$ are unique factorization domains and f splits in $\mathbb{C}[x, y]$ as a product of two irreducible polynomials of degree two, we conclude that f is irreducible in $\mathbb{R}[x, y]$. On the other hand, $Z(f) = \{(1, 0), (-1, 0)\}$ is the union of two points. Therefore $f \in \mathbb{R}[x, y]$ is irreducible but $Z(f) \subset \mathbb{A}^2$ is reducible.

Exercise 8. [**Har**, Exercise 2.1]
a is homogeneous and hence defines a cone in $\mathbb{A}^{n+1}$. The polynomial f vanishes on all the elements of this cone (including 0 since f has positive degree) so $f^t \in a$ for some $t > 0$ by the usual Nullstellensatz.

Exercise 9. [**Har**, Exercise 2.2]
(iii) implies (i) is trivial because all monomials x_i^d belong to S_d. (i) implies (ii): If $Z(a)$ is empty, then in $\mathbb{A}^{n+1}$, $Z(a)$ is either empty or $(0, \dots, 0)$, so $\sqrt{a}$ must be S or the irrelevant ideal. (ii) implies (iii): $\sqrt{a}$ contains x_i, so there is some m s.t. $x_i^m \in a$ for all i, so a contains $S_{m(n+1)}$ as any monomial of degree $m(n + 1)$ must have x_i^m as a factor for some i.

Exercise 10. [**Har**, Exercise 2.3] (a),(b),(c),(e) are clear. For (d), clearly $I(Z(a))$ contains $\sqrt{a}$. Since $Z(a)$ is nonempty, any nonzero homogeneous polynomial vanishing on it must have positive degree. By 2.1, this implies that $f^t \in a$. Therefore $I(Z(a))$ is contained in $\sqrt{a}$ as it is a homogeneous ideal.

Exercise 11. [**Har**, Exercise 2.4] (a) Follows from 2.3d,e, and 2.2.
(b) If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2) \supset I(Y_1)I(Y_2)$. Therefore if $I(Y)$ is prime, $I(Y)$ must be either $I(Y_1)$ or $I(Y_2)$, so Y is Y_1 or Y_2. On the other hand if Y is not prime, then $ab \in I(Y)$, with $a \notin I(Y)$, $b \notin I(Y)$. Therefore Y is the union of the proper subsets $Y \cap Z(a)$, $Y \cap Z(b)$ and is therefore not irreducible.
(c) $I(\mathbb{P}^n) = 0$ which is a prime ideal.

Exercise 12. [**Har**, Exercise 2.9] Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Consider the homeomorphism

$$\begin{array}{rcl} \phi_0 : U_0 = \mathbb{P}^n \setminus \{x_0 = 0\} & \longrightarrow & \mathbb{A}^n \\ [x_0 : ... : x_n] & \longmapsto & (\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}) \end{array}$$

Finally, let $\overline{Y}$ be the projective closure of Y.
Let $F \in I(\overline{Y})$, then $f(y_1, ..., y_n) = F(1, x_0, ..., x_n)$ where $y_i = \frac{x_i}{x_0}$ vanishes on $Y = \overline{Y} \cap U_0$. We get that $f \in I(Y)$ and $x_0^s \beta(f) = F$ for some s. Therefore, $F \in (\beta(I(Y)))$, where β is the homogeneization with respect to x_0.
Now let $F \in \beta(I(Y))$, then $F = g_1\beta(f_1) + ... + g_r\beta(f_r)$ for some $f_1, ..., f_r \in I(Y)$, that is $F = g_1 x_0^{s_1} f_1(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}) + ... + g_r x_0^{s_r} f_r(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0})$. Hence $F \in I(\overline{Y})$.
Let $Y \subset \mathbb{A}^3$ be the affine twisted cubic. Then $I(Y) = (x^3 - z, x^2 - y)$ while $I(\overline{Y}) = (xz - y^2, yw - z^2, xw - yz)$. Note that $I(\overline{Y})$ can not be generated by two elements because $\overline{Y} \subset \mathbb{P}^3$ is not a scheme-theoretic complete intersection.

Exercise 13. [**Har**, Exercise 2.10]
(a) Obvious.
(b) They have the same ideal, which is prime if and only if they are irreducible.
(c) By ex. 2.6 of [**Har**], we have $S(Y)) = dim(Y) + 1$, and $dim(Y) = dim(S(Y))$, hence the claim.

Exercise 14. [**Har**, Exercise 2.13]
We assume that $\mathbb{P}^2$ is isomorphic to its image which is easy to check, so that curves in the image of $\mathbb{P}^2$ correspond to curves in $\mathbb{P}^2$.
The map is given by

$$(x_0 : x_1 : x_2) \to (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_2x_0).$$

Any curve in $\mathbb{P}^2$ is defined by some polynomial $f(x_0 : x_1 : x_2) = 0$, f homogeneous, and therefore also by the polynomial $f(x_0; x_1; x_2)^2 = g(x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_2x_0)$ for some polynomial g. Then some factor of this polynomial g defines a suitable hypersurface containing the image of the curve Z.

Exercise 15. [**Har**, Exercise 2.14]
The image of ψ is the set Y defined by the equations of the form $x_{ab}x_{cd} = x_{ac}x_{bd}$. Proof: the image is clearly contained in Y . Conversely if $(x_{00} : x_{10} : \cdots : x_{rs}) \in Y$ then we may assume that x_{00} is nonzero. But then the point is the image of $(x_{00} : x_{10} : \cdots : x_{r0}) \times (x_{00} : x_{01} : \cdots : x_{0s}) \in \mathbb{P}^r \times \mathbb{P}^s$.

Exercise 15. [**Har**, Exercise 2.15]

a) $(a_0 : a_1) \times (b_0 : b_1) = (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1) = (w : x : y : z)$, and the image of $\mathbb{P}^1 \times \mathbb{P}^1$ is then the subvariety $xy - zw = 0$ as in Ex. 15.

b) Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we can take the two families of lines to correspond to point $\times$ line and line $\times$ point. We check that these are lines inside $Q \subset P3$; for example the image of $(a_0 : a_1) \times \mathbb{P}^1$ is the set of points $(w : x : y : z) \in \mathbb{P}^3$ with $a_1w = a_0y, a_1x = a_0z$. THis is the intersection of two projective planes in $\mathbb{P}^3$, that is a line.

c) For example, the closed subset $x = y$ of Q is not one of these lines, and it is closed in the Zariski topology of Q, hence Q is not homemorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the product topology.

Exercise 16. [**Har**, Exercise 2.17]

a) By Ex. 1.8 of [**Har**], the intersection of q hypersurfaces has dimension at least $n - q$. If a is generated by q elements then $Z(Y)$ is the intersection of q hypersurfaces and therefore, by Ex. 2.8 of [**Har**], has dimension at least $n - q$.

b) If $I(Y)$ can be generated by r elements then Y is the intersection of their hypersurfaces.

c) Y can be seen as the intersection of $H_1 = Z(x^2 - wy)$ and $H_2 = Z(y^3 + wz^2 - 2xyz)$ as one can obtain powers of the three quadrics that generate $I(Y)$ by combining H_1 and H_2. For example $(xy - wz)^2 = w(y^3 + wz^2 - 2xyz) + y^2(x^2 - wy)$ and $(y^2 - xz)^2 = y(y^3 + wz^2 - 2xyz) + z^2(x^2 - wy)$, and similarly for the third. By the way, $I(Y)$ has no homogeneous elements of degree 0 or 1 and we know the space of homogeneous elements of degree 2 is 3 dimensional, so any set of generators must have at least 3 elements.

d)I think that this is still unknown.

Bibliography

[PhE] PH. ELLIA, *Un'introduzione light alla geometria algebrica*, IntroLight-GA-2006.pdf.

[Har] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Ku] E. KUNZ, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, first edition, 1984.