Geometry of Calogero-Moser spaces

Cédric Bonnafé

CNRS (UMR 5149) - Université de Montpellier

Poitiers, November 2016

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- $\mathcal{C} = \{c : \mathrm{Ref}(W)/\!\sim \;\longrightarrow \mathbb{C}\}$
- We fix $c \in \mathcal{C}$

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Easy fact.

$\mathbb{C}[V]^W, \mathbb{C}[V^*]^W \subset Z_c$.

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- Poisson bracket:

$$\begin{array}{rccc} \{,\}: & Z_c \times Z_c & \longrightarrow & Z_c \\ & (z, z') & \longmapsto & \lim_{t \to 0} \dfrac{[z, z']_{\mathbf{H}_{t,c}}}{t} \end{array}$$

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Conjecture (Broué-Malle-Michel, 1993)

If γ is a d-core and $n = |\gamma| + dr$, then there exists a Deligne-Lusztig variety $\mathbf{X}_\gamma(r)$ for G such that:

- $\rho_\lambda \mid H_c^\bullet(\mathbf{X}_\heartsuit(r), \overline{\mathbb{Q}}_\ell) \Longleftrightarrow \heartsuit_d(\lambda) = \gamma$.
- $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{GL}_n(\mathbb{F}_q)}\big(H_c^\bullet(\mathbf{X}_\gamma(r))\big) \simeq \mathrm{Hecke}_{\mathsf{params}}(G(d, 1, r))$

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Theorem (Haiman, $\sim$ 2000)

If γ is a d-core and $n = |\gamma| + dr$, then there exists an irreducible component $\mathcal{Z}_1(\mathfrak{S}_n)^\zeta_\gamma$ of $\mathcal{Z}_1(\mathfrak{S}_n)^\zeta$ such that:

- $z_\lambda \in \mathcal{Z}_1(\mathfrak{S}_n)^\zeta_\gamma \Longleftrightarrow \heartsuit_d(\lambda) = \gamma$.
- $\mathcal{Z}_1(\mathfrak{S}_n)^\zeta_\gamma$ is diffeo. (conj. isom.) to $\mathcal{Z}_{\text{params}}(G(d, 1, r))$.

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(b) $\mathcal{Z}_c$ *is a symplectic singularity (as defined by Beauville).*

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Corollary (G.-K., B.-G., Bellamy 2008)

Assume that W is irreducible.
Then $\mathcal{Z}_0 = (V \times V^*)/W$ admits a symplectic resolution if and only if $W = G(d,1,n) = \mathfrak{S}_n \ltimes (\boldsymbol{\mu}_d)^n \subset \mathbf{GL}_n(\mathbb{C})$ or $W = G_4 \subset \mathbf{GL}_2(\mathbb{C})$.

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Question. What about the general case?

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Example (B.). (C) and (EC) are true if $\dim_{\mathbb{C}}(V) = 1$.

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$\Rightarrow T_0(\mathcal{Z}_c)^* = \mathfrak{m}_0/\mathfrak{m}_0^2$ inherits from $\{,\}$ a structure of Lie algebra!

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