Kazhdan-Lusztig theory and Ariki's Theorem

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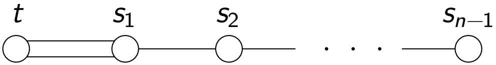

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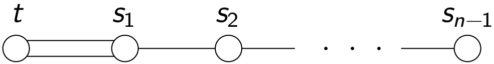

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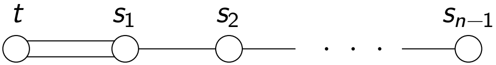

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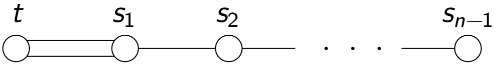

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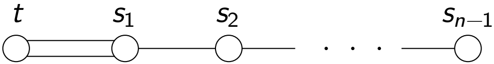

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$$\begin{cases} T_x T_y = T_{xy} & \text{if } \ell(xy) = \ell(x) + \ell(y) \\ (T_t - Q)(T_t + Q^{-1}) = 0 & \\ (T_{s_i} - q)(T_{s_i} + q^{-1}) = 0 & \text{if } 1 \leqslant i \leqslant n-1 \end{cases}$$

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Hypothesis and notation

- $Q_0^2 = -q_0^{2d}$, $d \in \mathbb{Z}$
- $e =$ order of q_0^2, $e > 2$.

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Uglov has constructed an involution $^{-} : \mathcal{F}_r \to \mathcal{F}_r$ and there exists a *unique* $G(\lambda, r) \in \mathcal{F}_r$ such that

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$(|\lambda, r\rangle)_{\lambda \in Bip}$ is called the standard basis
$(G(\lambda, r))_{\lambda \in Bip}$ is called the Kashiwara-Lusztig canonical basis

Ariki's Theorem (Ariki, Uglov, Geck-Jacon). *Assume that* $r \equiv d \mod e$. *There exists a subset* $\mathrm{Bip}_{e,r}(n)$ *of* $\mathrm{Bip}(n)$ *and a bijection*

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REMARK - $d^r_{\lambda\mu}(v)$ is "computable"

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- $\mathrm{Bip}_{d_0,e}(n) = \{$FLOTW bipartitions$\}$ (Jacon). Here, $d_0 \equiv d \mod e$ and $d_0 \in \{0, 1, 2, \ldots, e-1\}$.

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Theorem (Kazhdan-Lusztig, 1979). *For each* $w \in W_n$*, there exists a* **unique** $C_w^\theta \in \mathcal{H}_n$ *such that*

$$\begin{cases} \overline{C}_w^\theta = C_w^\theta \\ C_w^\theta \equiv T_w \mod \mathcal{H}_n^{<_\theta 0} \end{cases}$$

Case $n = 2$ (write $s = s_1$)

$$
\begin{aligned}
C_1^\theta &= 1, \quad C_t^\theta = T_t + Q^{-1}, \quad C_s^\theta = T_s + q^{-1} \\
C_{st}^\theta &= T_{st} + Q^{-1} T_s + q^{-1} T_t + Q^{-1} q^{-1} \\
C_{ts}^\theta &= T_{ts} + Q^{-1} T_s + q^{-1} T_t + Q^{-1} q^{-1} \\
C_{sts}^\theta &= T_{sts} + q^{-1}(T_{st} + T_{ts}) + \\
&\begin{cases} q^{-2} T_t + Q^{-1} q^{-1} (1 + q^2)(T_s + q^{-1}) & \text{if } \theta > 1 \\ Q^{-1} q^{-1} T_s + Q^{-1} q^{-1} (1 - Q^2)(T_t + Q^{-1}) & \text{if } 0 < \theta < 1 \end{cases} \\
C_{tst}^\theta &= T_{tst} + Q^{-1}(T_{st} + T_{ts}) + \\
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C_{w_0}^\theta &= T_{w_0} + Q^{-1} T_{sts} + q^{-1} T_{tst} + Q^{-1} q^{-1} (T_{st} + T_{ts}) \\
&\quad + Q^{-2} q^{-1} T_s + Q^{-1} q^{-2} T_t + Q^{-2} q^{-2}
\end{aligned}
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A θ-*left cell* is an equivalence class for the relation $\sim_L^\theta$.

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Assume that Lusztig's conjectures P1, P2,..., P15 hold. If $r \equiv d \mod e$ *and* $r < \theta < r + e$, *then* $D_\lambda^\theta \neq 0$ *if and only if* $\lambda \in \mathrm{Bip}_{e,r}(n)$. *So the map*

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is bijective and coincides with the map in Ariki's Theorem. Moreover, the decomposition map is given by

$$\mathbf{d}_n[KS_\lambda^\theta] = \sum_{\mu \in \mathrm{Bip}_{e,r}(n)} d_{\lambda\mu}^r(1)[D_\mu^\theta].$$

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- **Question** - It seems reasonable to expect that, if $\mathcal{C} \leqslant_L \mathcal{C}'$, then $\boldsymbol{\lambda}(\mathcal{C}') \trianglelefteq_r \boldsymbol{\lambda}(\mathcal{C})$.

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