On Kazhdan-Lusztig cells in type B

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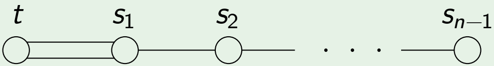

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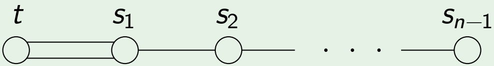

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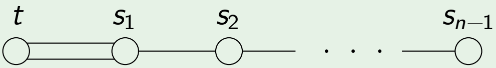

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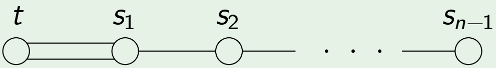

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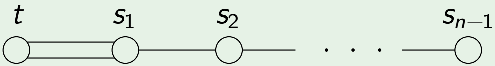

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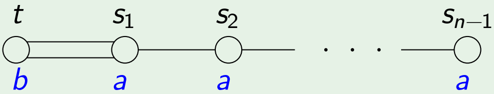

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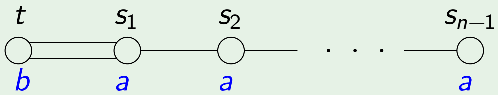

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- For simplification, a, $b > 0$.

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- **Involution:** $\overline{e^{\gamma}} = e^{-\gamma}$, $\overline{T}_w = T_{w^{-1}}^{-1}$

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If $w \in W_n$, there exists a unique $C_w \in \mathcal{H}_n$ such that

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$$C_{s_1 t s_1} = T_{s_1 t s_1} + q^{-1}(T_{s_1 t} + T_{t s_1}) + q^{-2} T_t$$
$$+Q^{-1} q^{-1}(T_{s_1} + q^{-1}) \times \begin{cases} (1+q^2) & \text{if } b > a, \\ 1 & \text{if } b = a, \\ (1-Q^2) & \text{if } b < a \end{cases}$$

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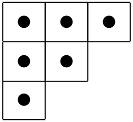

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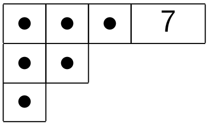

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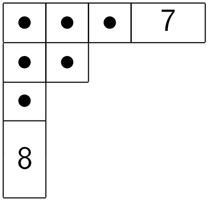

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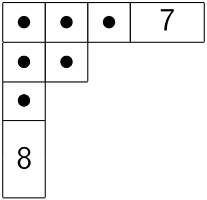

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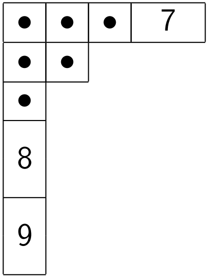

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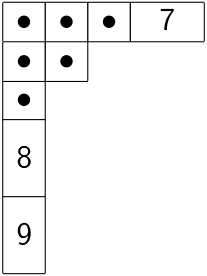

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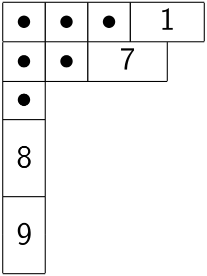

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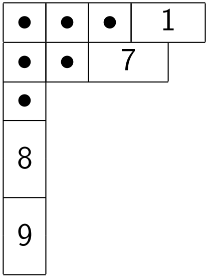

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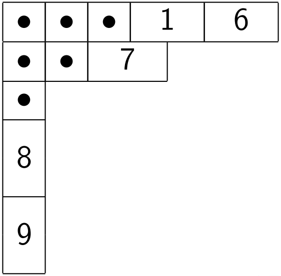

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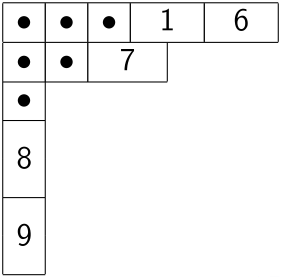

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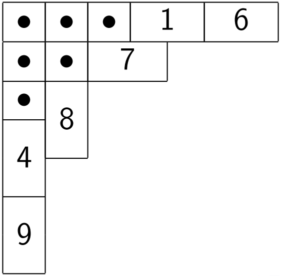

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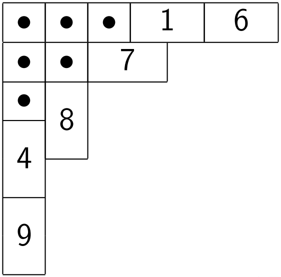

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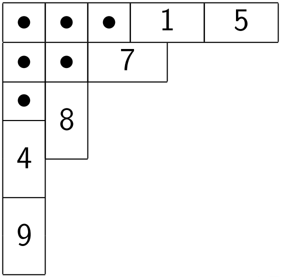

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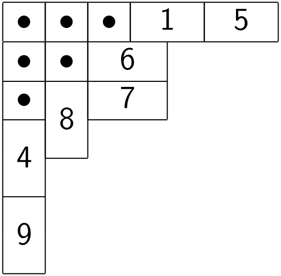

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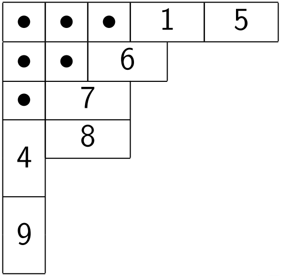

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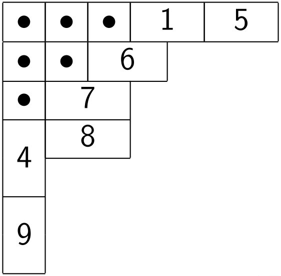

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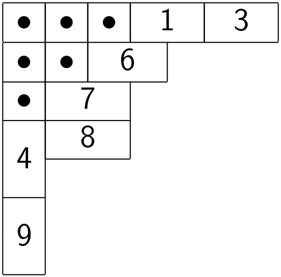

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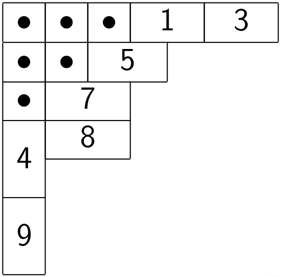

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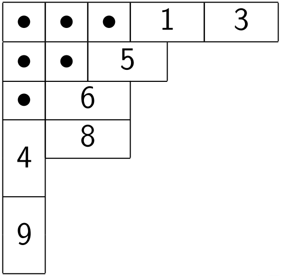

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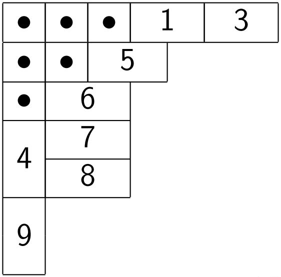

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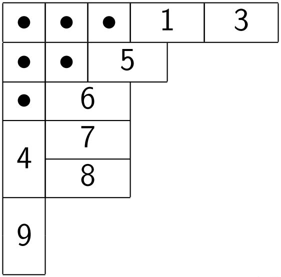

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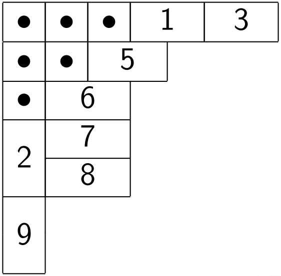

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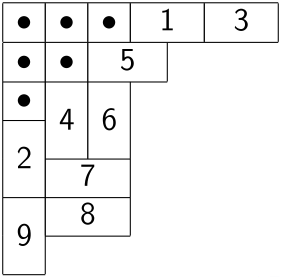

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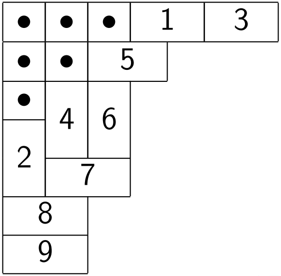

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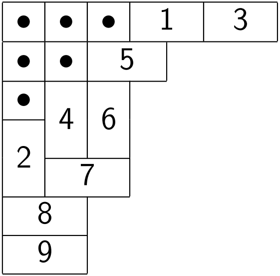

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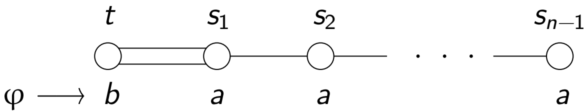

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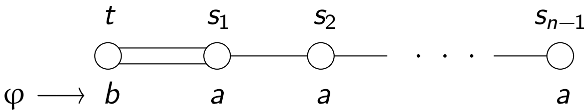

Conjecture A (Geck-Iancu-Lam-B. 2003)

Assume a, $b > 0$ and assume that $0 \leqslant ra < b < (r+1)a$. Then:

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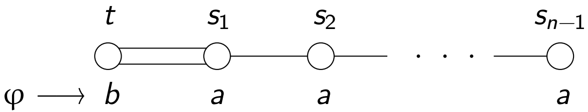

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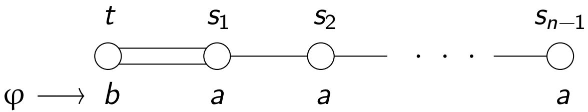

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REMARK - It should be possible to remove the hypothesis on b in the previous corollary using work of Pietraho.

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Lemma (1)

Let $w \in W_n$, let $i \in I_{n-1}^{+}$ and assume that one of the following holds:

(1) $i \geqslant 2$ *and* $w(i) < w(i-1) < w(i+1)$,

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Then $w \sim_R ws_i$.

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Lemma (2)

Let $w \in W_n$ *and let* $i \in I_{n-1}^{+}$ *be such that* $b \geqslant ia$ *and* $w(i)w(i+1) < 0$. *Then* $w \sim_R ws_i$.

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Let $w \in W_n$ and let $i \in I^+_{n-1}$ be such that $b \leqslant ia$ and $|w(1)| > |w(2)| > \cdots > |w(i+1)|$. Then $w \sim_R wt$.

Lemma 2 is implied by

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Let $l \in \{1, \dots, n-1\}$ and assume that $b \geqslant (n-1)a$. Then the coefficient of $C_{a_l \sigma_{[l+1,n]}}$ in $C_t C_{s_{n-1} \cdots s_{l+1} s_l s_1 s_2 \cdots s_{l-1} a_l \sigma_{[l+1,n]}}$ is non-zero!

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