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On a Theorem of Shintani

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Let χ be an irreducible character of $G_d = \mathbf{GL}_n(\mathbb{F}_{q^d})$ invariant under the automorphism ϕ of G_d induced by the field automorphism $\mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$, $x \mapsto x^q$, and let e be a divisor of d. By a theorem of Shintani, there exists an extension $\tilde{\chi}_e$ of χ to $G_d \rtimes \langle \phi^e \rangle$ whose Shintani descent to G_e is, up to a sign ε, an irreducible character of G_e. It is shown in this paper that $\tilde{\chi}_e$ may always be chosen such that $\varepsilon = 1$. With this particular choice, $\tilde{\chi}_e$ is the restriction of $\tilde{\chi}_1$. Our methods rely on the work of Digne and Michel on Deligne–Lusztig theory for nonconnected reductive groups.

Let $\mathbf{G}^\circ = \mathbf{GL}_n(\mathbb{F})^d$, where $\mathbb{F}$ is an algebraic closure of a finite field and where n and d are natural numbers. The symmetric group $\mathfrak{S}_d$ acts on $\mathbf{G}^\circ$ by permutations of the components of $\mathbf{G}^\circ$. We denote by $\mathbf{G}$ the semidirect product $\mathbf{G} = \mathbf{G}^\circ \rtimes \mathfrak{S}_d$. It is a nonconnected reductive group, with neutral component $\mathbf{G}^\circ$. We denote by F_0: $\mathbf{G} \to \mathbf{G}$ the natural split Frobenius endomorphism on $\mathbf{G}$ (acting trivially on $\mathfrak{S}_d$), and we choose an element $\sigma \in \mathfrak{S}_d$. Let F: $\mathbf{G} \to \mathbf{G}$ denote the Frobenius endomorphism defined by $F(g) = {}^\sigma F_0(g)$.

In this paper we discuss the irreducible characters of $\mathbf{G}^F$ (the unipotent characters of $\mathbf{G}^F$ were described in [B]). We first prove that there exists a Jordan decomposition of characters (this result is well-known for $\mathbf{G}^\circ$); moreover, this decomposition commutes with Lusztig generalized induction (cf. (3.2.1)). We also prove that all the irreducible characters of $\mathbf{G}^F$ are linear combinations of generalized Deligne–Lusztig characters (this gener-

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alizes the well-known result of G. Lusztig and B. Srinivasan [LS, Theorem 3.2)] about irreducible characters of the general linear group over a finite field).

As an application of these results, we obtain new results about Shintani descent in the case of the general linear group. In [S], Shintani proved that any irreducible characters of the finite group $G_d = \mathbf{GL}_n(\mathbb{F}_{q^d})$ stable under the automorphism ϕ induced by the field automorphism $\mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$, $x \mapsto x^q$ can be extended to $G_d\langle\phi\rangle$ in such a way that its Shintani descent is, up to sign, an irreducible character of $G_1 = \mathbf{GL}_n(\mathbb{F}_q)$. In Theorem 4.3.1 we prove that this sign can always be chosen to be equal to 1 and get precise formulas for the corresponding extension. As a consequence, we obtain that the Shintani descent of this particular extension to G_e is an irreducible character of G_e (where e divides d).

0. NOTATION

0.1. *General Notation*

Let $\mathbb{F}$ be an algebraic closure of a finite field. We denote by p its characteristic. We also fix a power q of p, and we denote by $\mathbb{F}_q$ the subfield of $\mathbb{F}$ with q elements. All algebraic varieties and all algebraic groups will be considered over $\mathbb{F}$. If $\mathbf{H}$ is an algebraic group (over $\mathbb{F}$), we will denote by $\mathbf{H}^\circ$ its connected component containing 1. If $\mathbf{H}$ is endowed with an $\mathbb{F}_q$-structure, we also define

$$\varepsilon_{\mathbf{H}^\circ} = (-1)^{\mathbb{F}_q\text{-rank}(\mathbf{H}^\circ)}.$$

Let ℓ be a prime number different from p. We denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the ℓ-adic field $\mathbb{Q}_\ell$. If G is a finite group, all representations and all characters of G will be considered over $\overline{\mathbb{Q}}_\ell$. For instance, a G-module is a $\overline{\mathbb{Q}}_\ell G$-module of finite dimension. We will denote by $\operatorname{Irr} G$ the set of irreducible characters of G.

If n is a positive integer, we denote by $\mathbf{GL}_n$ the group of invertible matrices with coefficients in $\mathbb{F}$, and if $g \in \mathbf{GL}_n$, we will denote by $g^{(q)}$ the matrix obtained from g by raising all coefficients to the qth power. We will denote by $\mathbf{T}_n$ the split maximal torus of $\mathbf{GL}_n$ consisting of diagonal matrices and by $\mathbf{B}_n$ the rational Borel subgroup of $\mathbf{GL}_n$ consisting of upper triangular matrices.

0.2. *The Problem*

Let r be a positive integer and let $d_1, \ldots, d_r$ and $n_1, \ldots, n_r$ also be positive integers. Throughout this paper $\mathbf{G}^\circ$ will denote the following connected reductive group:

$$\mathbf{G}^\circ = \prod_{i=1}^{r} \underbrace{(\mathbf{GL}_{n_i} \times \cdots \times \mathbf{GL}_{n_i})}_{d_i \text{ times}}.$$

We endow $\mathbf{G}^\circ$ with the split Frobenius endomorphism

$$\begin{array}{rccc} F_0: & \mathbf{G}^\circ & \to & \mathbf{G}^\circ \\ & (g_{i1}, \ldots, g_{id_i})_{1 \le i \le r} & \mapsto & \left(g_{i1}^{(q)}, \ldots, g_{id_i}^{(q)}\right)_{1 \le i \le r} \end{array}.$$

We will denote by $\mathbf{T}_0^\circ$ and $\mathbf{B}_0^\circ$ the maximal torus and the Borel subgroup of $\mathbf{G}^\circ$, defined, respectively, by

$$\mathbf{T}_0^\circ = \prod_{i=1}^{r} \underbrace{(\mathbf{T}_{n_i} \times \cdots \times \mathbf{T}_{n_i})}_{d_i \text{ times}}$$

and

$$\mathbf{B}_0^\circ = \prod_{i=1}^{r} \underbrace{(\mathbf{B}_{n_i} \times \cdots \times \mathbf{B}_{n_i})}_{d_i \text{ times}}.$$

The group $\mathfrak{S} = \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_r}$ acts on $\mathbf{G}^\circ$ in the natural way. More explicitly, if $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathfrak{S}$ and if $(g_{i1}, \ldots, g_{id_i})_{1 \le i \le r} \in \mathbf{G}^\circ$, we put

$${}^{\sigma}(g_{i1}, \ldots, g_{id_i})_{1 \le i \le r} = (g_{i\sigma_i^{-1}(1)}, \ldots, g_{i\sigma_i^{-1}(d_i)})_{1 \le i \le r}.$$

The elements of $\mathfrak{S}$ induce automorphisms of $\mathbf{G}^\circ$, which stabilize $\mathbf{T}_0^\circ$ and $\mathbf{B}_0^\circ$, so they are quasi-semisimple (cf. [DM2, Definition 1.1(i)]). In fact, they are all quasi-central (cf. [DM2, Definition-Theorem 1.15] and [B, Lemma 7.1.1]).

We extend the Frobenius endomorphism F_0 to $\mathbf{G}^\circ \rtimes \mathfrak{S}$ by letting F_0 act trivially on $\mathfrak{S}$. We fix once and for all an element $\sigma \in \mathfrak{S}$, and we denote by F the Frobenius endomorphism on $\mathbf{G}^\circ \rtimes \mathfrak{S}$ given by

$$F(g) = \sigma F_0(g)\sigma^{-1} = {}^{\sigma}F_0(g)$$

for all $g \in \mathbf{G}^\circ \rtimes \mathfrak{S}$.

We will denote by $\mathbf{G}$ an F-stable subgroup of $\mathbf{G}^\circ \rtimes \mathfrak{S}$ containing $\mathbf{G}^\circ$. Hence $\mathbf{G}$ is a reductive group with neutral component $\mathbf{G}^\circ$. Moreover, there exists an F-stable (that is, a σ-stable) subgroup A of $\mathfrak{S}$ such that

$$\mathbf{G} = \mathbf{G}^\circ \rtimes A.$$

Thus we have $\mathbf{G}^F = \mathbf{G}^{\circ F} \rtimes A^F = \mathbf{G}^{\circ F} \rtimes A^{\sigma}$.

Problem. Parametrize the irreducible characters of $\mathbf{G}^F$.

For this purpose we can make the following hypothesis without loss of generality:

HYPOTHESIS. *The Frobenius endomorphism F acts trivially on $\mathbf{G}/\mathbf{G}^\circ$, that is, A is contained in the centralizer of σ in $\mathfrak{S}$. Consequently,*

$$\mathbf{G}^F = \mathbf{G}^{\circ F} \rtimes A.$$

Remark 0. Let $N = d_1 n_1 + \cdots + d_r n_r$. Then $\mathbf{G}^\circ$ is isomorphic to a rational Levi subgroup $\mathbf{H}^\circ$ of a parabolic subgroup of $\mathbf{GL}_N$ (endowed with the split Frobenius endomorphism $g \mapsto g^{(q)}$), and $\mathbf{G}$ is isomorphic to a rational subgroup $\mathbf{H}$ of the normalizer of $\mathbf{H}^\circ$ in $\mathbf{GL}_N$, containing $\mathbf{H}^\circ$ and such that all elements of $\mathbf{H}/\mathbf{H}^\circ$ are rational. Conversely, if $\mathbf{H}$ is such a rational subgroup of $\mathbf{GL}_N$, then there exist positive integers r, $d_1, \dots, d_r, n_1, \dots, n_r$; an element σ of $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_r}$; and a subgroup A of $\mathfrak{S}^\sigma$ such that $\mathbf{H}$ is isomorphic to the group $\mathbf{G}$ constructed as above. In particular, if $\mathbf{L}$ is an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$ (cf. [B, Definitions 6.1.1 and 6.1.2] for the definitions of parabolic subgroups and Levi subgroups of nonconnected reductive groups), then all of the results proved for $\mathbf{G}$ hold in $\mathbf{L}$.

1. JORDAN DECOMPOSITION OF CHARACTERS OF $\mathbf{G}^F$

1.1. *Dual of* $\mathbf{G}$

Let $(\mathbf{G}^{\circ *}, \mathbf{T}_0^{\circ *}, F^*)$ be a dual triple of $(\mathbf{G}^\circ, \mathbf{T}_0^\circ, F)$. The elements α of $\mathfrak{S}$ induce automorphisms α^* of $\mathbf{G}^{\circ *}$. The group $\mathfrak{S}^*$ of automorphisms of $\mathbf{G}^{\circ *}$ induced by $\mathfrak{S}$ is isomorphic to the opposite group of $\mathfrak{S}$. We extend the action of F^* to $\mathbf{G}^{\circ *} \rtimes \mathfrak{S}^*$ so that it acts on $\mathfrak{S}^*$ by conjugation by σ^{*-1}. We denote by $\mathbf{G}^*$ the semidirect product $\mathbf{G}^{\circ *} \rtimes A^*$, where A^* is the image of A under the preceding anti-isomorphism. In particular, $\mathbf{G}^{*\circ} = \mathbf{G}^{\circ *}$!

1.2. *Lusztig Series of* $\mathbf{G}^F$

Let s be a semisimple element of $\mathbf{G}^{*\circ F^*}$. We denote by (s) (or $(s)_{\mathbf{G}^{*F^*}}$ if confusion is possible) the $\mathbf{G}^{*F^*}$-conjugacy class of s and by $(s)^\circ$ (or $(s)^\circ_{\mathbf{G}^{*\circ F^*}}$) the $\mathbf{G}^{*\circ F^*}$-conjugacy class of s.

DEFINITION 1.2.1. The *Lusztig series* $\mathscr{E}(\mathbf{G}^F, (s))$ of $\mathbf{G}^F$ associated with s (or (s)) is the set of irreducible characters of $\mathbf{G}^F$ occurring in some $\mathrm{Ind}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \gamma^\circ$, where γ° is an element of a usual Lusztig series $\mathscr{E}(\mathbf{G}^{\circ F}, (s')^\circ)$ with $s' \in (s)$.

The characters of the Lusztig series $\mathscr{E}(\mathbf{G}^F, 1)$ are called *unipotent*; this definition agrees with definitions given in [DM2, Section 5] or [B, Definition 6.4.1] (cf. [B, Lemma 6.4.2]).

The following lemma follows immediately from the definitions:

LEMMA 1.2.2. *Let s be a semisimple element of $\mathbf{G}^{*\circ F^*}$, γ° be an element of $\mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$, and $\alpha \in A$. Then ${}^\alpha\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, ({}^{\alpha^{*-1}}s)^\circ)$.*

COROLLARY 1.2.3.

$$\operatorname{Irr} \mathbf{G}^F = \bigcup_{(s)} \mathscr{E}(\mathbf{G}^F, (s)),$$

where (s) runs over the set of $\mathbf{G}^{*F^*}$-classes of semisimple elements of $\mathbf{G}^{*\circ F^*}$. Moreover, this union is disjoint.

Proof. The equality follows easily from the corresponding fact for $\mathbf{G}^{\circ F}$. Let us prove now that the union is disjoint. Let s and t be two semisimple elements of $\mathbf{G}^{*\circ F^*}$ and let γ be an irreducible character of $\mathbf{G}^F$ belonging to both $\mathscr{E}(\mathbf{G}^F, (s))$ and $\mathscr{E}(\mathbf{G}^F, (t))$. Then by definition there exist irreducible characters γ_1° and γ_2° of $\mathbf{G}^{\circ F}$ occurring in the restriction of γ to $\mathbf{G}^{\circ F}$ such that $\gamma_1^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (s')^\circ)$ and $\gamma_2^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (t')^\circ)$, where $s' \in (s)$ and $t' \in (t)$.

But by Clifford theory there exists $\alpha \in A$ such that $\gamma_2^\circ = {}^\alpha\gamma_1^\circ$. It follows from Lemma 1.2.2 and from the fact that Corollary 1.2.3 holds in $\mathbf{G}^\circ$ that $t' \in ({}^{\alpha^{*-1}}s')^\circ$, so $t \in (s)$. ∎

COROLLARY 1.2.4. *Let s be a semisimple element in $\mathbf{G}^{*\circ F^*}$, and let $\gamma \in \mathscr{E}(\mathbf{G}^F, (s))$.*

(a) *Let γ° be an irreducible component of the restriction of γ to $\mathbf{G}^{\circ F}$, and let t be a semisimple element of $\mathbf{G}^{*\circ F^*}$ such that $\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (t)^\circ)$. Then $t \in (s)$.*

(b) *There exists an irreducible component of the restriction of γ to $\mathbf{G}^{\circ F}$ belonging to $\mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$.*

Proof. (a) is a reformulation of Corollary 1.2.3, and (b) is an easy consequence of (a) and of Lemma 1.2.2. ∎

1.3. *Nice Elements*

Let s be a semisimple element of $\mathbf{G}^{*\circ F^*}$. The centralizer of s in $\mathbf{G}^{*\circ}$ is connected and is a Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$. The image of $C_{G^*}(s)$ by the morphism

$$C_{\mathbf{G}^*}(s) \to \mathbf{G}^* \to A^*$$

is denoted by $A^*(s)$. Then the $\mathbf{G}^{*\circ}$-conjugacy class of s is stable under $A^*(s)$. If we denote by $\mathbf{G}^{*\circ\, A^*(s)}$ the group of fixed points of $A^*(s)$ on $\mathbf{G}^{*\circ}$, then the $\mathbf{G}^{*\circ}$-conjugacy class of s in $\mathbf{G}^{*\circ}$ meets $\mathbf{G}^{*\circ\, A^*(s)}$ in a single $\mathbf{G}^{*\circ\, A^*(s)}$-conjugacy class because $A^*(s)$ acts by permutations on the components of $\mathbf{G}^{*\circ}$. This conjugacy class is F^*-stable and $\mathbf{G}^{*\circ\, A^*(s)}$ is connected, so there exists an F^*-stable element t in the $\mathbf{G}^{*\circ}$-conjugacy class of s centralized by $A^*(s)$. Moreover, $C_{\mathbf{G}^{*\circ}}(s)$ is connected, so $t \in (s)^\circ$. It also implies that $A^*(t)$ contains $A^*(s)$. Because they are conjugate under A^*, they are equal.

Definition 1.3.1. The element s is said to be *nice* (or $\mathbf{G}^*$-*nice*) if $A^*(s)$ centralizes s.

The preceding discussion shows that there exists a nice element in every semisimple $\mathbf{G}^{*\circ\, F^*}$-conjugacy class. If s is a nice element of $\mathbf{G}^{*\circ\, F^*}$ and if $\alpha^* \in A^*$ is such that ${}^{\alpha^*}(s)^\circ = (s)^\circ$, then $\alpha^* \in A^*(s)$.

1.4. *The Group* $\mathbf{G}(s)$

Until the end of this section, we fix a nice semisimple element s in $\mathbf{G}^{*\circ\, F^*}$. Let $A(s)$ be the subgroup of A corresponding to $A^*(s)$. The group $C_{\mathbf{G}^{*\circ}}(s) = C_{\mathbf{G}^*}(s)^\circ$ if an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$. Let $\mathbf{G}^\circ(s)$ be an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^\circ$ dual to $C_{\mathbf{G}^{*\circ}}(s)$; we can assume that $A(s)$ normalizes $\mathbf{G}^\circ(s)$. We define $\mathbf{G}(s)$ to be the semidirect product

$$\mathbf{G}(s) = \mathbf{G}^\circ(s) \rtimes A(s). \tag{1.4.1}$$

Because $A(s)$ acts on $\mathbf{G}^\circ$ by permutations of the components, there exists a parabolic subgroup of $\mathbf{G}^\circ$ that has $\mathbf{G}^\circ(s)$ as a Levi subgroup and is stable under $A(s)$. Hence, $\mathbf{G}(s)$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}$. Moreover, $\mathbf{G}(s)^\circ = \mathbf{G}^\circ(s)$.

With the semisimple element s is associated a linear character $\hat{s}^\circ$ of $\mathbf{G}^\circ(s)^F$ (cf. [DM1, Proposition 13.30]). Since s is centralized by $A^*(s)$, the character $\hat{s}^\circ$ is invariant by $A(s)$, so it extends to a character $\hat{s}$ of $\mathbf{G}(s)^F$, where $\hat{s}(\alpha) = 1$ for $\alpha \in A(s)$.

1.5. *A Lemma*

Let $\gamma^\circ(s)$ be a unipotent character of $\mathbf{G}^\circ(s)^F$. By [B, Theorem 7.3.2 and Definition 7.3.3], there exists a canonical extension $\tilde{\gamma}(s)$ of $\gamma^\circ(s)$ to $\mathbf{G}^\circ(s)^F \rtimes A(s, \gamma^\circ(s))$, where $A(s, \gamma^\circ(s))$ is the stabilizer of $\gamma^\circ(s)$ in $A(s)$.

Lemma 1.5.1. *$\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ \rtimes A(s,\gamma^\circ(s))}_{\mathbf{G}^\circ(s)\rtimes A(s,\gamma^\circ(s))}(\tilde{\gamma}(s) \otimes \hat{s})$ is an irreducible character of the group $\mathbf{G}^{\circ\, F} \rtimes A(s, \gamma^\circ(s))$. Its restriction to $\mathbf{G}^{\circ\, F}$ is the irreducible character $\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ}_{\mathbf{G}^\circ(s)}(\gamma^\circ(s) \otimes \hat{s}^\circ)$ which belongs to $\mathscr{E}(\mathbf{G}^{\circ\, F}, (s)^\circ)$.*

Remark. By [B, Theorem 7.3.2], the unipotent character $\tilde{\gamma}(s)$ of $\mathbf{G}^\circ(s)^F \rtimes A(s, \gamma^\circ(s))$ is a uniform function, that is, a linear combination of generalized Deligne-Lusztig characters. Hence the class function $\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}^\circ(s) \rtimes A(s,\gamma^\circ(s))}^{\mathbf{G}^\circ \rtimes A(s,\gamma^\circ(s))}(\tilde{\gamma}(s) \otimes \hat{s})$ is independent of the choice of a parabolic subgroup of $\mathbf{G}$ having $\mathbf{G}^\circ(s) \rtimes A(s, \gamma^\circ(s))$ as Levi subgroup. That is why the Lusztig functor is denoted without reference to the parabolic subgroup (the notion of a Lusztig functor for disconnected reductive groups has been defined in [DM2], and slightly generalized for the purpose of this article in [B]).

Proof of Lemma 1.5.1. To simplify notation, we can assume that $A = A(s, \gamma^\circ(s))$. Let

$$\tilde{\gamma} = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}(s)}^{\mathbf{G}}(\tilde{\gamma}(s) \otimes \hat{s})$$

and

$$\gamma^\circ = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}^\circ(s)}^{\mathbf{G}^\circ}(\gamma^\circ(s) \otimes \hat{s}^\circ).$$

It follows from [DM2, Corollary 2.4] that the restriction of $\tilde{\gamma}$ to $\mathbf{G}^{\circ F}$ is equal to γ°. Moreover, by [LS, Theorem 3.2], γ° is irreducible and lies in $\mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$. So we need only prove that $\tilde{\gamma}$ is a character of $\mathbf{G}^F$.

Let $\mathbf{P}(s)$ be a parabolic subgroup of $\mathbf{G}(s)$ that has $\mathbf{G}(s)$ as Levi subgroup, and let $\mathbf{U}$ be its unipotent radical. We define

$$\mathbf{Y}_\mathbf{U} = \{g \in \mathbf{G} | g^{-1}F(g) \in \mathbf{U}\}$$

and

$$\mathbf{Y}_\mathbf{U}^\circ = \{g \in \mathbf{G}^\circ | g^{-1}F(g) \in \mathbf{U}\}.$$

Let $H_c^i(\mathbf{Y}_\mathbf{U})$ be the ith cohomology group with compact support with coefficients in the constant sheaf $\overline{\mathbb{Q}}_\ell$ (where $i \in \mathbb{N}$). The group $\mathbf{G}^F$ (respectively, $\mathbf{G}(s)^F$) acts on $\mathbf{Y}_\mathbf{U}$ by left (respectively, right) translation. Hence $H_c^i(\mathbf{Y}_\mathbf{U})$ inherits the structure of a $\mathbf{G}^F$-module-$\mathbf{G}(s)^F$. Let V be an irreducible $\mathbf{G}(s)^F$-module affording $\tilde{\gamma}(s)$ as character. Then the virtual $\mathbf{G}^F$-module

$$\sum_{i \in \mathbb{N}} (-1)^i H_c^i(\mathbf{Y}_\mathbf{U}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V$$

affords $\tilde{\gamma}$ as (virtual) character. We have similar results for $\mathbf{G}^{\circ F}$. We denote by V° the restriction of V to $\mathbf{G}^{\circ F}$.

By [DM1, Theorem 13.25(i)] there exists j in $\mathbb{N}$ such that

$$H_c^i(\mathbf{Y}_\mathbf{U}^\circ) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ = 0$$

if $i \neq j$ and such that

$$H_c^j(\mathbf{Y}_{\mathbf{U}}^\circ) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ$$

is irreducible (in [DM1], the statement and the proof of Theorem 13.25 are not entirely correct; a precise value for j is given, and it is not clear that this value is correct. However, the existence of j satisfying the above conditions has been established in a revised version of their book). Moreover, $(-1)^j = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}$. But by [DM2, Proof of Proposition 2.3] we have

$$H_c^i(\mathbf{Y}_{\mathbf{U}}) = \overline{\mathbb{Q}}_\ell \mathbf{G}^F \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^{\circ F}} H_c^i(\mathbf{Y}_{\mathbf{U}}^\circ)$$

as a $\mathbf{G}^F$-module. Hence we have

$$H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ = 0$$

for all $i \neq j$. But

$$\begin{aligned} H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ &= \left(H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F\right) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ \\ &= H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \left(\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ\right) \\ &= H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \operatorname{Ind}_{\mathbf{G}^\circ(s)^F}^{\mathbf{G}(s)^F} V^\circ. \end{aligned}$$

Since V is a direct summand of the $\mathbf{G}(s)^F$-module $\operatorname{Ind}_{\mathbf{G}^\circ(s)^F}^{\mathbf{G}(s)^F} V^\circ$, it follows that

$$H_c^i(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V = 0$$

if $i \neq j$ and that $\tilde{\gamma}$ is the character of the module

$$H_c^j(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V.$$

∎

1.6. *Clifford Theory*

Let $\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$. By [LS, Theorem 3.2] there exists a unique unipotent character $\gamma^\circ(s)$ of $\mathbf{G}^\circ(s)^F$ such that

$$\gamma^\circ = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}^\circ(s)}^{\mathbf{G}^\circ}(\gamma^\circ(s) \otimes \hat{s}^\circ). \tag{1.6.1}$$

Let $A(\gamma^\circ)$ be the stabilizer of γ° in A. Its dual $A^*(\gamma^\circ)$ stabilizes the $\mathbf{G}^{*\circ F^*}$-conjugacy class of s and hence is contained in $A^*(s)$. By duality $A(\gamma^\circ)$ is contained in $A(s)$. The uniqueness of $\gamma^\circ(s)$ implies that $A(\gamma^\circ)$ is the stabilizer $A(s, \gamma^\circ(s))$ of $\gamma^\circ(s)$ in $A(s)$.

We denote by $\tilde{\gamma}(s)$ the canonical extension of $\gamma^\circ(s)$ to $\mathbf{G}^\circ(s)^F \rtimes A(\gamma^\circ)$ (as defined in [B, Definition 7.3.3]). We put

$$\tilde{\gamma} = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ \rtimes A(\gamma^\circ)}_{\mathbf{G}^\circ(s)\rtimes A(\gamma^\circ)}(\tilde{\gamma}(s) \otimes \hat{s}). \tag{1.6.2}$$

Then, by Lemma 1.5.1, $\tilde{\gamma}$ is an irreducible character of $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ extending γ°.

DEFINITION 1.6.3. The irreducible character $\tilde{\gamma}$ of $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ will be called the *canonical* extension of γ°.

If ξ is an irreducible character of $A(\gamma^\circ)$, then by Clifford theory $\tilde{\gamma} \otimes \xi$ is an irreducible character of $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$, and $\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}\rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi)$ is an irreducible character of $\mathbf{G}^F$ (where ξ is lifted to $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ in the natural way). Moreover,

$$\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}}\gamma^\circ = \sum_{\xi\in \mathrm{Irr}\, A(\gamma^\circ)} \xi(1)\, \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}\rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi). \tag{1.6.4}$$

1.7. *Jordan Decomposition*

Let γ be an irreducible character in $\mathscr{E}(\mathbf{G}^F, (s))$. By Corollary 1.2.4 there exists an irreducible character $\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$ occurring in the restriction of γ to $\mathbf{G}^{\circ F}$. Let $\tilde{\gamma}$ be the canonical extension of γ° to $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ defined in Definition 1.6.3. Then by Clifford theory there exists a unique irreducible character ξ of $A(\gamma^\circ)$ such that

$$\gamma = \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}\rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi).$$

Let $\gamma^\circ(s)$ be the unipotent character of $\mathbf{G}^\circ(s)^F$ satisfying (1.6.1), and let $\tilde{\gamma}(s)$ be its canonical extension to $\mathbf{G}^\circ(s)^F \rtimes A(\gamma^\circ)$ (recall that $A(\gamma^\circ)$ is the stabilizer of $\gamma^\circ(s)$ in $A(s)$). Then

$$\gamma(s) = \mathrm{Ind}^{\mathbf{G}(s)^F}_{\mathbf{G}^\circ(s)^F\rtimes A(\gamma^\circ)}(\tilde{\gamma}(s) \otimes \xi)$$

is an irreducible character of $\mathbf{G}(s)^F$ and is unipotent by definition. It follows from [B, Propositions 6.3.2 and 6.3.3] that

$$\gamma = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}}_{\mathbf{G}(s)}(\gamma(s) \otimes \hat{s}). \tag{1.7.1}$$

Remark. The remark following Lemma 1.5.1 shows that the Lusztig functor $R^{\mathbf{G}}_{\mathbf{G}(s)}$ does not depend on the choice of a parabolic subgroup of $\mathbf{G}$ that has $\mathbf{G}(s)$ as Levi subgroup.

THEOREM 1.7.2 (Jordan Decomposition). *With the above notation the map*

$$\nabla_{\mathbf{G},s}\colon \mathscr{E}\big(\mathbf{G}^F,(s)\big) \to \mathscr{E}\big(\mathbf{G}(s)^F,1\big)$$
$$\gamma \mapsto \gamma(s)$$

is well-defined and bijective. The inverse map is given by Formula (1.7.1).

Proof. First we have to prove that $\nabla_{\mathbf{G},s}$ is well defined. There is one ambiguity in the construction of $\gamma(s)$: in the first step, we chose an irreducible character $\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)$ occurring in the restriction of γ to $\mathbf{G}^{\circ F}$. If δ° is another element of the Lusztig series $\mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)$ occurring in the restriction of γ to $\mathbf{G}^{\circ F}$, then there exists $\alpha \in A$ such that $\delta^\circ = {}^\alpha\gamma^\circ$. But both lie in $\mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)$, so we have $\alpha \in A(s)$. If we construct $\delta^\circ(s)$, $\tilde{\delta}(s)$, and $\delta(s)$ in the same way as $\gamma^\circ(s)$, $\tilde{\gamma}(s)$, and $\gamma(s)$, respectively, then $\delta^\circ(s) = {}^\alpha\gamma^\circ(s)$ (by uniqueness), so $\tilde{\delta}(s) = {}^\alpha\tilde{\gamma}(s)$, and so $\delta(s) = {}^\alpha\delta(s) = \delta(s)$ because $\alpha \in A(s)$. Thus $\nabla_{\mathbf{G},s}$ is well defined.

$\nabla_{\mathbf{G},s}$ is injective by Formula (1.7.1) and surjective by Lemma 1.5.1, which proves that Formula (1.7.1) always defines an element of $\mathscr{E}(\mathbf{G}^F,(s))$. ∎

2. UNIFORM FUNCTIONS

In [B, Formula 7.3.1 and Theorem 7.3.2] the unipotent characters of $\mathbf{G}^F$ are described as linear combinations of generalized Deligne–Lusztig characters. It is possible using Formula (1.7.1) to describe all of the irreducible characters of $\mathbf{G}^F$ as linear combinations of generalized Deligne–Lusztig characters.

2.1. *Notation*

Let s be a nice semisimple element of $\mathbf{G}^{*\circ F^*}$.

We fix an F-stable and $A(s)$-stable Borel subgroup $\mathbf{B}_1^\circ(s)$ of $\mathbf{G}^\circ(s)$ and an F-stable and $A(s)$-stable maximal torus $\mathbf{T}_1^\circ(s)$ of $\mathbf{B}_1^\circ(s)$. We denote by $W(s)$ (respectively, $W^\circ(s)$) the Weyl group of $\mathbf{G}(s)$ (respectively, $\mathbf{G}^\circ(s)$) to $\mathbf{T}_1^\circ(s)$.

For each $\alpha \in A(s)$, we define $\mathbf{T}_1^\circ(s,\alpha)$ to be the semidirect product $\mathbf{T}_1^\circ(s) \rtimes \langle \alpha \rangle$. For each $w \in W^\circ(s)^\alpha$ (that is, the subgroup of $W^\circ(s)$ consisting of elements centralized by α), we denote by $\mathbf{T}_w(s,\alpha)$ the *quasi-maximal torus* of $\mathbf{G}^\circ(s) \rtimes \langle \alpha \rangle$ associated with w as in [DM2, Proposition 1.40] (for the definition of a quasi-maximal torus, cf. [B, Definition 6.1.3]). $\mathbf{T}_w(s,\alpha)$ is defined by the following property: $(\mathbf{T}_w(s,\alpha)^\alpha)^\circ$ is an F-stable maximal torus of $\mathbf{G}^\circ(s)^\alpha$ of type w with respect to $\mathbf{T}_1^\circ(s)^\alpha$.

The group $W^\circ(s)$ is a product of symmetric groups, and $A(s)$ and F act on $W^\circ(s)$ by permutations of the components (F acts on $W^\circ(s)$ as σ). By the argument used in [B, Sect. 7.3] we can associate canonically with each irreducible character χ° of $W^\circ(s)^F$ and each α in the stabilizer $A(s,\chi^\circ)$ of χ° in $A(s)$ an irreducible character $\tilde{\boldsymbol{\chi}}_\alpha$ of $W^\circ(s)^\alpha \rtimes \langle\sigma\rangle$.

2.2. *Irreducible Characters in $\mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)$ as Uniform Functions*

Let χ° be an irreducible character of $W^\circ(s)^F$. We define

$$R^\circ_{\chi^\circ}(s) = R^{\mathbf{G}^\circ}_{\chi^\circ}(s) = \frac{\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}}{|W^\circ(s)|}\sum_{w\in W^\circ(s)}\tilde{\boldsymbol{\chi}}_1(w\sigma)R^{\mathbf{G}^\circ}_{\mathbf{T}_w(s,1)}(\hat{s}^\circ). \quad (2.2.1)$$

PROPOSITION 2.2.2 (Lusztig–Srinivasan [LS, Theorem 3.2]). (a) *For all $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$, $R^\circ_{\chi^\circ}(s)$ is an irreducible character of $\mathbf{G}^{\circ F}$ in $\mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)$.*

(b) *The map*

$$\begin{aligned}\operatorname{Irr} W^\circ(s)^F &\to \mathscr{E}(\mathbf{G}^{\circ F},(s)^\circ)\\ \chi^\circ &\mapsto R^\circ_{\chi^\circ}(s)\end{aligned}$$

is bijective.

COROLLARY 2.2.3. (a) *If $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$ and $\alpha\in A(s)$, then ${}^\alpha R^\circ_{\chi^\circ}(s) = R^\circ_{{}^\alpha\chi^\circ}(s)$.*

(b) *If $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$, then $A(R^\circ_{\chi^\circ}(s)) = A(s,\chi^\circ)$.*

2.3. *Canonical Extensions as Uniform Functions*

Let χ° be an irreducible character of $W^\circ(s)^F$. We define a function $\tilde{R}_{\chi^\circ}(s)$ on $G^{\circ F}\rtimes A(s,\chi^\circ)$ by

$$\operatorname{Res}^{\mathbf{G}^{\circ F}\rtimes A(s,\chi^\circ)}_{\mathbf{G}^{\circ F}\cdot\alpha}\tilde{R}_{\chi^\circ}(s) = \frac{\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}}{|W^\circ(s)^\alpha|}\sum_{w\in W^\circ(s)^\alpha}\tilde{\boldsymbol{\chi}}_\alpha(w\sigma)\quad \operatorname{Res}^{\mathbf{G}^{\circ F}\rtimes\langle\alpha\rangle}_{\mathbf{G}^{\circ F}\cdot\alpha}R^{\mathbf{G}^\circ\rtimes\langle\alpha\rangle}_{\mathbf{T}_w(s,\alpha)}(\hat{s}) \quad (2.3.1)$$

for all $\alpha\in A(s,\chi^\circ)$.

PROPOSITION 2.3.2. *$\tilde{R}_{\chi^\circ}(s)$ is an irreducible character of $\mathbf{G}^{\circ F}\rtimes A(s,\chi^\circ)$ and is in fact the canonical extension of $R^\circ_{\chi^\circ}(s)$ (cf. Definition 1.6.3).*

Proof. This follows immediately from Formula (1.6.2), from [B, Theorem 7.3.2], and from [DM2, Proposition 2.3]. ∎

2.4. *Parameterization of* $\mathscr{E}(\mathbf{G}^F,(s))$

We denote by $\mathscr{I}(s)$ the set of pairs (χ°, ξ) where χ° is an irreducible character of $W^\circ(s)^F$ and ξ is an irreducible character of $A(s,\chi^\circ)$. The group $A(s)$ acts by conjugation on $\mathscr{I}(s)$, and we denote by $\bar{\mathscr{I}}(s)$ the set of orbits of $A(s)$ in $\mathscr{I}(s)$. Moreover, if $(\chi^\circ, \xi) \in \mathscr{I}(s)$, we denote by $\chi^\circ * \xi$ its orbit under $A(s)$.

For all $\chi^\circ * \xi \in \bar{\mathscr{I}}(s)$, we define

$$R^{\mathbf{G}}_{\chi^\circ * \xi}(s) = R_{\chi^\circ * \xi}(s) = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F} \rtimes A(s,\chi^\circ)}\Big(\tilde{R}_{\chi^\circ}(s) \otimes \xi\Big). \quad (2.4.1)$$

It follows from Corollary 2.2.2(a) that $R_{\chi^\circ * \xi}(s)$ only depends on the orbit of (χ°, ξ) under $A(s)$. Moreover, it follows from Clifford theory and from Corollary 2.2.2(b) that we have

LEMMA 2.4.2. *The map*

$$\begin{aligned} \bar{\mathscr{I}}(s) &\to \mathscr{E}\big(\mathbf{G}^F,(s)\big) \\ \chi^\circ * \xi &\mapsto R_{\chi^\circ * \xi}(s) \end{aligned}$$

is bijective.

By [B, Proposition 2.3.1], χ° has a canonical extension $\tilde{\chi}$ to the semidirect product $W^\circ(s) \rtimes A(s,\chi^\circ)$. By Clifford theory again we have

LEMMA 2.4.3. *The map*

$$\begin{aligned} \bar{\mathscr{I}}(s) &\to \operatorname{Irr} W(s)^F \\ \chi^\circ * \xi &\mapsto \operatorname{Ind}^{W(s)^F}_{W^\circ(s)^F \rtimes A(s,\chi^\circ)}(\tilde{\chi} \otimes \xi) \end{aligned}$$

is bijective.

Lemmas 2.4.2 and 2.4.3 imply the following:

THEOREM 2.4.4. *There is a well-defined bijection*

$$\begin{aligned} \operatorname{Irr} W(s)^F &\to \mathscr{E}\big(\mathbf{G}^F,(s)\big) \\ \chi &\mapsto \mathbf{R}_\chi(s). \end{aligned}$$

Remark. If necessary, we will write $\mathbf{R}^{\mathbf{G}}_\chi(s)$ for the irreducible character $\mathbf{R}_\chi(s)$ of $\mathbf{G}^F$. By applying Theorem 2.4.4 in the case where $\mathbf{G} = \mathbf{G}(s)$ and $s = 1$, we obtain a bijection,

$$\begin{aligned} \operatorname{Irr} W(s)^F &\to \mathscr{E}\big(\mathbf{G}(s)^F, 1\big) \\ \chi &\mapsto \mathbf{R}^{\mathbf{G}(s)}_\chi(1), \end{aligned}$$

and it is easy to check that the following diagram is commutative:

$$\begin{array}{ccc} & \operatorname{Irr} W(s)^F & \\ \swarrow\sim & & \sim\searrow \\ \mathscr{E}(\mathbf{G}^F,(s)) & \xrightarrow[\nabla_{\mathbf{G},s}]{} & \mathscr{E}(\mathbf{G}(s)^F,1). \end{array} \tag{2.4.5}$$

2.5. *Induction from a Particular Subgroup of* **G**

Let $\mathbf{G}'$ be a subgroup of $\mathbf{G}$ containing $\mathbf{G}^\circ$. It is F-stable because F acts trivially on A. There exists a subgroup A' of A such that

$$\mathbf{G}' = \mathbf{G}^\circ \rtimes A'.$$

If we construct $\mathbf{G}'^*$ in the way we construct $\mathbf{G}$, then $\mathbf{G}'^*$ may be identified with a subgroup of $\mathbf{G}^*$. We can also construct $\mathbf{G}'(s)$ so that it is contained in $\mathbf{G}(s)$, and we denote by $W'(s)$ the Weyl group of $\mathbf{G}'(s)$ relative to $\mathbf{T}_1^\circ(s)$ so that $W'(s)$ is a subgroup of $W(s)$.

PROPOSITION 2.5.1. *Let χ' be an irreducible character of $W'(s)^F$. Suppose*

$$\operatorname{Ind}_{W'(s)^F}^{W(s)^F} \chi' = \sum_{\chi \in \operatorname{Irr} W(s)^F} n_\chi \chi.$$

Then

$$\operatorname{Ind}_{\mathbf{G}'^F}^{\mathbf{G}^F} \mathbf{R}_{\chi'}^{\mathbf{G}'}(s) = \sum_{\chi \in \operatorname{Irr} W(s)^F} n_\chi \mathbf{R}_\chi^{\mathbf{G}}(s).$$

3. LUSZTIG FUNCTORS

HYPOTHESIS. *Throughout this section, and only in this section, A will be assumed abelian.*

3.1. *Notation*

Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let $A_{\mathbf{L}}$ be the image of $\mathbf{L}$ through the composite morphism

$$\mathbf{L} \to \mathbf{G} \to \mathbf{G}/\mathbf{G}^\circ \to A$$

($A_{\mathbf{L}}$ is a subgroup of A). Because A is abelian, we can use the same argument as in [B, 7.6] to assume that $\mathbf{L}$ contains $A_{\mathbf{L}}$. Let $A_{\mathbf{L}}^*$ be the image of $A_{\mathbf{L}}$ under the anti-isomorphism $A \to A^*$.

Let $\mathbf{L}^{\circ *}$ be an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$ that is a dual of $\mathbf{L}^\circ$. We can choose $\mathbf{L}^{\circ *}$ to be $A^*_{\mathbf{L}}$-stable, and we define

$$\mathbf{L}^* = \mathbf{L}^{\circ *} \rtimes A^*_{\mathbf{L}}.$$

then $\mathbf{L}^*$ is an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ and $\mathbf{L}^{*\circ} = \mathbf{L}^{\circ *}$.

3.2. *Jordan Decomposition and Lusztig Functors*

Let s be a semisimple element in $\mathbf{L}^{*\circ F^*}$. We may assume that s is nice in $\mathbf{G}^*$. Then the subgroup $\mathbf{L}(s)$ of $\mathbf{L}$ following the construction of Section 1.4 can be chosen as a subgroup of $\mathbf{G}(s)$. The linear character of $\mathbf{L}(s)^F$ associated with s as defined in Section 1.4 is then the restriction of $\hat{s}$ to $\mathbf{L}(s)^F$. It results from this remark and from the transitivity of Lusztig induction functors (cf. [B, Proposition 6.3.3]) that the following diagram is commutative:

$$\begin{array}{ccc}
\mathscr{E}\big(\mathbf{L}^F, (s)_{\mathbf{L}^{*F^*}}\big) & \xrightarrow{\nabla_{\mathbf{L},s}} & \mathscr{E}\big(\mathbf{L}(s)^F, 1\big) \\
\Big\downarrow {\scriptstyle \varepsilon_{\mathbf{L}^\circ}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{L}}^{\mathbf{G}}} & & \Big\downarrow {\scriptstyle \varepsilon_{\mathbf{L}^\circ(s)}\varepsilon_{\mathbf{G}^\circ(s)} R_{\mathbf{L}(s)}^{\mathbf{G}(s)}} \\
\mathscr{E}\big(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}}\big) & \xrightarrow[\nabla_{\mathbf{G},s}]{} & \mathscr{E}\big(\mathbf{G}(s)^F, 1\big)
\end{array} \tag{3.2.1}$$

The description of the functor $R_{\mathbf{L}(s)}^{\mathbf{G}(s)}$ in [B, Theorem 7.6.1] thus provides a description of the functor $R_{\mathbf{L}}^{\mathbf{G}}$ via the commutative diagram (3.2.1).

4. SHINTANI DESCENT IN THE GENERAL LINEAR GROUP

In this section, we explain the link between the theory of irreducible characters of $\mathbf{G}^F$ and the theory of Shintani descent for the general linear group. For this purpose, we need to consider a particular case:

HYPOTHESIS AND NOTATIONS. *Throughout this section, we assume that* $r = 1$. *We will denote* $d = d_1$ *and* $n = n_1$ *for simplicity. We also assume that* $\sigma = (1, \ldots, d)$ *and that* A *is generated by* σ.

4.1. *The Group* $\mathbf{G}^F$

We denote by $\mathbf{G}_1$ the general linear group $\mathbf{GL}_n$, and we endow it with the split Frobenius endomorphism:

$$\begin{aligned} F_0\colon \mathbf{G}_1 &\to \mathbf{G}_1 \\ g &\mapsto g^{(q)}. \end{aligned}$$

We denote by ϕ_0 the automorphism of $\mathbf{G}_1^{F_0^d}$ induced by F_0. Then the map

$$\begin{aligned}\theta\colon \mathbf{G}_1^{F_0^d} &\to \mathbf{G}^{\circ F}\\ g &\mapsto \left(g, F_0(g), \dots, F_0^{d-1}(g)\right)\end{aligned}$$

is an isomorphism of groups and the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{G}_1^{F_0^d} & \xrightarrow{\ \theta\ } & \mathbf{G}^{\circ F} \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \sigma^{-1}} \\ \mathbf{G}_1^{F_0^d} & \xrightarrow[\ \theta\ ]{} & \mathbf{G}^{\circ F}. \end{array}$$

This implies that θ can be extended to an isomorphism denoted by

$$\begin{aligned}\tilde{\theta}\colon \mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle &\to \mathbf{G}^F\\ g\phi_0^k &\mapsto \theta(g)\sigma^{-k}\end{aligned}$$

for all $g \in \mathbf{G}_1^{F_0^d}$ and $k \in \mathbb{Z}$.

4.2. *Shintani Descent*

Let $g \in \mathbf{G}_1^{F_0}$. By Lang's theorem, there exists $x \in \mathbf{G}_1$ such that $g = x^{-1}F_0^d(x)$. Then $g' = F_0(x)x^{-1}$ belongs to $\mathbf{G}_1^{F_0^d}$, and the map that sends the conjugacy class of g in $\mathbf{G}_1^{F_0}$ to the ϕ_0-conjugacy class of g' in $\mathbf{G}_1^{F_0^d}$ is well-defined and is bijective. We denote it by

$$N_{F_0/F_0^d}\colon \mathrm{Cl}\left(\mathbf{G}_1^{F_0}\right) \to H^1\left(\phi_0, \mathbf{G}_1^{F_0^d}\right),$$

where $H^1(\phi_0, \mathbf{G}_1^{F_0^d})$ denotes the set of ϕ_0-conjugacy classes of $\mathbf{G}_1^{F_0^d}$ and $\mathrm{Cl}(\mathbf{G}_1^{F_0})$ denotes the set of conjugacy classes of $\mathbf{G}_1^{F_0}$. If we denote by $\mathscr{C}(\mathbf{G}_1^{F_0^d} \cdot \phi_0)$ (respectively, $\mathscr{C}(\mathbf{G}_1^{F_0})$) the space of class functions on $\mathbf{G}_1^{F_0^d} \cdot \phi_0$ obtained by restrictions from class functions on the group $\mathbf{G}_1^{F_0^d}\langle \phi_0 \rangle$ (respectively, $\mathbf{G}_1^{F_0}$), then $N_{F_0^d/F_0}$ induces an isomorphism

$$\mathrm{Sh}_{F_0^d/F_0}\colon \mathscr{C}\left(\mathbf{G}_1^{F_0^d} \cdot \phi_0\right) \to \mathscr{C}\left(\mathbf{G}_1^{F_0}\right),$$

called the *Shintani descent* from F_0^d to F_0.

We recall the following theorem:

THEOREM 4.2.1 (Shintani). *Let γ_1 be an irreducible character of $\mathbf{G}_1^{F_0^d}$ stable under ϕ_0. Then there exists an extension $\tilde{\gamma}_1$ of γ_1 to $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle$ such that $\mathrm{Sh}_{F_0^d/F_0}\tilde{\gamma}_1$ is, up to a sign, an irreducible character of $\mathbf{G}_1^{F_0}$.*

4.3. *Shintani Descent and Characters of* $\mathbf{G}^F$

We denote by θ^* and $\tilde{\theta}^*$ the isomorphisms of $\overline{\mathbb{Q}}_\ell$-vector spaces:

$$\theta^* : \mathscr{C}(\mathbf{G}^{\circ F}) \to \mathscr{C}\left(\mathbf{G}_1^{F_0^d}\right)$$

and

$$\tilde{\theta}^* : \mathscr{C}(\mathbf{G}^F) \to \mathscr{C}\left(\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle\right),$$

induced by θ and $\tilde{\theta}$, respectively.

Let γ° be an irreducible character of $\mathbf{G}^{\circ F}$, and let $\gamma_1 = \theta^*(\gamma^\circ)$. Then γ_1 is ϕ_0-stable if and only if γ° is σ-stable.

HYPOTHESIS. *From now on, we assume that* γ_1 *is* ϕ_0*-stable.*

Let s be a nice semisimple element of $\mathbf{G}^{*\circ F^*}$ such that $\gamma^\circ \in \mathscr{E}(\mathbf{G}^{\circ F}, (s)^\circ)$. Then $A(s) = A$ because γ_1 is ϕ_0-stable. Let χ° be the irreducible character of $W^\circ(s)$ (stable under F) such that $\gamma^\circ = R^\circ_{\chi^\circ}(s)$. Then $A(s, \chi^\circ) = A$.

THEOREM 4.3.1. *With the above notations, we have*

(a) *There exists a unique extension* $\tilde{\gamma}_1$ *of* γ_1 *to* $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle$ *such that* $\mathrm{Sh}_{F_0^d/F_0}\tilde{\gamma}_1$ *is an irreducible character of* $\mathbf{G}_1^{F_0}$*. We call it the Shintani extension of* γ_1.

(b) *We have* $\tilde{\gamma}_1 = \tilde{\theta}^*(\tilde{R}_{\chi^\circ}(s))$.

(c) *Let e be a divisor of d, and let* $\tilde{\gamma}_1^{(e)}$ *be the Shintani extension of* γ_1 *to* $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0^e \rangle$*. Then* $\tilde{\gamma}_1^{(e)}$ *is the restriction of* $\tilde{\gamma}_1$.

Remark. The result stated in (a) of Theorem 4.3.1 is slightly stronger than Shintani's. It was already known for characters of the principal series [DM3].

Proof. By Theorem 4.2.1, (a), (b), and (c) are immediate consequences of the following:

LEMMA 4.3.2. $\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e)$ *is a positive integer for all* $e \in \mathbb{Z}$.

Proof of Lemma 4.3.2. Let $e \in \mathbb{Z}$. We first prove that

$$\varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}} = \varepsilon_{\mathbf{G}^\circ(s)} \quad \text{and} \quad \varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}} = \varepsilon_{\mathbf{G}^\circ}. \tag{$\bigstar$}$$

Because $\mathbf{G}^\circ(s)$ is a direct product of groups of the same type as $\mathbf{G}^\circ$, it is sufficient to prove the result for $\mathbf{G}^\circ$. But $(\mathbf{T}_0^\circ)^\sigma$ is a maximal split subtorus of $\mathbf{G}^\circ$, so it is a maximal split subtorus of $(\mathbf{G}^\circ)^{\sigma^e}$. That proves ($\bigstar$).

Let $\tilde{\boldsymbol{\chi}}_e$ be the irreducible character of $W^\circ(s)^{\sigma^e} \rtimes \langle \sigma \rangle$ associated with χ° as in Section 2.1 (it was denoted $\tilde{\boldsymbol{\chi}}_{\sigma^e}$, but we just want to have simpler notations).

Then, by formulas (2.3.1) and (★), we have

$$\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e) = \frac{\varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}} \varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}}}{\left|W^\circ(s)^{\sigma^e}\right|} \sum_{w \in W^\circ(s)^{\sigma^e}} \tilde{\chi}_e(w\sigma) R^{\mathbf{G}^\circ \rtimes \langle \sigma^e \rangle}_{\mathbf{T}_w(s,\sigma^e)}(\hat{s})(\sigma^e).$$

Using [DM2, Theorem 4.13], we get

$$\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e) = \frac{\varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}} \varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}}}{\left|W^\circ(s)^{\sigma^e}\right|} \sum_{w \in W^\circ(s)^{\sigma^e}} \tilde{\chi}_e(w\sigma) \dim R^{(\mathbf{G}^\circ)^{\sigma^e}}_{(\mathbf{T}_w(s,\sigma^e)^\circ)^{\sigma^e}}. \quad (1)$$

But this last formula gives the degree of an irreducible character of $((\mathbf{G}^\circ)^{\sigma^e})^F$ (cf. [LS, Theorem 3.2]). ∎

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