Aiguille du Midi

- Departure: **12h55** at the *Amis de la Nature*
- Consequently: please be ready to pick your lunch bag at the restaurant at **12h30**
- **VERY COLD!!!**
- **Sun glasses...**
- Pay *individually* in Chamonix for the lift (around 42 euros)
- The bus is paid by the conference
Calogero-Moser cells
(joint work with R. Rouquier)

Cédric Bonnafé

CNRS (UMR 5149) - Université de Montpellier 2

Les Houches - Janvier 2011
What does Kazhdan-Lusztig theory do for you?
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  \[\implies\] $\mathcal{W} \to \mathbb{Z}\text{Irr} (\mathcal{W})$ (cell module)
  \[\implies\] If $\Gamma$ is a $c$-KL-two-sided cell, $\text{Irr}_{KL}^\Gamma (\mathcal{W}) \subset \text{Irr} (\mathcal{W})$ (c-KL-family)
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- Links with the geometry of flag varieties (singularities of Schubert cells)
- Representations of complex Lie algebras, of reductive groups in positive characteristic...
- Representations of finite reductive groups
- Unipotent classes
- Decomposition numbers of Hecke algebras at root of unity...
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**Strategy.** Use rational Cherednik algebras at $t = 0$ to construct partitions of $W$ (resp. $\Irr(W)$) into Calogero-Moser cells (resp. families), a map $W \to \mathbb{Z} \Irr(W)$ and attach to a cell $\Gamma$ a subset $\Irr_{CM}^\Gamma(W)$ of $\Irr(W)$. 
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Weakness.

Our construction depends on an "uncontrollable" choice. We don't get the partial order. Hard to compute (one month for completing $B_2$).

Strengths. Sharing many properties with KL-two-sided cells, the semicontinuity is trivial. Coincide with KL two-sided cells for $A_2$, $B_2$, $G_2$, very flexible (tons of notions of cells). Geometric methods.
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Set-up

\[ \text{dim} C V < \infty \]

\[ W \subset \text{GL}_C(V) \mid W \mid < \infty \]

\[ W = \langle \text{Ref}(W) \rangle, \text{where Ref}(W) = \{ s \in W \mid \text{codim} C \text{Ker}(s - \text{Id}_V) = 1 \} \]

\[ C = \{ c : \text{Ref}(W) / \sim \to C \} \]

\[ C[\text{C}] = \text{S}(C^*) = C[\text{C}_{\text{s \in \text{Ref}(W) / \sim}}] \]

If \( s \in \text{Ref}(W) \), let \( \alpha_s \in V^* \) and \( \alpha_s \lor s \in V \) be such that \( \text{Ker}(s - \text{Id}_V) = \text{Ker}(\alpha_s) \) and \( \text{Im}(s - \text{Id}_V) = C_{\alpha_s \lor s} \).

\[ \varepsilon : W \to C \times C, w \mapsto \det(w) \]
Set-up

- $\dim_{\mathbb{C}} V < \infty$
- $W \subset \text{GL}_{\mathbb{C}}(V)$
- $|W| < \infty$
- $W = \langle \text{Réf}(W) \rangle$, where
  \[ \text{Réf}(W) = \{ s \in W \mid \text{codim}_{\mathbb{C}} \text{Ker}(s - \text{Id}_V) = 1 \}. \]
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- $\mathcal{C} = \{ c : \text{Réf}(W)/\sim \rightarrow \mathbb{C} \}$
- $C_s : \mathcal{C} \rightarrow \mathbb{C}$, $c \mapsto c_s$
- $\mathbb{C}[\mathcal{C}] = S(\mathcal{C}^*) = \mathbb{C}[(C_s)_{s \in \text{Réf}(W)/\sim}]$
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  \]

- \( \varepsilon : W \rightarrow \mathbb{C}^\times, w \mapsto \det(w) \).
Rational Cherednik algebra

- \( H \) is the \( C[C] \)-algebra such that
  \[ H = \bigoplus_{s \in \text{Ref}(W)} (1 - \varepsilon(s)) C_s \langle x, \alpha_s \rangle \cdot \langle \alpha_s, y \rangle \langle \alpha_s, \alpha_s \rangle. \]

- Specialisation at \( c \in C \)
  \[ H_c = C_c \otimes C[C]. \]
Rational Cherednik algebra

- $H$ is the $\mathbb{C}[C]$-algebra such that

$$H = \text{vector space } \mathbb{C}[T] \otimes \mathbb{C}[C] \otimes \mathbb{C}[V] \otimes CW \otimes \mathbb{C}[V^*]$$

$$[x, y] = T \langle x, y \rangle + \sum_{s \in \text{Réf}(W)} (1 - \varepsilon(s)) c_s \frac{\langle x, \alpha_s \rangle \cdot \langle \alpha_s^\vee, y \rangle}{\langle \alpha_s, \alpha_s^\vee \rangle} s.$$
Rational Cherednik algebra at $t = 0$

- $H$ is the $\mathbb{C}[C]$-algebra such that

$$H \cong \mathbb{C}[C] \otimes \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]$$

vector space

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$$H_c = \mathbb{C}_c \otimes \mathbb{C}[\mathbb{C}] H.$$
Freeness

\[ P = C \left[ C \right] \otimes C \left[ V \right] \otimes W \otimes C \left[ V^* \right] \subset Z \left( H \right) =: Q\]

\(Q\) is a free \(P\)-module of rank \(|W|\) (in particular, \(Q\) is Cohen-Macaulay)

Satake isomorphism: the natural map \(Q \to \text{End}_H \left( H^e \right) = (e_{H^e})^\text{op}\) is an isomorphism (with \(e = \frac{1}{|W|} \sum w \in W w\))

\(H \cong \text{End}_Q \left( H^e \right)\)

Graduation(s)

\(N \times N\)-graduation: \(\text{deg} \left( N \times N \left( V \right) \right) = (1, 0), \text{deg} \left( N \times N \left( V^* \right) \right) = (0, 1), \text{deg} \left( N \times N \left( W \right) \right) = (0, 0), \text{deg} \left( N \times N \left( C \right) \right) = (1, 1)\).

Specialisation

\(H^c\) is not \(N \times N\)-graded \(Q\) and \(Q^c\) are normal integral domains
Some features (Etingof-Ginzburg 2002)

Freeness

- \( P = \mathbb{C}[C] \otimes \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W \subset Z(H) =: Q \)
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**Graduation(s)**

- \( \mathbb{N} \times \mathbb{N} \)-gradation: \( \deg_{\mathbb{N} \times \mathbb{N}}(V) = (1, 0), \deg_{\mathbb{N} \times \mathbb{N}}(V^*) = (0, 1), \deg_{\mathbb{N} \times \mathbb{N}}(W) = (0, 0), \deg_{\mathbb{N} \times \mathbb{N}}(C) = (1, 1). \)
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Freeness

- \( P = \mathbb{C}[\mathcal{C}] \otimes \mathbb{C}[\mathcal{V}]^W \otimes \mathbb{C}[\mathcal{V}^*]^W \subset \mathbb{Z}(H) =: Q \)
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Graduation(s)

- \( \mathbb{N} \times \mathbb{N} \)-graduation: \( \text{deg}_{\mathbb{N} \times \mathbb{N}}(\mathcal{V}) = (1, 0) \), \( \text{deg}_{\mathbb{N} \times \mathbb{N}}(\mathcal{V}^*) = (0, 1) \),
  \( \text{deg}_{\mathbb{N} \times \mathbb{N}}(W) = (0, 0) \), \( \text{deg}_{\mathbb{N} \times \mathbb{N}}(\mathcal{C}) = (1, 1) \).

Specialisation

- \( H_c \) is not \( \mathbb{N} \times \mathbb{N} \)-graded
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Example: specialisation at $c=0$

\[ H_0 = C[V \times V^*] \rtimes W \]

and

\[ Q_0 = C[V \times V^*] W \]

$H_0$ e $\simeq C[V \times V^*]$ (because $W$ does not contain any reflection... for its action on $V \times V^*$)
Example: specialisation at $c=0$

- $H_0 = \mathbb{C}[V \times V^*] \rtimes W$ and $Q_0 = \mathbb{C}[V \times V^*]^W$
Example: specialisation at $c=0$

- $H_0 = \mathbb{C}[V \times V^*] \rtimes W$ and $Q_0 = \mathbb{C}[V \times V^*]^W$
- $H_0 e \cong \mathbb{C}[V \times V^*]$
Example: specialisation at \( c=0 \)

- \( H_0 = \mathbb{C}[V \times V^*] \rtimes W \) and \( Q_0 = \mathbb{C}[V \times V^*]^W \)

- \( H_0 e \simeq \mathbb{C}[V \times V^*] \)

- \( \mathbb{C}(V \times V^*) \rtimes W = \text{End}_{\mathbb{C}(V \times V^*)^W}(\mathbb{C}(V \times V^*)) \)
Example: specialisation at \( c=0 \)

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- $\mathbb{C}[V \times V^*] \rtimes W = \text{End}_{\mathbb{C}[V \times V^*]^W}(\mathbb{C}[V \times V^*])$ (because $W$ does not contain any reflection... for its action on $V \times V^*$)
The $P = \mathbb{C}[\mathbb{C} \times V/W \times V^*/W]$-algebra $H$
The $P = \mathbb{C}[C \times V/W \times V^*/W]$-algebra $H$

- Relatively classical: take a point $p = (c, v, v^*) \in \mathcal{P} = C \times V/W \times V^*/W$ and view $C = C_{c,v,v^*}$ as a $P$-algebra via evaluation at $p$
The $P = \mathbb{C}[\mathbb{C} \times V/W \times V^*/W]$-algebra $H$

- Relatively classical: take a point $p = (c, \nu, \nu^*) \in P = \mathbb{C} \times V/W \times V^*/W$ and view $\mathbb{C} = \mathbb{C}_{c,\nu,\nu^*}$ as a $P$-algebra via evaluation at $p \mapsto H_{c,\nu,\nu^*} = \mathbb{C}_{c,\nu,\nu^*} \otimes_P H$.

**Restricted Cherednik algebra**

Take $\nu = 0$ and $\nu^* = 0$. You get

\[ \overline{H}_c \quad \mathring{=} \quad \mathbb{C}[V]^{co(W)} \otimes \mathbb{C} W \otimes \mathbb{C}[V^*]^{co(W)} . \]
The \( P = \mathbb{C}[\mathbb{C} \times V/W \times V^*/W] \)-algebra \( H \)

- Relatively classical: take a point 
  \( p = (c, v, v^*) \in \mathcal{P} = \mathbb{C} \times V/W \times V^*/W \) and view \( \mathbb{C} = \mathbb{C}_{c,v,v^*} \) as a \( P \)-algebra via evaluation at \( p \mapsto H_{c,v,v^*} = \mathbb{C}_{c,v,v^*} \otimes P H \).

**Restricted Cherednik algebra**

Take \( v = 0 \) and \( v^* = 0 \). You get

\[
\overline{H}_c = \mathbb{C}[V]^{\text{co}(W)} \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]^{\text{co}(W)}.
\]

This will lead to **Calogero-Moser families** (Gordon).
The $P = \mathbb{C}[\mathbb{C} \times V/W \times V^*/W]$-algebra $H$

- Relatively classical: take a point $p = (c, v, v^*) \in \mathcal{P} = \mathbb{C} \times V/W \times V^*/W$ and view $\mathbb{C} = \mathbb{C}_{c,v,v^*}$ as a $P$-algebra via evaluation at $p \mapsto H_{c,v,v^*} = \mathbb{C}_{c,v,v^*} \otimes_P H$.

**Restricted Cherednik algebra**

Take $v = 0$ and $v^* = 0$. You get

$$\overline{H}_c = \mathbb{C}[V]^{\text{co}(W)} \otimes CW \otimes \mathbb{C}[V^*]^{\text{co}(W)}.$$  

This will lead to Calogero-Moser families (Gordon).

- New (!): take $K = \text{Frac}(P)$
The $P = \mathbb{C}[\mathbb{C} \times V/W \times V^*/W]$-algebra $H$

- Relatively classical: take a point $p = (c, \nu, \nu^*) \in \mathcal{P} = \mathbb{C} \times V/W \times V^*/W$ and view $\mathbb{C} = \mathbb{C}_{c, \nu, \nu^*}$ as a $P$-algebra via evaluation at $p \mapsto H_{c, \nu, \nu^*} = \mathbb{C}_{c, \nu, \nu^*} \otimes_P H$.

### Restricted Cherednik algebra

Take $\nu = 0$ and $\nu^* = 0$. You get

$$\overline{H}_c \cong \mathbb{C}[V]^{co(W)} \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]^{co(W)}.$$  

This will lead to Calogero-Moser families (Gordon).

- New (!): take $K = \text{Frac}(P)$  
  This will lead to Calogero-Moser cells (B.-Rouquier)
Take $c \in \mathbb{C}$ and let $H - c = C W \otimes C[V^*] \co W \subset H_c$.

If $\chi \in \text{Irr}(W)$, let $M_c(\chi) = H_c \otimes H_{-c} \tilde{V}_\chi$.

Then (Gordon):

$M_c(\chi)$ is indecomposable with a unique simple quotient $L_c(\chi)$.

$\text{Irr}(W) \rightarrow \text{Irr}(H_c)$, $\chi \mapsto L_c(\chi)$ is a bijection.

$K_0(H_c) \cong \mathbb{Z} \text{Irr}(W)$.

\textbf{Calogero-Moser families}

The $c$-Calogero-Moser families are the subsets of $\text{Irr}(W)$ corresponding to the blocks of $H_c$. 
Restricted Cherednik algebra (Gordon, 2003)

Take $c \in \mathbb{C}$ and let

$$\overline{H}_c = \mathbb{C}W \otimes \mathbb{C}[^*V]^{\text{co}(W)} = W \times \mathbb{C}[^*V]^{\text{co}(W)} \subset \overline{H}_c.$$
Restricted Cherednik algebra (Gordon, 2003)

Take \( c \in \mathbb{C} \) and let

\[
\overline{H}_c^{-} = \mathbb{C}W \otimes \mathbb{C}[V^*]^{co(W)} = W \times \mathbb{C}[V^*]^{co(W)} \subset \overline{H}_c.
\]

If \( \chi \in \text{Irr}(W) \), let \( \mathcal{M}_c(\chi) = \overline{H}_c \otimes_{\overline{H}_c} \overline{V}_\chi \).
Take $c \in \mathbb{C}$ and let

$$\overline{H}_c = \mathbb{C}W \otimes \mathbb{C}[V^*]^{co(W)} = W \times \mathbb{C}[V^*]^{co(W)} \subset \overline{H}_c.$$ 

Si $\chi \in \text{Irr}(W)$, let $M_c(\chi) = \overline{H}_c \otimes_{\overline{H}_c} \overline{V}_\chi$. Then (Gordon):

- $M_c(\chi)$ is indecomposable with a unique simple quotient $L_c(\chi)$. 

Calogero-Moser families

The $c$-Calogero-Moser families are the subsets of $\text{Irr}(W)$ corresponding to the blocks of $H_c$. 
Take \( c \in \mathbb{C} \) and let

\[
\overline{H}_c^-=\mathbb{C}W \otimes \mathbb{C}[V^*]^{\text{co}(W)} = W \times \mathbb{C}[V^*]^{\text{co}(W)} \subset \overline{H}_c.
\]

Si \( \chi \in \text{Irr}(W) \), let \( M_c(\chi) = \overline{H}_c \otimes \overline{H}_c^- \tilde{V}_\chi \). Then (Gordon):

- \( M_c(\chi) \) is indecomposable with a unique simple quotient \( L_c(\chi) \).
- \( \text{Irr}(W) \longrightarrow \text{Irr}(\overline{H}_c) \), \( \chi \mapsto L_c(\chi) \) is a bijection.
Restricted Cherednik algebra (Gordon, 2003)

Take $c \in \mathbb{C}$ and let

$$\overline{H}_c = \mathbb{C}W \otimes \mathbb{C}[V^*]^\text{co}(W) = W \ltimes \mathbb{C}[V^*]^\text{co}(W) \subset \overline{H}_c.$$  

Si $\chi \in \text{Irr}(W)$, let $M_c(\chi) = \overline{H}_c \otimes_{\overline{H}_c} \tilde{V}_\chi$. Then (Gordon):

- $M_c(\chi)$ is indecomposable with a unique simple quotient $L_c(\chi)$.
- $\text{Irr}(W) \longrightarrow \text{Irr}(\overline{H}_c)$, $\chi \mapsto L_c(\chi)$ is a bijection.

$$\mathcal{H}_0(\overline{H}_c) \cong \mathbb{Z} \text{Irr}(W).$$
Restricted Cherednik algebra (Gordon, 2003)

Take \( c \in \mathbb{C} \) and let

\[
\mathcal{H}_c = \mathbb{C} W \otimes \mathbb{C}[V^*]^{\text{co}(W)} = W \ltimes \mathbb{C}[V^*]^{\text{co}(W)} \subset \mathcal{H}_c.
\]

Si \( \chi \in \text{Irr}(W) \), let \( \mathcal{M}_c(\chi) = \mathcal{H}_c \otimes_{\mathcal{H}_c} \tilde{V}_\chi \). Then (Gordon):

- \( \mathcal{M}_c(\chi) \) is indecomposable with a unique simple quotient \( \mathcal{L}_c(\chi) \).
- \( \text{Irr}(W) \longrightarrow \text{Irr}(\mathcal{H}_c), \chi \mapsto \mathcal{L}_c(\chi) \) is a bijection.

\[
\mathcal{K}_0(\mathcal{H}_c) \cong \mathbb{Z} \text{ Irr}(W).
\]

Calogero-Moser families

The \( c \)-Calogero-Moser families are the subsets of \( \text{Irr}(W) \) corresponding to the blocks of \( \mathcal{H}_c \).
Restricted Cherednik algebra (continued)

\[ \overline{Q}_{\text{c}} = C_{\text{c}, 0, 0} \otimes P_{Q} \]

Then \( \overline{Q}_{\text{c}} \subseteq Z(\mathcal{H}_{\text{c}}) \) but, in general, the inclusion is strict.

However (Müller) \( \text{Idem}_{\text{pr}}(\overline{Q}_{\text{c}}) = \text{Idem}_{\text{pr}}(Z(\mathcal{H}_{\text{c}})) \).

Theorem (B.-Rouquier 2010)

If \( F \) is a Calogero-Moser family corresponding to a primitive idempotent \( b \in \text{Idem}_{\text{pr}}(\overline{Q}_{\text{c}}) \), then

\[ \dim C_{\overline{Q}_{\text{c}}} b = \sum_{\chi \in \text{Irr}(W)} \chi(1) \times 2. \]
Restricted Cherednik algebra (continued)

Let $\bar{Q}_c = \mathbb{C}_{c,0,0} \otimes_P Q$. Then

$$\bar{Q}_c \subseteq Z(\overline{H}_c)$$
Restricted Cherednik algebra (continued)

Let \( \bar{Q}_c = \mathbb{C}_{c,0,0} \otimes P \bar{Q} \). Then

\[
\bar{Q}_c \subseteq Z(\bar{H}_c)
\]

but, in general, the inclusion is strict.
Let $\bar{Q}_c = \mathbb{C}_{c,0,0} \otimes_P Q$. Then

$$\bar{Q}_c \subseteq Z(\overline{H}_c)$$

but, in general, the inclusion is strict. However (Müller)

$$\text{Idem}_{pr}(\bar{Q}_c) = \text{Idem}_{pr}(Z(\overline{H}_c)).$$

**Theorem (B.-Rouquier 2010)**

If $\mathcal{F}$ is a Calogero-Moser family corresponding to a primitive idempotent $b \in \text{Idem}_{pr}(\bar{Q}_c)$, then

$$\dim_{\mathbb{C}} \bar{Q}_c b = \sum_{\chi \in \text{Irr}(W)} \chi(1)^2.$$
Recall that $H \simeq \text{End}_Q(H^e)$. So $KH \simeq \text{End}_KQ(KH^e)$. But $Q$ is an integral domain (and normal) so $KQ = \text{Frac}(Q) =: L$. Moreover, $[L:K] = |W|$ and $\dim K(KH^e) = |W|^2$. So $\dim L(KH^e) = |W|$ and $KH \simeq \text{Mat}_{|W|}(L)$. NOT SPLIT
Recall that

\[ H \simeq \text{End}_Q(He). \]
Recall that

\[ H \simeq \text{End}_Q(H_e). \]

So

\[ KH \simeq \text{End}_{KQ}(KH_e). \]
Recall that

\[ H \cong \text{End}_Q(He). \]

So

\[ KH \cong \text{End}_{KQ}(KHe). \]

But \( Q \) is an integral domain (and normal) so

\[ KQ = \text{Frac}(Q) \]
Recall that

\[ H \cong \text{End}_Q(He). \]

So

\[ KH \cong \text{End}_{KQ}(KH e). \]

But \( Q \) is an integral domain (and normal) so

\[ KQ = \text{Frac}(Q) =: L. \]

Moreover,

\[ [L : K] = |W| \quad \text{and} \quad \dim_K(KHe) = |W|^2. \]
Recall that
\[ H \cong \text{End}_Q(He). \]

So
\[ \KH \cong \text{End}_{\mathbb{K}Q}(\KH e). \]

But \( Q \) is an integral domain (and normal) so
\[ \mathbb{K}Q = \text{Frac}(Q) =: \mathbb{L}. \]

Moreover,
\[ [\mathbb{L} : \mathbb{K}] = |W| \quad \text{and} \quad \dim_{\mathbb{K}}(\KH e) = |W|^2. \]

So \( \dim_{\mathbb{L}}(\KH e) = |W| \) and

\[ \KH \cong \text{Mat}_{|W|}(\mathbb{L}). \]
Recall that
\[ H \simeq \text{End}_Q(He). \]
So
\[ KH \simeq \text{End}_{KQ}(KH_0). \]
But \( Q \) is an integral domain (and normal) so
\[ KQ = \text{Frac}(Q) =: L. \]
Moreover,
\[ [L : K] = |W| \quad \text{and} \quad \dim_K(KH_0) = |W|^2. \]
So \( \dim_L(KH_0) = |W| \) and
\[ KH \simeq \text{Mat}_{|W|}(L). \]

\textbf{NOT SPLIT}
How to split $KH \simeq \text{Mat}_{|W|}(L)$?

Let $M$ be the Galois closure of the extension $L/K$ and let $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. Then, as $[L:K] = |W|$, one gets $|G/H| = |W|$. Then $M \otimes_K L \to \bigoplus_{g H \in G/H} M$ and $\text{Irr}(M_H) \longleftrightarrow G/H$. $|\text{Irr}(M_H)| = |G/H| = |W|$.
How to split $KH \cong \text{Mat}_{|W|}(L)$?

Let $M$ be the Galois closure of the extension $L/K$ and let

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Then, as $[L : K] = |W|$, one gets

$$|G/H| = |W|.$$ 

Then

$$M \otimes_K L \quad \longrightarrow \quad \bigoplus_{gH \in G/H} M$$

$$m \otimes_K l \quad \longrightarrow \quad \bigoplus_{gH \in G/H} mg(l).$$
How to split $KH \simeq \text{Mat}_{|W|}(L)$?

Let $M$ be the Galois closure of the extension $L/K$ and let

$$G = \text{Gal}(M/K) \quad \text{and} \quad H = \text{Gal}(M/L).$$

Then, as $[L : K] = |W|$, one gets

$$|G/H| = |W|.$$ 

Then

$$M \otimes_K L \quad \mapsto \quad \bigoplus_{gH \in G/H} M$$

and

$$m \otimes_K l \quad \mapsto \quad \bigoplus_{gH \in G/H} mg(l).$$

So

$$MH \simeq \bigoplus_{gH \in G/H} \text{Mat}_{|W|}(M)$$

and

$$\text{Irr}(MH) \longleftrightarrow G/H.$$
How to split \( KH \simeq \text{Mat}_{|W|}(L) \)?

Let \( M \) be the \textit{Galois closure} of the extension \( L/K \) and let

\[
G = \text{Gal}(M/K) \quad \text{and} \quad H = \text{Gal}(M/L).
\]

Then, as \([L : K] = |W|\), one gets

\[
|G/H| = |W|.
\]

Then

\[
M \otimes_K L \quad \mapsto \quad \bigoplus_{gH \in G/H} M
\]

\[
m \otimes_K l \quad \mapsto \quad \bigoplus_{gH \in G/H} mg(l).
\]

So

\[
MH \simeq \bigoplus_{gH \in G/H} \text{Mat}_{|W|}(M)
\]

and

\[
\text{Irr}(MH) \leftrightarrow G/H.
\]

\[
|\text{Irr}(MH)| = |G/H| = |W|
\]
$G/H$ and $W$?
$G/H$ and $W$?

Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \hookrightarrow$ evaluation at $c = 0$. 
$G/H$ and $W$?

Let $p_0 = \ker(P \to \mathbb{C}[V/W \times V^*/W]) \iff$ evaluation at $c = 0$. Recall that $Q_0 = Q/p_0Q \simeq \mathbb{C}[(V \times V^*)/W]$. 
Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \leftrightarrow$ evaluation at $c = 0$. Recall that $Q_0 = Q/p_0Q \cong \mathbb{C}[(V \times V^*)/W]$. So

$$q_0 = p_0Q \in \text{Spec}(Q)$$

and we fix a prime ideal $\tau_0$ of $R$ lying above $q_0$. 

Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \leftrightarrow$ evaluation at $c = 0$. Recall that $Q_0 = Q/p_0Q \simeq \mathbb{C}[(V \times V^*)/W]$. So

$$q_0 = p_0Q \in \text{Spec}(Q)$$

and we fix a prime ideal $\mathfrak{r}_0$ of $R$ lying above $q_0$.

$$D_0 = G^{\text{dec}}_{\mathfrak{r}_0} = \{ g \in G \mid g(\mathfrak{r}_0) = \mathfrak{r}_0 \}$$

$$I_0 = G^{\text{in}}_{\mathfrak{r}_0} = \{ g \in G \mid \forall \ r \in R, \ g(r) \equiv r \ \text{mod} \ \mathfrak{r}_0 \}$$
Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \iff$ evaluation at $c = 0$. Recall that $Q_0 = Q/p_0Q \simeq \mathbb{C}[(V \times V^*)/W]$. So

\[ q_0 = p_0Q \in \text{Spec}(Q) \]

and we fix a prime ideal $\mathfrak{r}_0$ of $R$ lying above $q_0$.

\[ D_0 = G_{\mathfrak{r}_0}^{\text{dec}} = \{ g \in G \mid g(\mathfrak{r}_0) = \mathfrak{r}_0 \} \]

\[ l_0 = G_{\mathfrak{r}_0}^{\text{in}} = \{ g \in G \mid \forall r \in R, \ g(r) \equiv r \mod \mathfrak{r}_0 \} \]

Then

\[ \text{Gal}(k_R(\mathfrak{r}_0)/k_P(p_0)) \simeq D_0/l_0. \]
Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \iff\text{evaluation at } c = 0.$

Recall that $Q_0 = Q/p_0Q \simeq \mathbb{C}[(V \times V^*)/W].$ So

$$q_0 = p_0Q \in \text{Spec}(Q)$$

and we fix a prime ideal $r_0$ of $R$ lying above $q_0.$

$$D_0 = G_{r_0}^{\text{dec}} = \{g \in G \mid g(r_0) = r_0\}$$
$$l_0 = G_{r_0}^{\text{in}} = \{g \in G \mid \forall r \in R, \ g(r) \equiv r \ mod \ r_0\}$$

Then

$$\text{Gal}(k_R(r_0)/k_P(p_0)) \simeq D_0/l_0.$$ 

But, since $p_0Q \in \text{Spec}(Q),$ we also have:

- $G = D_0 \cdot H = H \cdot D_0$
- $l_0 = 1$
- $k_R(r_0)/k_P(p_0)$ is the Galois closure of $k_Q(q_0)/k_P(p_0).$
Let $p_0 = \text{Ker}(P \to \mathbb{C}[V/W \times V^*/W]) \leftrightarrow$ evaluation at $c = 0$.
Recall that $Q_0 = Q/p_0Q \simeq \mathbb{C}[(V \times V^*)/W]$. So

$$q_0 = p_0Q \in \text{Spec}(Q)$$

and we fix a prime ideal $r_0$ of $R$ lying above $q_0$.

$$D_0 = G^\text{dec}_{r_0} = \{g \in G \mid g(r_0) = r_0\}$$

$$l_0 = G^\text{in}_{r_0} = \{g \in G \mid \forall r \in R, \ g(r) \equiv r \mod r_0\}$$

Then

$$\text{Gal}(k_R(r_0)/k_P(p_0)) \simeq D_0/l_0.$$ But, since $p_0Q \in \text{Spec}(Q)$, we also have:

- $G = D_0 \cdot H = H \cdot D_0$
- $l_0 = 1$
- $k_R(r_0)/k_P(p_0)$ is the Galois closure of $k_Q(q_0)/k_P(p_0)$.

But

$$k_P(p_0) \simeq \mathbb{C}(V \times V^*)^W \times W \subset k_Q(q_0) = \mathbb{C}(V \times V^*)^\Delta W \subset \mathbb{C}(V \times V^*).$$
$G/H$ and $W$? (continued)
So

\[ D_0 \simeq (W \times W)/\Delta Z(W) \]

\[ D_0 \cap H \simeq \Delta W/\Delta Z(W) \]

and so

\[ W \overset{\sim}{\longrightarrow} (W \times W)/\Delta W \]

\[ K_0(MH) \simeq \mathbb{Z}W \] and the decomposition map (modulo \( \bar{r}c \)) gives a map

\[ K_0(MH) \longrightarrow K_0(Hc) \]

\[ \mathbb{Z} \rightarrow \mathbb{Z} \operatorname{Irr} W \]
So

\[ D_0 \cong (W \times W)/\Delta Z(W) \]

\[ D_0 \cap H \cong \Delta W/\Delta Z(W) \]

and so

\[ W \leftrightarrow (W \times W)/\Delta W \leftrightarrow D_0/(D_0 \cap H) \]
So

\[ D_0 \cong (W \times W)/\Delta Z(W) \]

\[ D_0 \cap H \cong \Delta W/\Delta Z(W) \]

and so

\[ W \xlongleftarrow{\sim} (W \times W)/\Delta W \xlongleftarrow{\sim} D_0/(D_0 \cap H) \xlongleftarrow{\sim} G/H \]
So

\[ D_0 \simeq (W \times W)/\Delta Z(W) \]

\[ D_0 \cap H \simeq \Delta W/\Delta Z(W) \]

and so

\[ W \hookrightarrow (W \times W)/\Delta W \leftrightarrow D_0/(D_0 \cap H) \hookrightarrow G/H \hookrightarrow \text{Irr}(\text{MH}). \]
So

\[ D_0 \cong (W \times W)/\Delta Z(W) \]

\[ D_0 \cap H \cong \Delta W/\Delta Z(W) \]

and so

\[ W \xrightarrow{\sim} (W \times W)/\Delta W \xrightarrow{\sim} D_0/(D_0 \cap H) \xrightarrow{\sim} G/H \xrightarrow{\sim} \text{Irr}(\text{MH}). \]

\[ \mathcal{K}_0(\text{MH}) \cong \mathbb{Z} W \]

and the decomposition map (modulo \( \overline{\tau}_c \)) gives a map

\[
\begin{array}{ccc}
\mathcal{K}_0(\text{MH}) & \xrightarrow{\text{dec}_c} & \mathcal{K}_0(\overline{H}_c) \\
\mathbb{Z} W & \xrightarrow{\overline{\tau}_c} & \mathbb{Z} \text{Irr } W
\end{array}
\]
Calogero-Moser cells

Let $\bar{p}_c = \text{Ker}(P \to C_c, 0, 0)$, so that $H_c = H / \bar{p}_c H$. Let $\bar{r}_c$ be a prime ideal of $R$ lying above $\bar{p}_c$. Then $P / \bar{p}_c \simeq C_c, 0, 0 \simeq R / \bar{r}_c$. So $R H / \bar{r}_c H \simeq H_c$.

Calogero-Moser cells (first definition)

A $\Gamma$-Calogero-Moser two-sided cell is a subset of $W$ associated with a block of $R / \bar{r}_c H$.

If $\Gamma$ is a $\Gamma$-Calogero-Moser cell, we denote by $\text{Irr}_{CM}\Gamma(W)$ the subset of $\text{Irr}(W) \sim \leftrightarrow \text{Irr}(H_c)$ associated with the corresponding block of $H_c$. 
Calogero-Moser cells

Let \( \bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0}) \),
Calogero-Moser cells

Let $\bar{p}_c = \text{Ker}(P \to C_{c,0,0})$, so that

$$\bar{H}_c = H/\bar{p}_c H.$$
Let $\bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0})$, so that

$$\bar{H}_c = H/\bar{p}_c H.$$ 

Let $\bar{r}_c$ be a prime ideal of $R$ lying above $\bar{p}_c$. 

Calogero-Moser cells (first definition)

A $c$-Calogero-Moser two-sided cell is a subset of $W$ associated with a block of $R/\bar{r}_c H$.

If $\Gamma$ is a $c$-Calogero-Moser cell, we denote by $\text{Irr}_{CM}(\Gamma, W)$ the subset of $\text{Irr}(W) \sim \leftrightarrow \text{Irr}(H_c)$ associated with the corresponding block of $H_c$. 

Calogero-Moser cells

Let \( \bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0}) \), so that

\[
\bar{H}_c = H/\bar{p}_cH.
\]

Let \( \bar{r}_c \) be a prime ideal of \( R \) lying above \( \bar{p}_c \). Then \( P/\bar{p}_c \cong \mathbb{C}_{c,0,0} \cong R/\bar{r}_c \).
Calogero-Moser cells

Let $\bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0})$, so that

$$\bar{H}_c = H/\bar{p}_c H.$$  

Let $\bar{r}_c$ be a prime ideal of $R$ lying above $\bar{p}_c$. Then $P/\bar{p}_c \cong \mathbb{C}_{c,0,0} \cong R/\bar{r}_c$. So

$$RH/\bar{r}_c H \cong \bar{H}_c.$$
Calogero-Moser cells

Let $\bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0})$, so that

$$\overline{H}_c = H/\bar{p}_c H.$$ 

Let $\bar{r}_c$ be a prime ideal of $R$ lying above $\bar{p}_c$. Then $P/\bar{p}_c \simeq \mathbb{C}_{c,0,0} \simeq R/\bar{r}_c$. So

$$RH/\bar{r}_c H \simeq \overline{H}_c.$$ 

Calogero-Moser cells (first definition)

A $c$-Calogero-Moser two-sided cell is a subset of $W$ associated with a block of $R_{\bar{r}_c} H$. 
Calogero-Moser cells

Let \( \overline{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0}) \), so that

\[
\overline{H}_c = \frac{H}{\overline{p}_c H}.
\]

Let \( \overline{r}_c \) be a prime ideal of \( R \) lying above \( \overline{p}_c \). Then \( P/\overline{p}_c \cong \mathbb{C}_{c,0,0} \cong R/\overline{r}_c \). So

\[
R\overline{H}/\overline{r}_c H \cong \overline{H}_c.
\]

Calogero-Moser cells (first definition)

A c-Calogero-Moser two-sided cell is a subset of \( W \) associated with a block of \( R_{\overline{r}_c} H \).

If \( \Gamma \) is a c-Calogero-Moser cell, we denote by \( \text{Irr}^{\text{CM}}_{\Gamma}(W) \) the subset of \( \text{Irr}(W) \cong \text{Irr}(\overline{H}_c) \) associated with the corresponding block of \( \overline{H}_c \).
Calogero-Moser cells

Let $\bar{p}_c = \text{Ker}(P \to \mathbb{C}_{c,0,0})$, so that

$$\bar{H}_c = H/\bar{p}_c H.$$ 

Let $\bar{r}_c$ be a prime ideal of $R$ lying above $\bar{p}_c$. Then $P/\bar{p}_c \cong \mathbb{C}_{c,0,0} \cong R/\bar{r}_c$. So

$$RH/\bar{r}_c H \cong \bar{H}_c.$$ 

**Calogero-Moser cells (first definition)**

A $c$-Calogero-Moser two-sided cell is a subset of $W$ associated with a block of $R_{\bar{r}_c} H$.

If $\Gamma$ is a $c$-Calogero-Moser cell, we denote by $\text{Irr}_{\bar{r}_c}^{\text{CM}}(W)$ the subset of $\text{Irr}(W) \leftrightarrow \text{Irr}(\bar{H}_c)$ associated with the corresponding block of $\bar{H}_c$. 
Conjecture (Gordon-Martino 2006, almost true)

If \((W, S)\) is a Coxeter system and if \(c\) is real-valued, then the \(c\)-Calogero-Moser families coincide with the \(c\)-Kazhdan-Lusztig families.

True in types \(A, D\), dihedral; type \(B\) and some \(c\)'s, type \(F_4\) for generic \(c\)'s (Gordon-Martino, Bellamy,...)

Conjecture (B.-Rouquier 2010)

If \((W, S)\) is a Coxeter system and if \(c\) is real-valued, then there exists a prime ideal \(\bar{r}_c\) lying above \(\bar{p}_c\) such that the \(c\)-Calogero-Moser cells coincide with the \(c\)-Kazhdan-Lusztig two-sided cells. Moreover, if \(\Gamma\) is a CM/KL-cell, then \(\text{Irr}_{CM} \Gamma(W) = \text{Irr}_{KL} \Gamma(W)\).
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Conjecture (B.-Rouquier 2010)
If \((W, S)\) is a Coxeter system and if \(c\) is real-valued, then there exists a prime ideal \(\overset{\sim}{\tau}_c\) lying above \(\overset{\sim}{p}_c\) such that the \(c\)-Calogero-Moser cells coincide with the \(c\)-Kazhdan-Lusztig two-sided cells. Moreover, if \(\Gamma\) is a CM/KL-cell, then
\[
\text{Irr}^\text{CM}_\Gamma(W) = \text{Irr}^\text{KL}_\Gamma(W).
\]
B.-Rouquier (2010):

\[ \Gamma = \sum_{\chi \in \text{Irr} \ CM} \chi(1) \]

Assume that all reflections in \( W \) have order 2. Then:

- If \( F \) is a CM-family, then \( F \in \text{is a CM-family} (\varepsilon = \det) \)
- If moreover \( -1 \in W \) (write \( w_0 = -1 \)) and if \( \Gamma \) is a CM-cell, then \( w_0 \Gamma = \Gamma w_0 \) are CM-cells and \( \text{Irr} \ CM w_0 \Gamma (W) = \text{Irr} \ CM \Gamma (W) \)

Generic, left, right cells...

Part of the semicontinuity properties are trivial.
B.-Rouquier (2010): $|\Gamma| = \sum_{\chi \in \text{Irr}_{\Gamma}^{\text{CM}}(W)} \chi(1)^2$

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  - If moreover $-1 \in \mathcal{W}$ (write $w_0 = -1$) and if $\Gamma$ is a CM-cell, then $w_0\Gamma = \Gamma w_0$ are CM-cells and $\text{Irr}_{w_0\Gamma}^{CM}(\mathcal{W}) = \text{Irr}_{\Gamma}^{CM}(\mathcal{W}) \varepsilon$
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- Generic, left, right cells...

- Part of the semicontinuity properties are trivial.
Departure: 12h55 at the Amis de la Nature

Consequently: please be ready to pick your lunch bag at the restaurant at 12h30

VERY COLD!!!

Sun glasses...

Pay individually in Chamonix for the lift (around 42 euros)

The bus is paid by the conference