Geometry of Calogero-Moser spaces

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Set-up
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- $\dim_{\mathbb{C}} V = n < \infty$
- $W < \text{GL}_{\mathbb{C}}(V)$, $|W| < \infty$.
- $\text{Ref}(W) = \{ s \in W \mid \text{codim}_{\mathbb{C}} V^s = 1 \}$
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Hypothesis. $W = \langle \text{Ref}(W) \rangle$
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**Hypothesis.** \( W = \langle \text{Ref}(W) \rangle \)

(i.e. \( V/W \simeq \mathbb{C}^n \))
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**Hypothesis.** $W = \langle \text{Ref}(W) \rangle$

(i.e. $V/W \simeq \mathbb{C}^n$)

- $\mathcal{C} = \{ c : \text{Ref}(W)/\sim \rightarrow \mathbb{C}\}$

- We fix $c \in \mathcal{C}$
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$$H_c = \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \quad \text{(as a vector space)}$$

$$\forall y \in V, \ \forall x \in V^*, \ [y, x] = \sum_{s \in \text{Ref}(W)} c_s \langle y, s(x) - x \rangle s$$
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Easy fact.

\[ \mathbb{C}[V]^W, \mathbb{C}[V^*]^W \subset Z_c. \]

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**Definition**

The **Calogero-Moser space** associated with the datum \((W, c)\) is the affine variety

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Theorem (Etingof-Ginzburg, 2002)
\(Z_c\) is an integrally closed domain, and is a free \(P\)-module of rank \(|W|\).
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$$\mathcal{Z}_0 = (V \times V^*)/W.$$
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Poisson bracket:

$$\{,\}: \quad Z_c \times Z_c \quad \longrightarrow \quad Z_c$$

$$\quad (z, z') \quad \longmapsto \quad \lim_{t \to 0} \frac{[z, z']_{H_{t,c}}}{t}$$
\[ \text{GL}_n(\mathbb{F}_q), \quad q = p^2, \ l \neq p. \]
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- \text{Fact (Fong-Srinivasan, 1980’s):}
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Conjecture (Broué-Malle-Michel, 1993)

If \( \gamma \) is a \( d \)-core and \( n = |\gamma| + dr \), then there exists a Deligne-Lusztig variety \( X_\gamma(r) \) for \( G \) such that:

• \( \rho_\lambda \mid H^*_c(X_\heartsuit(r), \overline{Q}_\ell) \iff \heartsuit_d(\lambda) = \gamma \).

• \( \text{End}_{\overline{Q}_\ell \text{GL}_n(\mathbb{F}_q)}(H^*_c(X_\gamma(r))) \cong \text{Hecke}_{\text{params}}(G(d, 1, r)) \)
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• Fact (Haiman, 2000):
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**Theorem (Haiman, \sim 2000)**

If $\gamma$ is a $d$-core and $n = |\gamma| + dr$, then there exists an irreducible component $\mathcal{Z}_1(\mathfrak{S}_n)^{\zeta}_\gamma$ of $\mathcal{Z}_1(\mathfrak{S}_n)^{\zeta}$ such that:

• $z_\lambda \in \mathcal{Z}_1(\mathfrak{S}_n)^{\zeta}_\gamma \iff \bigodot_d(\lambda) = \gamma$.

• $\mathcal{Z}_1(\mathfrak{S}_n)^{\zeta}_\gamma$ is diffeo. (conj. isom.) to $\mathcal{Z}_{\text{params}}(G(d, 1, r))$. 
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(b) \(Z_c\) is a symplectic singularity (as defined by Beauville).
Symplectic resolutions
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**Theorem (Ginzburg-Kaledin 2004, Namikawa 2007)**

\[ Z_0 = \left( V \times V^* \right) / W \] admits a *symplectic resolution* if and only if there exists \( c \in \mathcal{C} \) such that \( Z_c \) is smooth.
Symplectic resolutions

Theorem (Ginzburg-Kaledin 2004, Namikawa 2007)

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$Z_c$ is smooth if and only if all the simple $H_c$-modules have dimension $|W|$. 
**Symplectic resolutions**

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**Theorem (Brown-Gordon, 2003)**

\( \mathcal{Z}_c \) is smooth if and only if all the simple \( \mathbf{H}_c \)-modules have dimension \( |W| \).

**Corollary (G.-K., B.-G., Bellamy 2008)**

Assume that \( W \) is irreducible.

Then \( \mathcal{Z}_0 = (V \times V^*)/W \) admits a **symplectic resolution** if and only if \( W = G(d, 1, n) = \mathfrak{S}_n \rtimes (\mu_d)^n \subset \text{GL}_n(\mathbb{C}) \) or \( W = G_4 \subset \text{GL}_2(\mathbb{C}) \).
Theorem (Ginzburg-Kaledin, 2004)

Assume that \( \tilde{Z}_0 \rightarrow Z_0 \) is a symplectic resolution.
Cohomology

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Assume that \( \tilde{Z}_0 \to Z_0 \) is a symplectic resolution.

(1) \( H^{2i+1}(\tilde{Z}_0, \mathbb{C}) = 0 \)
### Theorem (Ginzburg-Kaledin, 2004)

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Assume that $\tilde{Z}_0 \to Z_0$ is a symplectic resolution.

1. $H^{2i+1}(\tilde{Z}_0) = 0$;
2. $H^{2\bullet}(\tilde{Z}_0) \simeq \text{gr}_F(Z(CW))$. 
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Conjecture EC (Ginzburg-Kaledin, 2004)

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(EC1) $H^{2i+1}_{\mathbb{C}^\times}(\tilde{Z}_0) = 0$;
## Cohomology

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### Conjecture EC (Ginzburg-Kaledin, 2004)

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(1) $H^{2i+1}Z_0(\tilde{Z}_0) = 0$;
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Theorem (Vasserot, 2001)

(EC) holds if $W = \mathfrak{S}_n$ ($\tilde{Z}_0 = \text{Hilb}_n(\mathbb{C}^2)$).
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Other cases. $W = W(B_2)$ or $G_4$ (Shan-B. 2016).
Theorem (Vasserot, 2001)

(EC) holds if $W = S_n (\tilde{Z}_0 = \text{Hilb}_n (\mathbb{C}^2))$.

Other cases. $W = W(B_2)$ or $G_4$ (Shan-B. 2016).

Question. What about the general case?
**Cohomology**

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**Conjecture EC (Ginzburg-Kaledin, 2004)**

Assume that $\tilde{Z}_0 \to Z_0$ is a symplectic resolution.

1. $H^{2i+1}_{C^\times}(\tilde{Z}_0) = 0$;
2. $H^{2\bullet}_{C^\times}(\tilde{Z}_0) \simeq \text{Rees}_F(Z(CW))$. 
Cohomology (smooth case)

Theorem (Ginzburg-Kaledin, 2004)

Assume that $\mathcal{Z}_c$ is smooth.

1. $H^{2i+1}(\mathcal{Z}_c) = 0$;
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Conjecture EC (Ginzburg-Kaledin, 2004)

Assume that $\mathcal{Z}_c$ is smooth.

1. $H^{2i+1}_{C^\times}(\mathcal{Z}_c) = 0$;
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Conjecture C (Rouquier-B.)

(C1) \( H^{2i+1}(\mathcal{Z}_c) = 0; \)

(C2) \( H^{2\bullet}(\mathcal{Z}_c) \simeq \text{gr}_\mathcal{F}(\text{Im } \Omega_c). \)

Conjecture EC (Rouquier-B.)

(EC1) \( H^{2i+1}_{C\times}(\mathcal{Z}_c) = 0; \)

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Cohomology (general case)

Conjecture C (Rouquier-B.)

(C1) $H^{2i+1}(\mathcal{Z}_c) = 0$
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Conjecture EC (Rouquier-B.)

(EC1) $H^{2i+1}_{\mathbb{C} \times}(\mathcal{Z}_c) = 0$
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Example (B.). (C) and (EC) are true if $\dim_{\mathbb{C}}(V) = 1$. 
Example of a symplectic singularity
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- \( W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle \)
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- Easy fact: \( \dim_{\mathbb{C}} T_0(\mathcal{Z}_c) = 8. \)
- Let \( m_0 \) denote the maximal ideal of \( \mathcal{Z}_c \) corresponding to 0.
Example of a symplectic singularity

- \( W = \mathcal{W}(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle \)
- Let \( a = c_s \) and \( b = c_t \).
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  \[ \mathcal{Z}_c \hookrightarrow \mathbb{C}^8, \quad \dim \mathcal{Z}_c = 4 \]
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- Easy fact: \( \dim_{\mathbb{C}} T_0(\mathcal{Z}_c) = 8 \).
- Let \( m_0 \) denote the maximal ideal of \( Z_c \) corresponding to 0. Then \( \{m_0, m_0\} \subset m_0 \) because \( \{0\} \) is a symplectic leaf.
Example of a symplectic singularity

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- Easy fact: \( \dim_{\mathbb{C}} T_0(\mathcal{Z}_c) = 8 \).
- Let \( m_0 \) denote the maximal ideal of \( \mathcal{Z}_c \) corresponding to 0. Then
  \( \{ m_0, m_0 \} \subset m_0 \) because \( \{0\} \) is a symplectic leaf.
  \[ \Rightarrow T_0(\mathcal{Z}_c)^* = m_0/m_0^2 \text{ inherits from } \{,\} \text{ a structure of Lie algebra!} \]
Example (continued)

- $W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle$
- $a = c_s = c_t$,
- $\mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c)$
- $\Rightarrow T_0(\mathcal{Z}_c)^* \text{ is a Lie algebra for } \{, \}$
Example (continued)

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- Computation (thanks to MAGMA): \( T_0(\mathcal{Z}_c)^* \simeq \mathfrak{sl}_3(\mathbb{C}) \) (!)
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- Computation (thanks to MAGMA): $T_0(\mathcal{Z}_c)^* \simeq \mathfrak{sl}_3(\mathbb{C})$ (!)
- So $\mathcal{Z}_c \hookrightarrow \mathfrak{sl}_3(\mathbb{C})^* \simeq \mathfrak{sl}_3(\mathbb{C})$ (trace form).
Example (continued)

- \( W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle \)
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- Computation (thanks to MAGMA): \( T_0(\mathcal{Z}_c)^* \cong \mathfrak{sl}_3(\mathbb{C}) \) (!)
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- Computation (MAGMA):
  \[ TC_0(\mathcal{Z}_c) = \{ M \in \mathfrak{sl}_3(\mathbb{C}) | M^2 = 0 \} \]
Example (continued)

- $W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle$
- $a = c_s = c_t$,
- $\mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c)$
- $\Rightarrow T_0(\mathcal{Z}_c)^\ast$ is a Lie algebra for $\{,\}$
- Computation (thanks to MAGMA): $T_0(\mathcal{Z}_c)^\ast \simeq \mathfrak{sl}_3(\mathbb{C})$ (!)
- So $\mathcal{Z}_c \hookrightarrow \mathfrak{sl}_3(\mathbb{C})^\ast \simeq \mathfrak{sl}_3(\mathbb{C})$ (trace form).
- Computation (MAGMA):
  $TC_0(\mathcal{Z}_c) = \{ M \in \mathfrak{sl}_3(\mathbb{C}) \mid M^2 = 0 \} = \overline{O}_{min}$. 
Example (continued)

- \( W = W(B_2) = \langle s, t \mid s^2 = t^2 = (st)^4 = 1 \rangle \)
- \( a = c_s = c_t, \)
- \( \mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c) \)
- \( \Rightarrow T_0(\mathcal{Z}_c)^* \) is a Lie algebra for \( \{, \} \)
- Computation (thanks to MAGMA): \( T_0(\mathcal{Z}_c)^* \cong \mathfrak{sl}_3(\mathbb{C}) \) (!)
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- Computation (MAGMA):
  \[ TC_0(\mathcal{Z}_c) = \{ M \in \mathfrak{sl}_3(\mathbb{C}) \mid M^2 = 0 \} = \overline{\mathcal{O}}_{\text{min}}. \]
- So \( PTC_0(\mathcal{Z}_c) = \mathcal{O}_{\text{min}}/\mathbb{C}^\times \) is smooth
Example (continued)

- \( W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle \)
- \( a = c_s = c_t, \)
- \( \mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c) \)
- \( \Rightarrow T_0(\mathcal{Z}_c)^* \text{ is a Lie algebra for } \{, \} \)
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- So \( \mathcal{Z}_c \hookrightarrow \mathfrak{sl}_3(\mathbb{C})^* \simeq \mathfrak{sl}_3(\mathbb{C}) \) (trace form).
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- So \( PTC_0(\mathcal{Z}_c) = \mathcal{O}_{\text{min}}/\mathbb{C}^\times \) is smooth so Beauville classification theorem applies:
Example (continued)

- \( W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle \)
- \( a = c_s = c_t, \)
- \( \mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c) \)
- \( \Rightarrow T_0(\mathcal{Z}_c)^* \) is a Lie algebra for \( \{ , \} \)
- Computation (thanks to MAGMA): \( T_0(\mathcal{Z}_c)^* \simeq \mathfrak{sl}_3(\mathbb{C}) \ )
- So \( \mathcal{Z}_c \hookrightarrow \mathfrak{sl}_3(\mathbb{C})^* \simeq \mathfrak{sl}_3(\mathbb{C}) \ ) (trace form).
- Computation (MAGMA):
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- So \( PTC_0(\mathcal{Z}_c) = \mathcal{O}_{\text{min}}/\mathbb{C}^\times \) is smooth so Beauville classification theorem applies:

Conclusion (Juteau-B.)

The symplectic singularities \((\mathcal{Z}_c, 0)\) and \((\mathcal{O}_{\text{min}}, 0)\) are equivalent.
Example (continued)

- $W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 = 1 \rangle$
- $a = c_s = c_t$,
- $\mathcal{Z}_c \hookrightarrow \mathbb{C}^8 = T_0(\mathcal{Z}_c)$
- $\Rightarrow T_0(\mathcal{Z}_c)^* \text{ is a Lie algebra for } \{, \}$
- Computation (thanks to MAGMA): $T_0(\mathcal{Z}_c)^* \simeq \mathfrak{sI}_3(\mathbb{C})$ (!)
- So $\mathcal{Z}_c \hookrightarrow \mathfrak{sI}_3(\mathbb{C})^* \simeq \mathfrak{sI}_3(\mathbb{C})$ (trace form).
- Computation (MAGMA): $TC_0(\mathcal{Z}_c) = \{ M \in \mathfrak{sI}_3(\mathbb{C}) | M^2 = 0 \} = \overline{O}_{\text{min}}$.
- So $PTC_0(\mathcal{Z}_c) = \mathcal{O}_{\text{min}}/\mathbb{C}^\times$ is smooth so Beauville classification theorem applies:

Conclusion (Juteau-B.)

The symplectic singularities $(\mathcal{Z}_c, 0)$ and $(\overline{O}_{\text{min}}, 0)$ are equivalent. In particular, $\mathcal{Z}_c$ is not rationally smooth.