Introduction to Deligne-Lusztig Theory

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Eigenvalues of $F$
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  - Let $X = Y/\mu_{q+1} = P^1 \setminus P^1(F_q)$
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  - $0 = |X^F| = q - \text{Tr}(F, H^1_c(Y)_1)$. 
    - Lefschetz $\underline{\text{dim.}}$ $q$
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- By Schur's lemma, two eigenvalues on $H^1_c(Y)_{\theta_0}$: $\rho_+, \rho_-$
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  - Let $X = Y/\mu_{q+1} = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$
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  - By Lefschetz, $\rho_1 = 1$.

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$\Rightarrow \rho_+ = -\rho_-$. 
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  \[
  |Y^\xi F^2| = q^2 - q\lambda_1 - \sum_{\theta \neq 1} \text{action on } H_c^2
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$$= q^2 - 1 - (q - 1) \sum_{\theta} \theta(\xi)\lambda_\theta.$$
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- By Lefschetz Formula, we have, for all $\xi \in \mu_{q+1}$,
  \[
  |Y^{\xi F^2}| = \begin{cases} 
  q^2 & \text{if } \xi = -1 \\
  -q\lambda_1 - \sum_{\theta \neq 1} \theta(\xi)\lambda_\theta(q - 1) & \text{action on } H^2_c \\
  q^2 - 1 - (q - 1) \sum_\theta \theta(\xi)\lambda_\theta.
  \end{cases}
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- On the other hand, $|Y^{\xi F^2}| = \begin{cases} 
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$\Rightarrow$ So $\lambda_\theta = -\theta(-1)q$ if $\theta \neq 1$
Modular representations

Let $S$ be the Sylow subgroup of $\mu_{q+1}$. We identify $S \wedge$ and $(\mu_{q+1}) \wedge \ell$.

If $\alpha^2 \neq 1$, then $\{R_{\alpha}\}$ is a block of defect zero $\{R_{\alpha^0}\}$ and $\{R_{-\alpha^0}\}$ are two blocks of defect zero.

If $\theta$ is an $\ell$-regular linear character of $\mu_{q+1}$ such that $\theta^2 \neq 1$, then $\{R'_{\theta\eta}| \eta \in S \wedge\}$ is a block of defect $S$.

$\{1_G, St_G\} \cup \{R'_{\eta}| \eta \in S \wedge, \eta \neq 1\}$ is a principal block (defect $S$).
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2. \( \{ R'_{\theta_0}^+, R'_{\theta_0}^- \} \cup \{ R'_{\theta_0\eta} \mid \eta \in S^\wedge, \eta \neq 1 \} \) is a block of defect \( S \)
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Curiosities

Abhyankar's conjecture (Raynaud's Theorem):
A finite group $\Gamma$ is the Galois group of a Galois étale covering of $\mathbb{A}^1(F)$ if and only if it is generated by its Sylow $p$-subgroups.

Example: $\Gamma = \text{SL}_2(F_q)$, $Y \rightarrow \mathbb{A}^1(F)$ $(x, y) \mapsto -\rightarrow xy^{q^2} - yx^{q^2}$ $q = 7$, $Y/\{\pm 1\}$ is acted on by $\text{PSL}_2(F_7) \cong \text{GL}_3(F_2)$: it is the reduction modulo 7 of the Klein's quartic (whose group of automorphism is exactly $\text{PSL}_2(F_7)$, reaching Hurwitz' bound).
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  **Example:** $\Gamma = \text{SL}_2(\mathbb{F}_q)$, \( \mathcal{Y} \rightarrow \mathbb{A}^1(\mathbb{F}) \) with \( (x, y) \mapsto -yx^q - xy^q \). $\mathcal{Y}/\{\pm1\}$ is acted on by $\text{PSL}_2(\mathbb{F}_7) \cong \text{GL}_3(\mathbb{F}_2)$: it is the reduction modulo 7 of the Klein’s quartic (whose group of automorphism is exactly $\text{PSL}_2(\mathbb{F}_7)$, reaching Hurwitz’ bound).
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Example: $\Gamma = \text{SL}_2(\mathbb{F}_q)$,

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\begin{align*}
\mathbf{Y} & \longrightarrow \mathbb{A}^1(\mathbb{F}) \\
(x, y) & \longmapsto xy^{q^2} - yx^{q^2}
\end{align*}
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- $q = 7$, $\mathbf{Y}/\{\pm 1\}$ is acted on by $\text{PSL}_2(\mathbb{F}_7) \simeq \text{GL}_3(\mathbb{F}_2)$:
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- \( q = 7 \), \( Y/\{\pm 1\} \) is acted on by \( \text{PSL}_2(\mathbb{F}_7) \cong \text{GL}_3(\mathbb{F}_2) \): it is the reduction modulo 7 of the Klein’s quartic (whose group of automorphism is exactly \( \text{PSL}_2(\mathbb{F}_7) \), reaching Hurwitz’ bound).