# Aiguille du Midi

- Departure: 12h55 at the Amis de la Nature
- Consequently: please be ready to pick your lunch bag at the restaurant at 12h30
- VERY COLD!!!
- Sun glasses...
- Pay individually in Chamonix for the lift (around 42 euros)
- The bus is paid by the conference

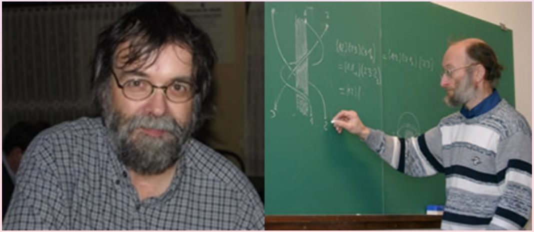
# Calogero-Moser cells

(joint work with R. Rouquier)

Cédric Bonnafé

CNRS (UMR 5149) - Université de Montpellier 2

Les Houches - Janvier 2011



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- $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$.

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$$[x, y] = T\langle x, y\rangle + \sum_{s \in \mathrm{Réf}(W)} (1 - \varepsilon(s))\ C_s\ \frac{\langle x, \alpha_s\rangle \cdot \langle \alpha_s^\vee, y\rangle}{\langle \alpha_s, \alpha_s^\vee\rangle}\ s.$$

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## Specialisation

- $\mathbf{H}_c$ is not $\mathbb{N} \times \mathbb{N}$-graded
- $Q$ and $Q_c$ are normal integral domains

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**Restricted Cherednik algebra**

Take $v = 0$ and $v^* = 0$. You get

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- New (!): take $\mathbf{K} = \mathrm{Frac}(P)$

# The $P = \mathbb{C}[\mathscr{C} \times V/W \times V^*/W]$-algebra $\mathbf{H}$

- Relatively classical: take a point $p = (c, v, v^*) \in \mathscr{P} = \mathscr{C} \times V/W \times V^*/W$ and view $\mathbb{C} = \mathbb{C}_{c,v,v^*}$ as a $P$-algebra via evaluation at $p \Longrightarrow \mathbf{H}_{c,v,v^*} = \mathbb{C}_{c,v,v^*} \otimes_P \mathbf{H}$.

**Restricted Cherednik algebra**

Take $v = 0$ and $v^* = 0$. You get

$$\overline{\mathbf{H}}_c \underbrace{=}_{vector\ \ space} \mathbb{C}[V]^{\mathrm{co}(W)} \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]^{\mathrm{co}(W)}.$$

This will lead to Calogero-Moser families (Gordon).

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**Calogero-Moser families**

The $c$-Calogero-Moser families are the subsets of $\mathrm{Irr}(W)$ corresponding to the blocks of $\overline{\mathbf{H}}_c$.

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but, in general, the inclusion is strict. However (Müller)

$$\mathrm{Idem}_{\mathrm{pr}}(\bar{Q}_c) = \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(\overline{\mathbf{H}}_c)).$$

**Theorem (B.-Rouquier 2010)**

*If $\mathscr{F}$ is a Calogero-Moser family corresponding to a primitive idempotent $b \in \mathit{Idem}_{pr}(\bar{Q}_c)$, then*

$$\dim_{\mathbb{C}} \bar{Q}_c b = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1)^2.$$

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But

$$k_P(\mathfrak{p}_0) \simeq \mathbb{C}(V \times V^*)^{W \times W} \subset k_Q(\mathfrak{q}_0) = \mathbb{C}(V \times V^*)^{\Delta W} \subset \mathbb{C}(V \times V^*).$$

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$$\mathscr{K}_0(\mathbf{MH}) \simeq \mathbb{Z}W$$

and the decomposition map (modulo $\overline{\mathfrak{r}}_c$) gives a map

$$\begin{array}{ccc} \mathscr{K}_0(\mathbf{MH}) & \longrightarrow & \mathscr{K}_0(\overline{\mathbf{H}}_c) \\ \| & & \| \\ \mathbb{Z}W & \xrightarrow{\overline{\mathrm{dec}}_c} & \mathbb{Z}\,\mathrm{Irr}\,W \end{array}$$

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If $\Gamma$ is a $c$-Calogero-Moser cell, we denote by $\mathrm{Irr}_\Gamma^{\mathrm{CM}}(W)$ the subset of $\mathrm{Irr}(W) \xleftrightarrow{\sim} \mathrm{Irr}(\overline{\mathbf{H}}_c)$ associated with the corresponding block of $\overline{\mathbf{H}}_c$.

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**Conjecture (Gordon-Martino 2006, almost true)**

If $(W, S)$ is a Coxeter system and if $c$ is real-valued, then the $c$-Calogero-Moser families coincide with the $c$-Kazhdan-Lusztig families.

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## Conjecture (B.-Rouquier 2010)

If $(W, S)$ is a Coxeter system and if $c$ is real-valued, then there exists a prime ideal $\bar{\mathfrak{r}}_c$ lying above $\bar{\mathfrak{p}}_c$ such that the $c$-Calogero-Moser cells coincide with the $c$-Kazhdan-Lusztig two-sided cells. Moreover, if $\Gamma$ is a CM/KL-cell, then

$$\mathrm{Irr}_\Gamma^{\mathrm{CM}}(W) = \mathrm{Irr}_\Gamma^{\mathrm{KL}}(W).$$

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# Aiguille du Midi

- Departure: 12h55 at the Amis de la Nature
- Consequently: please be ready to pick your lunch bag at the restaurant at 12h30
- VERY COLD!!!
- Sun glasses...
- Pay individually in Chamonix for the lift (around 42 euros)
- The bus is paid by the conference