Regular unipotent elements
Éléments unipotents réguliers

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Abstract : *We construct a map from the set of regular unipotent classes of a finite reductive group to the corresponding set in a Levi subgroup. A theorem of Digne, Lehrer and Michel on Lusztig restriction of characteristic functions of such classes gives in particular another (not explicit) construction of such a map. We conjecture these two maps coincide.*

Résumé : *Nous construisons une application entre l'ensemble des classes unipotentes régulières d'un groupe réductif fini et l'ensemble correspondant dans un de ses sous-groupes de Levi. Cela permet de préciser conjecturalement un théorème de Digne, Lehrer et Michel sur la restriction de Lusztig des fonctions caractéristiques de ces classes.*

Version française abrégée

Soit $\mathbf{G}$ un groupe réductif connexe défini sur une clôture algébrique d'un corps fini et soit $F : \mathbf{G} \to \mathbf{G}$ un endomorphisme de Frobenius munissant $\mathbf{G}$ d'une structure rationnelle sur un corps fini. Soit $\mathbf{L}$ un sous-groupe de Levi F-stable d'un sous-groupe parabolique $\mathbf{P}$ (non nécessairement F-stable) de $\mathbf{G}$. Dans [1], conjecture 5.2, Digne, Lehrer et Michel font la conjecture suivante :

Conjecture 1 : *Soit u un élément unipotent régulier de $\mathbf{G}^F$. Alors il existe un élément unipotent régulier v dans $\mathbf{L}^F$ tel que ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\gamma^{\mathbf{G}}_u = \gamma^{\mathbf{L}}_v$ où $\gamma^{\mathbf{G}}_u$ est la fonction caractéristique de la classe de $\mathbf{G}^F$-conjugaison de u multipliée par $|C_{\mathbf{G}^F}(u)|$ (idem pour $\gamma^{\mathbf{L}}_v$).*

Dans [1], ils prouvent que cette conjecture a lieu lorsque $\mathbf{P}$ est F-stable (proposition 5.3) et lorsque le centre de $\mathbf{G}$ est connexe (proposition 5.4). Dans ces deux cas, l'élément v est déterminé explicitement. Dans [2], ils prouvent que la conjecture a lieu lorsque p est bon et q assez grand (théorème 3.7). Cependant, dans ce dernier cas, l'élément v n'est pas explicité. Le problème est même plus sérieux : comment définir *a priori* un élément v dont la classe de $\mathbf{L}^F$-conjugaison ne dépend que de la classe de $\mathbf{G}^F$-conjugaison de u ?

Si on note $\mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ l'ensemble des classes de $\mathbf{G}^F$-conjugaison d'éléments unipotents réguliers de $\mathbf{G}^F$, le but de cette note est de construire *a priori*, lorsque p est bon pour $\mathbf{G}$, une application explicite $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) \longrightarrow \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$.

Conjecture 2 : *Supposons p bon pour $\mathbf{G}$. Soit u un élément unipotent régulier de $\mathbf{G}^F$ et soit $u_{\mathbf{L}} = \mathrm{res}_{\mathbf{L}}^{\mathbf{G}} u$. Alors ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \gamma_u^{\mathbf{G}} = \gamma_{u_{\mathbf{L}}}^{\mathbf{L}}$.*

L'application $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ est construite en utilisant des méthodes inspirées de la théorie des faisceaux-caractères, théorie utilisée dans [2] pour prouver la conjecture 1 lorsque p est bon et q assez grand. C'est pourquoi la conjecture 2 semble raisonnable. Cependant, même sous ces hypothèses, nous ne savons pas démontrer cette conjecture un peu plus précise.

L'application $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ ainsi construite vérifie quelques propriétés : elle ne dépend pas de $\mathbf{P}$ et elle est transitive, c'est-à-dire

$$\mathrm{res}_{\mathbf{M}}^{\mathbf{L}} \circ \mathrm{res}_{\mathbf{L}}^{\mathbf{G}} = \mathrm{res}_{\mathbf{M}}^{\mathbf{G}}$$

si $\mathbf{M}$ est un sous-groupe de Levi F-stable d'un sous-groupe parabolique de $\mathbf{L}$. D'autre part, si $z \in \mathbf{Z}(\mathbf{G})$ et si $l \in \mathbf{L}$ vérifie $l^{-1}F(l) = z$, alors, pour tout $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$, ${}^l\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ et

$$\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\, {}^l\mathcal{U} = {}^l(\mathrm{res}_{\mathbf{L}}^{\mathbf{G}} \mathcal{U}).$$

Pour pouvoir avoir une chance de satisfaire la conjecture 2, il est heureux que l'application $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ vérifie ces propriétés.

Au cours de cette note, nous démontrons le résultat suivant :

Soit v un élément unipotent de $\mathbf{L}$ tel que sa classe de $\mathbf{L}$-conjugaison supporte un système local cuspidal (cf. [3]). Alors le morphisme de groupes $C_{\mathbf{L}}(v) \hookrightarrow C_{\mathbf{G}}(v)$ induit un isomorphisme $A_{\mathbf{L}}(v) \simeq A_{\mathbf{G}}(v)$ où $A_{\mathbf{L}}(v) = C_{\mathbf{L}}(v)/C_{\mathbf{L}}^{\circ}(v)$ et idem pour $A_{\mathbf{G}}(v)$.

Ce résultat n'est pas nécessaire pour construire l'application $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$: il ne sert qu'à simplifier la preuve de la proposition 1.4 qui pourrait être obtenue en vérifiant au cas par cas. Cependant, il est vrai en toute généralité (c'est-à-dire dans le cas où v n'est pas forcément régulier et même lorsque la caractéristique du corps n'est pas bonne pour $\mathbf{G}$). Il généralise un résultat de Lusztig qui montrait la même propriété lorsque la caractéristique du corps est assez grande [4]. Il pourrait d'autre part permettre de lever l'ambiguité sur la définition des fonctions caractéristiques de faisceaux-caractères cuspidaux ce qui donnerait un point d'appui pour attaquer la conjecture 2.

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1. Preliminaries in general reductive groups

Let $\mathbf{G}$ be a connected reductive group over an algebraically closed field $\mathbb{F}$ of characteristic $p \geqslant 0$. We denote by $\mathbf{Z}(\mathbf{G})$ the center of $\mathbf{G}$. If $g \in \mathbf{G}$, $A_{\mathbf{G}}(g)$ will denote the finite group $C_{\mathbf{G}}(g)/C_{\mathbf{G}}^{\circ}(g)$. The $\mathbf{G}$-conjugacy class of g will be denoted by (g), or $(g)_{\mathbf{G}}$ if the group is not clear from the context.

Cuspidal local systems. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$. We fix a unipotent element v in $\mathbf{L}$. Recall the following result :

Proposition 1.1 (Lusztig [3]). *Assume that $(v)_{\mathbf{L}}$ supports a cuspidal local system $\mathcal{E}$. Then :*

(a) *v is a distinguished unipotent element of $\mathbf{L}$, that is, v is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$.*

(b) *$N_{\mathbf{G}}(\mathbf{L})$ stabilizes $(v)_{\mathbf{L}}$ and $\mathcal{E}$.*

(c) *$N_{\mathbf{G}}(\mathbf{L})$ is a reflection group.*

(d) *All parabolic subgroups of $\mathbf{G}$ having $\mathbf{L}$ as Levi subgroup are conjugate (under $N_{\mathbf{G}}(\mathbf{L})$).*

Corollary. *Assume that $(v)_{\mathbf{L}}$ supports a cuspidal local system. Then the morphism $C_{\mathbf{L}}(v) \hookrightarrow C_{\mathbf{G}}(v)$ induces an isomorphism $A_{\mathbf{L}}(v) \simeq A_{\mathbf{G}}(v)$.*

Remark - This corollary is proved by Lusztig in [4] for sufficiently large p ; his proof uses the classification of cuspidal local systems. Our result generalizes slightly Lusztig's one and our proof does not use the classification.

Proof - It is well-known that the morphism $A_{\mathbf{L}}(v) \to A_{\mathbf{G}}(v)$ is injective. So we consider $A_{\mathbf{L}}(v)$ as a subgroup of $A_{\mathbf{G}}(v)$. Let $g \in C_{\mathbf{G}}(v)$, then ${}^g\mathbf{Z}(\mathbf{L})^\circ$ is a maximal torus of $C^\circ_{\mathbf{G}}(v)$ (because v is distinguished in $\mathbf{L}$ by proposition 1.1, (a)) so there exists $h \in C^\circ_{\mathbf{G}}(v)$ such that ${}^g\mathbf{Z}(\mathbf{L})^\circ = {}^h\mathbf{Z}(\mathbf{L})^\circ$, hence $h^{-1}g \in N_{\mathbf{G}}(\mathbf{L})$. This proves that the group $A_{\mathbf{L}}(v)$ is normal in $A_{\mathbf{G}}(v)$ and that the character ζ is invariant under $A_{\mathbf{G}}(v)$ (*cf.* proposition 1.1, (b)). By Mackey formula, it is then sufficient to show that $\mathrm{Ind}_{A_{\mathbf{L}}(v)}^{A_{\mathbf{G}}(v)} \zeta$ is irreducible.

Let χ be an irreducible character of $A_{\mathbf{G}}(v)$. In [3], Lusztig attached to the pair (v, χ) a quadruple $(\mathbf{M}, w, \psi, \rho)$, well-defined up to $\mathbf{G}$-conjugacy, where $\mathbf{M}$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}$, w is a unipotent element of $\mathbf{M}$, ψ is an irreducible character of $A_{\mathbf{M}}(w)$ and ρ is an irreducible character of $N_{\mathbf{G}}(\mathbf{M})/\mathbf{M}$. Moreover, the quadruple $(\mathbf{M}, w, \psi, \rho)$ determines (v, χ) up to conjugacy.

According to [3], proposition 9.5, (v, χ) is associated to a pair of the form $(\mathbf{L}, v, \zeta, \rho)$ if and only if $\rho = \mathrm{sgn}$ is the sign character of $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ ($N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ is a reflection group by proposition 1.1, (c)). But by [5], 1.4, (IV), (v, χ) is associated to a pair of the form $(\mathbf{L}, v, \zeta, \rho)$ if and only if χ is an irreducible component of $\mathrm{Ind}_{A_{\mathbf{L}}(v)}^{A_{\mathbf{G}}(v)} \zeta$. That proves that $\mathrm{Ind}_{A_{\mathbf{L}}(v)}^{A_{\mathbf{G}}(v)}$ has only one irreducible component occuring with some multiplicity m. By [5], 1.2, (II), $m = \mathrm{sgn}(1) = 1$. ■

Regular unipotent class. If $\mathbf{L}$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}$, then the injective morphism $\mathbf{Z}(\mathbf{G}) \hookrightarrow \mathbf{Z}(\mathbf{L})$ induces a surjective morphism $h_{\mathbf{L}} : \mathbf{Z}(\mathbf{G})/\mathbf{Z}(\mathbf{G})^\circ \to \mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$. If there is any ambiguity, we will denote this map by $h_{\mathbf{L}}^{\mathbf{G}}$. The group $\mathbf{G}$ is said to be **cuspidal** if $h_{\mathbf{L}}$ is not an isomorphism for every Levi subgroups $\mathbf{L}$ of proper parabolic subgroups of $\mathbf{G}$.

Proposition 1.2. *Let $\mathbf{L}$ be a cuspidal Levi subgroup of a parabolic subgroup of $\mathbf{G}$. Then :*

(a) *If $\mathbf{M}$ is a cuspidal Levi subgroup of a parabolic subgroup of $\mathbf{G}$ and $\mathrm{Ker}\, h_{\mathbf{L}} = \mathrm{Ker}\, h_{\mathbf{M}}$, then $\mathbf{L}$ and $\mathbf{M}$ are $\mathbf{G}$-conjugate.*

(b) *$\mathbf{L}$ satisfies properties* (c) *and* (d) *of proposition 1.1.*

Proof - Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be a maximal torus of $\mathbf{B}$. We denote by Φ the root system of $\mathbf{G}$ relative to $\mathbf{T}$ and by Δ the basis of Φ associated to $\mathbf{B}$. If $I \subset \Delta$, we denote by $\mathbf{L}_I$ the Levi subgroup of the parabolic subgroup of $\mathbf{G}$ associated to I. We recall the following

Lemma 1.3 ([1], 1.5). *Let I and J be two subsets of Δ. Then $\mathrm{Ker}\, h_{\mathbf{L}_{I \cap J}} = (\mathrm{Ker}\, h_{\mathbf{L}_I}).(\mathrm{Ker}\, h_{\mathbf{L}_J})$.*

We may assume that $\mathbf{L} = \mathbf{L}_I$ and $\mathbf{M} = \mathbf{L}_J$ for some subsets I and J of Δ. By lemma 1.3, the map $\mathbf{Z}(\mathbf{L}_I)/\mathbf{Z}(\mathbf{L}_I)^\circ \to \mathbf{Z}(\mathbf{L}_{I\cap J})/\mathbf{Z}(\mathbf{L}_{I \cap J})^\circ$ is an isomorphism. Since $\mathbf{L}_I$ is cuspidal, we have $I \cap J = I$ so $I \subset J$. By the same argument, we have $J \subset I$. So $I = J$. As in [3], 9.4, the proposition follows immediately from this fact. ■

Proposition 1.4. *Assume that p is* **good** *for $\mathbf{G}$. Let $\mathbf{L}$ be a cuspidal Levi subgroup of a parabolic subgroup of $\mathbf{G}$ and let v be a regular unipotent element of $\mathbf{L}$. Then $A_{\mathbf{L}}(v) = A_{\mathbf{G}}(v)$.*

Proof - The group $A_{\mathbf{L}}(v)$ is isomorphic to $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ because p is good for $\mathbf{G}$. Without loss of generality, we may assume that $\mathbf{G}$ is semisimple, simply connected, and almost simple.

• *Case 1 : The group* $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ *is cyclic.* In this case, there exists an injective linear character $\zeta : A_{\mathbf{L}}(v) \to \overline{\mathbb{Q}}_\ell^\times$. The local system on $(v)_{\mathbf{L}}$ associated to ζ is cuspidal because $\mathbf{L}$ is cuspidal so the result follows from the corollary to proposition 1.1.

• *Case 2 : The group* $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ *is not cyclic.* In this case, $\mathbf{G}$ is of type D_{2n}, with $n \geqslant 2$ and $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ \simeq \mathbf{Z}(\mathbf{G}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover, $\mathbf{L}$ is of type $A_1 \times \cdots \times A_1$ ($(n+1)$ times) : this determines $\mathbf{L}$ up to $\mathbf{G}$-conjugacy. The classification being well-known in this case (because p is odd or, equivalently, good), one can check that $A_{\mathbf{G}}(v) \simeq \mathbf{Z}(\mathbf{G})$. ■

If K is a subgroup of $\mathbf{Z}(\mathbf{G})/\mathbf{Z}(\mathbf{G})^\circ$, we will denote by $\mathcal{L}(K)$ or $\mathcal{L}^{\mathbf{G}}(K)$ the set of Levi subgroups $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$ such that $\operatorname{Ker} h_{\mathbf{L}} \subset K$ and by $\mathcal{L}_{\min}(K)$ or $\mathcal{L}_{\min}^{\mathbf{G}}(K)$ the set of minimal elements of $\mathcal{L}(K)$. It follows immediately from lemma 1.3 that

Lemma 1.5. *Two elements of* $\mathcal{L}_{\min}(K)$ *are* $\mathbf{G}$*-conjugate.*

2. Reductive groups over finite fields

Until the end of this note, we assume that $\mathbb{F}$ is an algebraic closure of a finite field (thus $p > 0$) and that $\mathbf{G}$ is endowed with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. If $g \in \mathbf{G}^F$, we denote by $[g]$ or $[g]_{\mathbf{G}^F}$ the $\mathbf{G}^F$-conjugacy class of g and by $\gamma_g^{\mathbf{G}}$ the charateristic function of $[g]$ multiplied by $C_{\mathbf{G}^F}(g)$.

Proposition 2.1. *Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be two* F*-stable Levi subgroups of* (*non necessarily* F*-stable*) *parabolic subgroups of* $\mathbf{G}$ *which are geometrically conjugate and let* u *be a unipotent element of* $\mathbf{M}^F$*. Assume the following conditions holds :*

(a) *u is a distinguished element of* $\mathbf{M}$,

(b) $N_{\mathbf{G}}(\mathbf{M})$ *stabilizes the class* $(u)_{\mathbf{M}}$*, and*

(c) $A_{\mathbf{M}}(u) = A_{\mathbf{G}}(u)$.

Then $(u)_{\mathbf{G}^F} \cap \mathbf{M}'$ *is a single* $\mathbf{M}'^F$*-conjugacy class.*

Proof - Let g be an element of $\mathbf{G}$ such that $\mathbf{M}' = {}^g\mathbf{M}$. Then $n = g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{M})$. Since $N_{\mathbf{G}}(\mathbf{M})$ stabilizes $(u)_{\mathbf{M}}$ and $A_{\mathbf{M}}(u)$ is isomorphic to $A_{\mathbf{G}}(u)$, there exists $m \in \mathbf{M}$ such that $nm \in N_{\mathbf{G}}(\mathbf{M}) \cap C_{\mathbf{G}}^\circ(u)$. From Lang's theorem, there exists $x \in \mathbf{M}$ such that $x^{-1}nF(x)n^{-1} = nmn^{-1}$. Let $h = gx$. Then $h^{-1}F(h) = nm$ and $\mathbf{M}' = {}^h\mathbf{M}$. Let $u' = {}^hu$. Then $u' \in \mathbf{M}'$ and $u' \in (u)_{\mathbf{G}^F}$ since $h^{-1}F(h) \in C_{\mathbf{G}}^\circ(u)$. That proves that $(u)_{\mathbf{G}^F} \cap \mathbf{M}'$ is not empty. To prove proposition 2.1, it now suffices to prove the following

Lemma. $(u)_{\mathbf{G}^F} \cap \mathbf{M}$ *is a single* $\mathbf{M}^F$*-conjugacy class.*

Proof of lemma - Let u' be an element of $(u)_{\mathbf{G}^F} \cap \mathbf{M}$. Then, there exists an element $g \in \mathbf{G}^F$ such that $u' = {}^gu$. Since u is distinguished in $\mathbf{M}$, $\mathbf{Z}(\mathbf{M})^\circ$ and ${}^{g^{-1}}\mathbf{Z}(\mathbf{M})^\circ$ are maximal tori of $C_{\mathbf{G}}^\circ(u)$. Thus there exists $y \in C_{\mathbf{G}}^\circ(u)$ such that ${}^y\mathbf{Z}(\mathbf{M})^\circ = {}^{g^{-1}}\mathbf{Z}(\mathbf{M})^\circ$. Let $h = gy$. Then $h \in \mathbf{G}$, $u' = {}^hu$ and ${}^h\mathbf{Z}(\mathbf{M})^\circ = \mathbf{Z}(\mathbf{M})^\circ$, that is $h \in N_{\mathbf{G}}(\mathbf{M})$. Moreover, $h^{-1}F(h) = y^{-1}F(y) \in C_{\mathbf{G}}^\circ(u)$.

By hypothesis, $N_{\mathbf{G}}(\mathbf{M})$ stabilizes the class $(u)_{\mathbf{M}}$. Hence, there exists $m \in \mathbf{M}$ such that $u' = {}^mu$. Let $x = h^{-1}m$. We have $x \in C_{\mathbf{G}}(u)$ and $(hx)^{-1}F(hx) = m^{-1}F(m)$. Hence $m^{-1}F(m) = (yx)^{-1}F(yx)$.

Since $A_{\mathbf{M}}(u)$ is isomorphic to $A_{\mathbf{G}}(u)$, there exists $a \in C_{\mathbf{M}}(u)$ and $b \in C_{\mathbf{G}}^\circ(u)$ such that $yx = ba$. Thus $m^{-1}F(m) = a^{-1}(b^{-1}F(b))aa^{-1}F(a)$. This implies that $a^{-1}(b^{-1}F(b))a$ belongs to $C_{\mathbf{G}}^\circ(u) \cap \mathbf{M} = C_{\mathbf{M}}^\circ(u)$ and that $b^{-1}F(b)$ also belongs to $C_{\mathbf{M}}^\circ(u)$. It results from Lang's theorem that there exists

$c \in C^\circ_{\mathbf{M}}(u)$ such that $b^{-1}F(b) = c^{-1}F(c)$. So $m^{-1}F(m) = (ca)^{-1}F(ca)$, and $ca \in C_{\mathbf{M}}(u)$. Let $m' = ma^{-1}c^{-1}$. Then $F(m') = m'$ and $u' = {}^{m'}u$. ■

From now on, we make the following

Hypothesis : *p is good for* $\mathbf{G}$.

We will denote by $\mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ the set of $\mathbf{G}^F$-classes of regular unipotent elements. If $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ and $z \in H^1(F, \mathbf{Z}(\mathbf{G}))$ we denote by $\mathcal{U}_z$ the class ${}^g\mathcal{U}$ where $g \in \mathbf{G}$ is such that $g^{-1}F(g)$ is in $\mathbf{Z}(\mathbf{G})$ and represents z. Then $\mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) = \{\mathcal{U}_z \mid z \in H^1(F, \mathbf{Z}(\mathbf{G}))\}$, because p is good for $\mathbf{G}$.

Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Digne, Lehrer and Michel [1] made the following conjecture :

Conjecture 1 : *Let u be a regular unipotent element of $\mathbf{G}^F$. There exists a regular unipotent element $v \in \mathbf{L}^F$ such that ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\gamma^{\mathbf{G}}_u) = \gamma^{\mathbf{L}}_v$ where ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is the Lusztig restriction.*

They proved this conjecture in the following three cases (*cf.* [1] and [2]) :

A. Whenever $\mathbf{P}$ is F-stable (*cf.* [1], proposition 5.3). In this case, v is obtained as follows : the intersection of the $\mathbf{G}^F$-conjugacy class of u with $\mathbf{P}^F$ is a single $\mathbf{P}^F$-conjugacy class and v is an element of the projection (from $\mathbf{P}^F$ to $\mathbf{L}^F$ via the canonical map) of this class. Moreover, the $\mathbf{L}^F$-conjugacy class of v does not depend on $\mathbf{P}$. We denote by $\rho^{\mathbf{G}}_{\mathbf{L}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) \to \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$ the map obtained in this way. We have, for all $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ and $z \in H^1(F, \mathbf{Z}(\mathbf{G}))$,

$$(\star) \qquad \rho^{\mathbf{G}}_{\mathbf{L}}(\mathcal{U}_z) = (\rho^{\mathbf{G}}_{\mathbf{L}}(\mathcal{U}))_{h'_{\mathbf{L}}(z)}$$

where $h'_{\mathbf{L}} : H^1(F, \mathbf{Z}(\mathbf{G})) \to H^1(F, \mathbf{Z}(\mathbf{L}))$ is the canonical surjective morphism induced by $h_{\mathbf{L}}$.

B. Whenever the center of $\mathbf{L}$ is connected (*cf.* [1], proposition 5.4). In this case, there is only one $\mathbf{L}^F$-conjugacy class of regular unipotent elements in $\mathbf{L}^F$...

C. Whenever q is large enough (*cf.* [2], theorem 3.7) : the proof of this fact is particularly difficult and involves character sheaves theory. In this last case, the existence of v has been proved but it has not been explicitly described.

The purpose of this note is to make conjecture 1 precise, that is, to construct a map $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) \to \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$.

The set $\mathcal{L}^{\mathbf{L}}_{\min}(\{1\})$ is a single conjugacy class under $\mathbf{L}$ (*cf.* lemma 1.5) and is F-stable because the trivial subgroup $\{1\}$ of $\mathbf{Z}(\mathbf{L})/\mathbf{Z}(\mathbf{L})^\circ$ is F-stable. Hence, there is an F-stable element in $\mathcal{L}^{\mathbf{L}}_{\min}(\{1\})$. Moreover, by proposition 1.2, (b), we can choose en element $\mathbf{M}$ of $\mathcal{L}^{\mathbf{L}}_{\min}(\{1\})$ such that $\mathbf{M}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup $\mathbf{Q}$ of $\mathbf{L}$. Because $h^{\mathbf{L}}_{\mathbf{M}}$ is an isomorphism, the map $\rho^{\mathbf{L}}_{\mathbf{M}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F) \to \mathrm{Reg}_{\mathrm{uni}}(\mathbf{M}^F)$ is bijective (*cf.* $(\star)$).

Moreover, $\mathbf{M} \in \mathcal{L}^{\mathbf{G}}_{\min}(\mathrm{Ker}\, h_{\mathbf{L}})$. Hence, by the same argument, $\mathbf{M}$ is conjugate, in $\mathbf{G}$, to an F-stable Levi subgroup $\mathbf{M}_0$ of an F-stable parabolic subgroup $\mathbf{Q}_0$ of $\mathbf{G}$. Let $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ and let $v \in \rho^{\mathbf{G}}_{\mathbf{M}_0}\mathcal{U}$. By propositions 1.4 and 2.1, $[v]_{\mathbf{G}^F} \cap \mathbf{M}$ is a single $\mathbf{M}^F$-conjugacy class. We define :

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}(\mathcal{U}) = (\rho^{\mathbf{L}}_{\mathbf{M}})^{-1}([v]_{\mathbf{G}^F} \cap \mathbf{M}).$$

Then $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}(\mathcal{U})$ is well-defined because $\mathbf{M}$ and $\mathbf{M}_0$ are unique up to $\mathbf{L}^F$-conjugacy and $\mathbf{G}^F$-conjugacy respectively. So we have constructed a map $\mathrm{res}^{\mathbf{G}}_{\mathbf{L}} : \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F) \to \mathrm{Reg}_{\mathrm{uni}}(\mathbf{L}^F)$.

Conjecture 2 : *Let u be a regular unipotent element of $\mathbf{G}^F$ and let $u_{\mathbf{L}} \in \mathrm{res}_{\mathbf{L}}^{\mathbf{G}}([u]_{\mathbf{G}^F})$. Then ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\gamma_u^{\mathbf{G}}) = \gamma_{u_{\mathbf{L}}}^{\mathbf{L}}$.*

Proposition 2.2. (a) $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ *is independent of* $\mathbf{P}$.

(b) *If* $\mathbf{P}$ *is* F*-stable, then* $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}} = \rho_{\mathbf{L}}^{\mathbf{G}}$.

(c) *If* $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$, *then* $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\mathcal{U}_z = (\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}\mathcal{U})_{h'_{\mathbf{L}}(z)}$. *In particular,* $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ *is surjective.*

(d) *Let* $\mathbf{M}$ *be an* F*-stable Levi subgroup of a parabolic subgroup of* $\mathbf{G}$. *Assume that each regular unipotent element* v *of* $\mathbf{L}$ *satisfies* $A_{\mathbf{L}}(v) \simeq A_{\mathbf{G}}(v)$ *and that* $\mathbf{M}$ *is* $\mathbf{G}$*-conjugate to* $\mathbf{L}$. *If* $\mathcal{U} \in \mathrm{Reg}_{\mathrm{uni}}(\mathbf{G}^F)$ *and* $v \in \mathrm{res}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{U})$, *then* $\mathrm{res}_{\mathbf{M}}^{\mathbf{G}}(\mathcal{U}) = [v]_{\mathbf{G}^F} \cap \mathbf{M}$ *(cf. proposition 2.1).*

(e) *Let* $\mathbf{M}$ *be an* F*-stable Levi subgroup of a parabolic subgroup of* $\mathbf{L}$. *Then* $\mathrm{res}_{\mathbf{M}}^{\mathbf{G}} = \mathrm{res}_{\mathbf{M}}^{\mathbf{L}} \circ \mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$.

(a), (b) and (c) are easy but (d) and (e) are quite technical. Proofs of these facts will appear in a forthcoming paper.

Remark - If $\mathbf{G}$ is a rational Levi subgroup of a parabolic subgroup of a split reductive group all components of which are of type A, then each F-stable Levi subgroup $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$ is $\mathbf{G}$-conjugate to an F-stable Levi subgroup $\mathbf{M}$ of an F-stable parabolic subgroup of $\mathbf{G}$. Moreover we have $A_{\mathbf{L}}(v) \simeq A_{\mathbf{G}}(v)$ for all regular unipotent v of $\mathbf{L}$ because the natural morphism $\mathbf{Z}(\mathbf{G}) \to A_{\mathbf{G}}(v)$ is surjective. Hence the map $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ is described well by properties (b) and (d) of $\mathrm{res}_{\mathbf{L}}^{\mathbf{G}}$ given in proposition 2.2.

References

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