On a Theorem of Shintani

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Let $\chi$ be an irreducible character of $G_\ell = \text{GL}_n(F_\ell)$ invariant under the automorphism $\phi$ of $G_\ell$ induced by the field automorphism $F_\ell \to F_{\ell^d}$, $x \to x^d$, and let $e$ be a divisor of $d$. By a theorem of Shintani, there exists an extension $\tilde{\chi}_e$ of $\chi$ to $G_{\ell^d} \rtimes \langle \phi^e \rangle$ whose Shintani descent to $G_\ell$ is, up to a sign $\varepsilon$, an irreducible character of $G_\ell$. It is shown in this paper that $\tilde{\chi}_e$ may always be chosen such that $\varepsilon = 1$. With this particular choice, $\tilde{\chi}_e$ is the restriction of $\tilde{\chi}_e$. Our methods rely on the work of Digne and Michel on Deligne–Lusztig theory for nonconnected reductive groups.

Let $G^o = \text{GL}_n(F)^d$, where $F$ is an algebraic closure of a finite field and where $n$ and $d$ are natural numbers. The symmetric group $\Xi_d$ acts on $G^o$ by permutations of the components of $G^o$. We denote by $G$ the semidirect product $G = G^o \rtimes \Xi_d$. It is a nonconnected reductive group, with neutral component $G^o$. We denote by $F_\ell: G \to G$ the natural split Frobenius endomorphism on $G$ (acting trivially on $\Xi_d$), and we choose an element $\sigma \in \Xi_d$. Let $F: G \to G$ denote the Frobenius endomorphism defined by $F(g) = \sigma F_\ell(g)$.

In this paper we discuss the irreducible characters of $G^F$ (the unipotent characters of $G^F$ were described in [B]). We first prove that there exists a Jordan decomposition of characters (this result is well-known for $G^o$); moreover, this decomposition commutes with Lusztig generalized induction (cf. (3.2.1)). We also prove that all the irreducible characters of $G^F$ are linear combinations of generalized Deligne–Lusztig characters (this gener-

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alizes the well-known result of G. Lusztig and B. Srinivasan [LS, Theorem 3.2] about irreducible characters of the general linear group over a finite field).

As an application of these results, we obtain new results about Shintani descent in the case of the general linear group. In [S], Shintani proved that any irreducible characters of the finite group $G_d = \text{GL}_d(F_q)$ stable under the automorphism $\phi$ induced by the field automorphism $F_q \to F_{q^d}, x \to x^q$ can be extended to $G_d \langle \phi \rangle$ in such a way that its Shintani descent is, up to sign, an irreducible character of $G_1 = \text{GL}_1(F_q)$. In Theorem 4.3.1 we prove that this sign can always be chosen to be equal to 1 and get precise formulas for the corresponding extension. As a consequence, we obtain that the Shintani descent of this particular extension to $G_e$ is an irreducible character of $G_e$ (where $e$ divides $d$).

0. NOTATION

0.1. General Notation

Let $F$ be an algebraic closure of a finite field. We denote by $p$ its characteristic. We also fix a power $q$ of $p$, and we denote by $F_q$ the subfield of $F$ with $q$ elements. All algebraic varieties and all algebraic groups will be considered over $F$. If $H$ is an algebraic group (over $F$), we will denote by $H^\circ$ its connected component containing 1. If $H$ is endowed with an $F_q$-structure, we also define

$$e_H = (-1)^{e_H} \text{rank}(H).$$

Let $\ell$ be a prime number different from $p$. We denote by $\overline{Q}_\ell$ an algebraic closure of the $\ell$-adic field $Q_\ell$. If $G$ is a finite group, all representations and all characters of $G$ will be considered over $\overline{Q}_\ell$. For instance, a $G$-module is a $\overline{Q}_\ell G$-module of finite dimension. We will denote by $\text{Irr}_G$ the set of irreducible characters of $G$.

If $n$ is a positive integer, we denote by $\text{GL}_n$ the group of invertible matrices with coefficients in $F$, and if $g \in \text{GL}_n$, we will denote by $g^{(q)}$ the matrix obtained from $g$ by raising all coefficients to the $q$th power. We will denote by $T_n$ the split maximal torus of $\text{GL}_n$ consisting of diagonal matrices and by $B_n$ the rational Borel subgroup of $\text{GL}_n$ consisting of upper triangular matrices.

0.2. The Problem

Let $r$ be a positive integer and let $d_1, \ldots, d_r$ and $n_1, \ldots, n_r$ also be positive integers. Throughout this paper $G^r$ will denote the following
connected reductive group:

\[ G^o = \prod_{i=1}^{r} (GL_{n_i} \times \cdots \times GL_{n_i}) \]

We endow \( G^o \) with the split Frobenius endomorphism

\[ F_0: \quad (g_{i1}, \ldots, g_{id_i})_{1 \leq i \leq r} \mapsto (g_{i1}^{(q)}, \ldots, g_{id_i}^{(q)})_{1 \leq i \leq r}. \]

We will denote by \( T_0^o \) and \( B_0^o \) the maximal torus and the Borel subgroup of \( G^o \), defined, respectively, by

\[ T_0^o = \prod_{i=1}^{r} (T_{n_i} \times \cdots \times T_{n_i}) \]

and

\[ B_0^o = \prod_{i=1}^{r} (B_{n_i} \times \cdots \times B_{n_i}). \]

The group \( \Xi = \Xi_{d_1} \times \cdots \times \Xi_{d_r} \) acts on \( G^o \) in the natural way. More explicitly, if \( \sigma = (\alpha_1, \ldots, \alpha_r) \in \Xi \) and if \((g_{i1}, \ldots, g_{id_i})_{1 \leq i \leq r} \in G^o\), we put

\[ \sigma (g_{i1}, \ldots, g_{id_i})_{1 \leq i \leq r} = (g_{i1\sigma_i^{-1}(1)}, \ldots, g_{id_i\sigma_i^{-1}(d_i)})_{1 \leq i \leq r}. \]

The elements of \( \Xi \) induce automorphisms of \( G^o \), which stabilize \( T_0^o \) and \( B_0^o \), so they are quasi-semisimple (cf. [DM 2, Definition 1.1(ii)]). In fact, they are all quasi-central (cf. [DM 2, Definition-Theorem 1.15] and [B, Lemma 7.1.1]).

We extend the Frobenius endomorphism \( F_0 \) to \( G^o \rtimes \Xi \) by letting \( F_0 \) act trivially on \( \Xi \). We fix once and for all an element \( \sigma \in \Xi \), and we denote by \( F \) the Frobenius endomorphism on \( G^o \rtimes \Xi \) given by

\[ F(g) = \sigma F_0(g) \sigma^{-1} = \sigma F_0(g) \]

for all \( g \in G^o \rtimes \Xi \).

We will denote by \( G \) an \( F \)-stable subgroup of \( G^o \rtimes \Xi \) containing \( G^o \). Hence \( G \) is a reductive group with neutral component \( G^o \). Moreover, there exists an \( F \)-stable (that is, a \( \sigma \)-stable) subgroup \( A \) of \( \Xi \) such that

\[ G = G^o \rtimes A. \]

Thus we have \( G^F = G^o^F \rtimes A^F = G^o^F \rtimes A^\sigma \).
Problem. Parametrize the irreducible characters of $G^F$.

For this purpose we can make the following hypothesis without loss of generality:

Hypothesis. The Frobenius endomorphism $F$ acts trivially on $G/G^\circ$, that is, $A$ is contained in the centralizer of $\sigma$ in $\Xi$. Consequently,

$$G^F = G^\circ F \rtimes A.$$

Remark 0. Let $N = d_1 n_1 + \cdots + d_r n_r$. Then $G^\circ$ is isomorphic to a rational Levi subgroup $H^\circ$ of a parabolic subgroup of $GL_N$ (endowed with the split Frobenius endomorphism $g \mapsto g^{(q)}$), and $G$ is isomorphic to a rational subgroup $H$ of the normalizer of $H^\circ$ in $GL_N$, containing $H^\circ$ and such that all elements of $H/H^\circ$ are rational. Conversely, if $H$ is such a rational subgroup of $GL_N$, then there exist positive integers $r, d_1, \ldots, d_r, n_1, \ldots, n_r$; an element $\sigma$ of $\Xi_{d_1} \times \cdots \times \Xi_{d_r}$; and a subgroup $A$ of $\Xi^\sigma$ such that $H$ is isomorphic to the group $G$ constructed as above. In particular, if $L$ is an $F$-stable Levi subgroup of a parabolic subgroup of $G$ (cf. [B, Definitions 6.1.1 and 6.1.2] for the definitions of parabolic subgroups and Levi subgroups of nonconnected reductive groups), then all of the results proved for $G$ hold in $L$.

1. JORDAN DECOMPOSITION OF CHARACTERS OF $G^F$

1.1. Dual of $G$

Let $(G^{\circ*}, T_0^{\circ*}, F^*)$ be a dual triple of $(G^\circ, T_0^\circ, F)$. The elements $\alpha$ of $\Xi$ induce automorphisms $\alpha^*$ of $G^{\circ*}$. The group $\Xi^*$ of automorphisms of $G^{\circ*}$ induced by $\Xi$ is isomorphic to the opposite group of $\Xi$. We extend the action of $F^*$ to $G^{\circ*} \rtimes \Xi^*$ so that it acts on $\Xi^*$ by conjugation by $\sigma^{*-1}$. We denote by $G^*$ the semidirect product $G^{\circ*} \rtimes A^*$, where $A^*$ is the image of $A$ under the preceding anti-isomorphism. In particular, $G^{\circ o} = G^{o*}!$

1.2. Lusztig Series of $G^F$

Let $s$ be a semisimple element of $G^{o o F^*}$. We denote by $(s)$ (or $(s)_{G^F}$, if confusion is possible) the $G^{o F^*}$-conjugacy class of $s$ and by $(s)^F$ (or $(s)^F_{G^F}$) the $G^{o o F^*}$-conjugacy class of $s$.

Definition 1.2.1. The Lusztig series $\mathscr{S}(G^F, (s))$ of $G^F$ associated with $s$ (or $(s)$) is the set of irreducible characters of $G^F$ occurring in some $\text{Ind}^G_{G^F} \gamma^o$, where $\gamma^o$ is an element of a usual Lusztig series $\mathscr{S}(G^{o F^*}, (s)^F)$ with $s^F \in (s)$. 

The characters of the Lusztig series $\mathcal{E}(G^F, 1)$ are called unipotent; this
definition agrees with definitions given in [DM 2, Section 5] or [B, Definition 6.4.1] (cf. [B, Lemma 6.4.2]).

The following lemma follows immediately from the definitions:

**Lemma 1.2.2.** Let $s$ be a semisimple element of $G^{* o F^*}$, $\gamma^o$ be an element
of $\mathcal{E}(G^F, (s)^\circ)$, and $\alpha \in A$. Then $\alpha^o \gamma^o \in \mathcal{E}(G^F, (\alpha^{* -1} s)^\circ)$.

**Corollary 1.2.3.** \[ \text{Irr} G^F = \bigcup_{(s)} \mathcal{E}(G^F, (s)), \]
where $(s)$ runs over the set of $G^{* o F^*}$-classes of semisimple elements of $G^{* o F^*}$. Moreover, this union is disjoint.

**Proof.** The equality follows easily from the corresponding fact for $G^{* o F}$. Let us prove now that the union is disjoint. Let $s$ and $t$ be two semisimple elements of $G^{* o F^*}$ and let $\gamma$ be an irreducible character of $G^F$ belonging to both $\mathcal{E}(G^F, (s))$ and $\mathcal{E}(G^F, (t))$. Then by definition there exist irreducible characters $\gamma_1^o$ and $\gamma_2^o$ of $G^{* o F}$ occurring in the restriction of $\gamma$ to $G^{* o F}$ such that $\gamma_1^o \in \mathcal{E}(G^{* o F}, (s)^\circ)$ and $\gamma_2^o \in \mathcal{E}(G^{* o F}, (t)^\circ)$, where $s' \in (s)$ and $t' \in (t)$.

But by Clifford theory there exists $\alpha \in A$ such that $\gamma_2^o = \alpha^o \gamma_1^o$. It follows from Lemma 1.2.2 and from the fact that Corollary 1.2.3 holds in $G^F$ that $t' \in (s)^\circ$, so $t \in (s)$.

**Corollary 1.2.4.** Let $s$ be a semisimple element in $G^{* o F^*}$, and let $\gamma \in \mathcal{E}(G^F, (s))$.

(a) Let $\gamma^o$ be an irreducible component of the restriction of $\gamma$ to $G^{* o F}$, and let $t$ be a semisimple element of $G^{* o F^*}$ such that $\gamma^o \in \mathcal{E}(G^{* o F}, (t)^\circ)$.

(b) There exists an irreducible component of the restriction of $\gamma$ to $G^{* o F}$ belonging to $\mathcal{E}(G^{* o F}, (s)^\circ)$.

**Proof.** (a) is a reformulation of Corollary 1.2.3, and (b) is an easy consequence of (a) and of Lemma 1.2.2.

1.3. **Nice Elements**

Let $s$ be a semisimple element of $G^{* o F^*}$. The centralizer of $s$ in $G^{* o}$ is connected and is a Levi subgroup of a parabolic subgroup of $G^{* o}$. The image of $C_{G^*}(s)$ by the morphism

\[ C_{G^*}(s) \rightarrow G^* \rightarrow A^n \]
Let \( G \) be a semisimple group. By Definition 7.3.3, there exists a canonical extension \( g_s \) where \( s \in G \). Moreover, a parabolic subgroup of \( G \) centralizes \( A \). If we denote by \( G^\circ \) a \( \mathbb{F} \)-conjugacy class because \( s \in G \) is such that \( g_s = (s)^\circ \), then the \( \mathbb{F} \)-conjugacy class of \( s \) in \( G^\circ \) is connected, so there exists an \( \mathbb{F} \)-stable element \( t \) in the \( G^\circ \)-conjugacy class of \( s \) centralized by \( A^\circ \). Moreover, \( C_{G^\circ}(s) \) is connected.

**Definition 1.3.1.** The element \( s \) is said to be nice (or \( G^\circ \)-nice) if \( A^\circ(s) \) centralizes \( s \).

The preceding discussion shows that there exists a nice element in every semisimple \( G^\circ \)-conjugacy class. If \( s \) is a nice element of \( G^\circ \) and if \( \alpha^\circ \in A^\circ \) is such that \( \alpha^\circ(s)^\circ = (s)^\circ \), then \( \alpha^\circ \in A^\circ(s) \).

### 1.4. The Group \( G(s) \)

Until the end of this section, we fix a nice semisimple element \( s \) in \( G^\circ \). Let \( A(s) \) be the subgroup of \( A \) corresponding to \( A^\circ(s) \). The group \( C_{G^\circ}(s) = C_{G^\circ}(s)^\circ \) if an \( \mathbb{F} \)-stable Levi subgroup of a parabolic subgroup of \( G^\circ \). Let \( G^\circ(s) \) be an \( \mathbb{F} \)-stable Levi subgroup of a parabolic subgroup of \( G^\circ \) dual to \( C_{G^\circ}(s) \); we can assume that \( A(s) \) normalizes \( G^\circ(s) \). We define \( G(s) \) to be the semidirect product

\[
G(s) = G^\circ(s) \rtimes A(s). 
\]

(1.4.1)

Because \( A(s) \) acts on \( G^\circ \) by permutations of the components, there exists a parabolic subgroup of \( G^\circ \) that has \( G^\circ(s) \) as a Levi subgroup and is stable under \( A(s) \). Hence, \( G(s) \) is a Levi subgroup of a parabolic subgroup of \( G \). Moreover, \( G(s)^\circ = G^\circ(s) \).

With the semisimple element \( s \) is associated a linear character \( \hat{s}^\circ \) of \( G^\circ(s)^\circ \) (cf. [DM 1, Proposition 13.30]). Since \( s \) is centralized by \( A^\circ(s) \), the character \( \hat{s}^\circ \) is invariant by \( A(s) \), so it extends to a character \( \hat{s} \) of \( G(s)^\circ \), where \( \hat{s}(\alpha) = 1 \) for \( \alpha \in A(s) \).

### 1.5. A Lemma

Let \( \gamma^\circ(s) \) be a unipotent character of \( G^\circ(s)^\circ \). By [B, Theorem 7.3.2 and Definition 7.3.3], there exists a canonical extension \( \hat{\gamma}(s) \) of \( \gamma^\circ(s) \) to \( G^\circ(s)^\circ \rtimes A(s, \gamma^\circ(s)) \), where \( A(s, \gamma^\circ(s)) \) is the stabilizer of \( \gamma^\circ(s) \) in \( A(s) \).

**Lemma 1.5.1.** \( e_{G^\circ(s)} e_{G^\circ(s)} R_{G^\circ(s)}(A(s, \gamma^\circ(s)), \hat{\gamma}(s) \otimes \hat{s}^\circ) \) is an irreducible character of the group \( G^\circ(s)^\circ \rtimes A(s, \gamma^\circ(s)) \). Its restriction to \( G^\circ(s) \) is the irreducible character \( e_{G^\circ(s)} e_{G^\circ(s)} R_{G^\circ(s)}(\gamma^\circ(s) \otimes \hat{s}^\circ) \) which belongs to \( \pi(G^\circ(s)^\circ) \).
Remark. By [B, Theorem 7.3.2], the unipotent character \( \tilde{\gamma}(s) \) of \( G^u(s)^F \times A(s, \gamma^o(s)) \) is a uniform function, that is, a linear combination of generalized Deligne-Lusztig characters. Hence the class function 
\[ e_{G^u(s)} e_{G^o} R_{G^u(s)}^G (\tilde{\gamma}(s) \otimes \hat{s}) \] 
is independent of the choice of a parabolic subgroup of \( G \) having \( G^u(s) \times A(s, \gamma^o(s)) \) as Levi subgroup. That is why the Lusztig functor is denoted without reference to the parabolic subgroup. (The notion of a Lusztig functor for disconnected reductive groups has been defined in [DM2], and slightly generalized for the purpose of this article in [B]).

Proof of Lemma 1.5.1. To simplify notation, we can assume that \( A = A(s, \gamma^o(s)) \). Let
\[ \tilde{\gamma} = e_{G^u(s)} e_{G^o} R_{G^u(s)}^G (\tilde{\gamma}(s) \otimes \hat{s}) \]
and
\[ \gamma^o = e_{G^u(s)} e_{G^o} R_{G^u(s)}^G (\gamma^o(s) \otimes \hat{s}) . \]

It follows from [DM2, Corollary 2.4] that the restriction of \( \tilde{\gamma} \) to \( G^oF \) is equal to \( \gamma^o \). Moreover, by [LS, Theorem 3.2], \( \gamma^o \) is irreducible and lies in \( \mathcal{B}(G^oF, (s)^F) \). So we need only prove that \( \tilde{\gamma} \) is a character of \( G^F \).

Let \( P(s) \) be a parabolic subgroup of \( G(s) \) that has \( G(s) \) as Levi subgroup, and let \( U \) be its unipotent radical. We define
\[ Y_u = \{ g \in G | g^{-1}F(g) \in U \} \]
and
\[ Y_u^o = \{ g \in G^o | g^{-1}F(g) \in U \} . \]

Let \( H^i_c(Y_u) \) be the \( i \)th cohomology group with compact support with coefficients in the constant sheaf \( \mathbb{Q}_\nu \) (where \( i \in \mathbb{N} \)). The group \( G^F \) (respectively, \( G(s)^F \)) acts on \( Y_u \) by left (respectively, right) translation. Hence \( H^i_c(Y_u) \) inherits the structure of a \( G^F \)-module-\( G(s)^F \). Let \( V \) be an irreducible \( G(s)^F \)-module affording \( \tilde{\gamma}(s) \) as character. Then the virtual \( G^F \)-module
\[ \sum_{i \in \mathbb{N}} (-1)^i H^i_c(Y_u) \otimes_{\mathfrak{g}, G(s)^F} V \]
affords \( \tilde{\gamma} \) as (virtual) character. We have similar results for \( G^oF \). We denote by \( V^o \) the restriction of \( V \) to \( G^oF \).

By [DM1, Theorem 13.25(i)] there exists \( j \) in \( \mathbb{N} \) such that
\[ H^j_c(Y_u^o) \otimes_{\mathfrak{g}, G(s)^F} V^o = 0 \]
if \( i \neq j \) and such that

\[
H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} V^o
\]

is irreducible (in [DM1], the statement and the proof of Theorem 13.25 are not entirely correct; a precise value for \( j \) is given, and it is not clear that this value is correct. However, the existence of \( j \) satisfying the above conditions has been established in a revised version of their book). Moreover, \((-1)^j = e_{G^*(s)^F} e_{G^*} \). But by [DM2, Proof of Proposition 2.3] we have

\[
H^j_i(Y^u_0) = \mathcal{O}_{sG^*(s)^F} \otimes_{\pi_{jG^*(s)^F}} H^j_i(Y^u_0)
\]

as a \( G^F \)-module. Hence we have

\[
H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} V^o = 0
\]

for all \( i \neq j \). But

\[
H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} V^o = \left( H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} \mathcal{O}_{sG} \right) \otimes_{\pi_{jG^*(s)^F}} V^o
\]

\[
= H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} \left( \mathcal{O}_{sG} \otimes_{\pi_{jG^*(s)^F}} V^o \right)
\]

\[
= H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} \text{Ind}_{G^*(s)^F}^{G^*(s)^F} V^o.
\]

Since \( V \) is a direct summand of the \( G^*(s)^F \)-module \( \text{Ind}_{G^*(s)^F}^{G^*(s)^F} V^o \), it follows that

\[
H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} V = 0
\]

if \( i \neq j \) and that \( \tilde{\gamma} \) is the character of the module

\[
H^j_i(Y^u_0) \otimes_{\pi_{jG^*(s)^F}} V.
\]

\[\Box\]

1.6. Clifford Theory

Let \( \gamma^o \in \mathcal{Z}(G^{oF}, (s)^o) \). By [LS, Theorem 3.2] there exists a unique unipotent character \( \gamma^o(s) \) of \( G^*(s)^F \) such that

\[
\gamma^o = e_{G^*(s)^F} e_{G^*} R^{G}_{G^*(s)^F}(\gamma^o(s) \otimes \hat{\mathfrak{s}}^o).
\] 

(16.1)

Let \( \mathcal{A}(\gamma^o) \) be the stabilizer of \( \gamma^o \) in \( \mathcal{A} \). Its dual \( \mathcal{A}^*(\gamma^o) \) stabilizes the \( G^{oF^*} \)-conjugacy class of \( s \) and hence is contained in \( \mathcal{A}^*(s) \). By duality \( \mathcal{A}(\gamma^o) \) is contained in \( \mathcal{A}(s) \). The uniqueness of \( \gamma^o(s) \) implies that \( \mathcal{A}(\gamma^o) \) is the stabilizer \( \mathcal{A}(s, \gamma^o(s)) \) of \( \gamma^o(s) \) in \( \mathcal{A}(s) \).
We denote by $\tilde{\gamma}(s)$ the canonical extension of $\gamma^o(s)$ to $G^o(s)^F \rtimes A(\gamma^o)$ (as defined in [B, Definition 7.3.3]). We put

$$\tilde{\gamma} = e_{G^o(s)} e_{G^o} R_{G^o(s)}^{G^o \rtimes A(\gamma^o)}(\tilde{\gamma}(s) \otimes \hat{s}).$$

(1.6.2)

Then, by Lemma 1.5.1, $\tilde{\gamma}$ is an irreducible character of $G^o \rtimes A(\gamma^o)$ extending $\gamma^o$.

**Definition 1.6.3.** The irreducible character $\tilde{\gamma}$ of $G^o \rtimes A(\gamma^o)$ will be called the canonical extension of $\gamma^o$.

If $\xi$ is an irreducible character of $A(\gamma^o)$, then by Clifford theory $\tilde{\gamma} \otimes \xi$ is an irreducible character of $G^o \rtimes A(\gamma^o)$, and $\text{Ind}_{G^o \rtimes A(\gamma^o)}^{G^o \rtimes A(\gamma^o)}(\tilde{\gamma} \otimes \xi)$ is an irreducible character of $G^F$ (where $\xi$ is lifted to $G^o \rtimes A(\gamma^o)$ in the natural way). Moreover,

$$\text{Ind}_{G^o \rtimes A(\gamma^o)}^{G^o \rtimes A(\gamma^o)}(\tilde{\gamma} \otimes \xi) = \sum_{\xi \in \text{Irr}(G^o \rtimes A(\gamma^o))} \xi(1) \text{Ind}_{G^o \rtimes A(\gamma^o)}^{G^o \rtimes A(\gamma^o)}(\tilde{\gamma} \otimes \xi).$$

(1.6.4)

### 1.7. Jordan Decomposition

Let $\gamma$ be an irreducible character in $E(G^F, (s))$. By Corollary 1.2.4 there exists an irreducible character $\gamma^o \in E(G^o \times F, (s))^F$ occurring in the restriction of $\gamma$ to $G^o \times F$. Let $\tilde{\gamma}$ be the canonical extension of $\gamma^o$ to $G^o \rtimes A(\gamma^o)$ defined in Definition 1.6.3. Then by Clifford theory there exists a unique irreducible character $\xi$ of $A(\gamma^o)$ such that

$$\gamma = \text{Ind}_{G^o \rtimes A(\gamma^o)}^{G^o \rtimes A(\gamma^o)}(\tilde{\gamma} \otimes \xi).$$

Let $\gamma^o(s)$ be the unipotent character of $G^o(s)^F$ satisfying (1.6.1), and let $\tilde{\gamma}(s)$ be its canonical extension to $G^o(s)^F \rtimes A(\gamma^o)$ (recall that $A(\gamma^o)$ is the stabilizer of $\gamma^o(s)$ in $A(s)$). Then

$$\gamma(s) = \text{Ind}_{G^o(s)^F \rtimes A(\gamma^o)}^{G^o(s)^F \rtimes A(\gamma^o)}(\tilde{\gamma}(s) \otimes \xi)$$

is an irreducible character of $G(s)^F$ and is unipotent by definition. It follows from [B, Propositions 6.3.2 and 6.3.3] that

$$\gamma = e_{G^o(s)} e_{G^o} R_{G^o(s)}^{G^o \rtimes A(\gamma^o)}(\gamma(s) \otimes \hat{s}).$$

(1.7.1)

**Remark.** The remark following Lemma 1.5.1 shows that the Lusztig functor $R_{G^o(s)}^{G^o}$ does not depend on the choice of a parabolic subgroup of $G$ that has $G(s)$ as Levi subgroup.
Theorem 1.7.2 (Jordan Decomposition). With the above notation the map

\[ \nabla_{g,s} : \mathcal{E}(G^F, (s)) \to \mathcal{E}(G(s)^F, 1) \]

\[ \gamma \mapsto \gamma(s) \]

is well-defined and bijective. The inverse map is given by Formula (1.7.1).

Proof. First we have to prove that \( \nabla_{g,s} \) is well defined. There is one ambiguity in the construction of \( \gamma(s) \): in the first step, we choose an irreducible character \( \gamma^o \in \mathcal{E}(G^o F, (s)^o F) \) occurring in the restriction of \( \gamma \) to \( G^o F \). If \( \delta^o \) is another element of the Lusztig series \( \mathcal{E}(G^o F, (s)^o F) \) occurring in the restriction of \( \gamma \) to \( G^o F \), then there exists \( \alpha \in \mathbb{A} \) such that \( \delta^o = \gamma^o \). But both lie in \( \mathcal{E}(G^o F, (s)^o F) \), so we have \( \alpha \in A(s) \). If we construct \( \delta^o(s), \delta(s) \), and \( \delta(s) \) in the same way as \( \gamma^o(s), \gamma(s) \), and \( \gamma(s) \), respectively, then \( \delta^o(s) = \gamma^o(s) \) (by uniqueness), so \( \delta(s) = \gamma(s) \), and so \( \delta(s) = \delta(s) = \delta(s) \) because \( \alpha \in A(s) \). Thus \( \nabla_{g,s} \) is well defined.

\( \nabla_{g,s} \) is injective by Formula (1.7.1) and surjective by Lemma 1.5.1, which proves that Formula (1.7.1) always defines an element of \( \mathcal{E}(G^F, (s)) \). \( \square \)

2. Uniform Functions

In [B, Formula 7.3.1 and Theorem 7.3.2] the unipotent characters of \( G^F \) are described as linear combinations of generalized Deligne–Lusztig characters. It is possible using Formula (1.7.1) to describe all of the irreducible characters of \( G^F \) as linear combinations of generalized Deligne–Lusztig characters.

2.1. Notation

Let \( s \) be a nice semisimple element of \( G^{\ast o F^+} \).

We fix an \( F \)-stable and \( A(s) \)-stable Borel subgroup \( B^+_1(s) \) of \( G^s(s) \) and an \( F \)-stable and \( A(s) \)-stable maximal torus \( T^+_1(s) \) of \( B^+_1(s) \). We denote by \( W(s) \) (respectively, \( W^o(s) \)) the Weyl group of \( G(s) \) (respectively, \( G^o(s) \)) to \( T^+_1(s) \).

For each \( \alpha \in A(s) \), we define \( T^+_1(s, \alpha) \) to be the semidirect product \( T^+_1(s) \rtimes \langle \alpha \rangle \). For each \( w \in W^o(s)^{\alpha} \) (that is, the subgroup of \( W^o(s) \) consisting of elements centralized by \( \alpha \)), we denote by \( T_w(s, \alpha) \) the quasi-maximal torus of \( G^o(s) \rtimes \langle \alpha \rangle \) associated with \( w \) as in [D-M 2, Proposition 1.40] (for the definition of a quasi-maximal torus, cf. [B, Definition 6.1.3]). \( T_w(s, \alpha) \) is defined by the following property: \( (T_w(s, \alpha)^o)^F \) is an \( F \)-stable maximal torus of \( G^o(s)^\alpha \) of type \( w \) with respect to \( T^+_1(s)^\alpha \).
The group $W^o(s)$ is a product of symmetric groups, and $A(s)$ and $F$ act on $W^o(s)$ by permutations of the components ($F$ acts on $W^o(s)$ as $\sigma$). By the argument used in [B, Sect. 7.3] we can associate canonically with each irreducible character $\chi^o$ of $W^o(s)$ and each $\alpha$ in the stabilizer $A(s, \chi^o)$ of $\chi^o$ in $A(s)$ an irreducible character $\tilde{\chi}_\alpha$ of $W^o(s) \rtimes \langle \sigma \rangle$.

2.2. Irreducible Characters in $\mathcal{E}(G^o, (s))^F$ as Uniform Functions

Let $\chi^o$ be an irreducible character of $W^o(s)$. We define

$$R^o_{\chi^o}(s) = R^{G^o}_{\chi^o}(s) = \frac{\delta_{G^o}(s) \delta_{G^o}}{|W^o(s)|} \sum_{w \in W^o(s)} \tilde{\chi}_1(w\sigma) R^{G^o}_{T_{\chi^o}(s, \alpha)}(s).$$

(2.2.1)

**Proposition 2.2.2** (Lusztig–Srinivasan [LS, Theorem 3.2]). (a) For all $\chi^o \in \text{Irr} W^o(s)$, $R^o_{\chi^o}(s)$ is an irreducible character of $G^o$ in $\mathcal{E}(G^o, (s))^F$.

(b) The map

$$\text{Irr } W^o(s)^F \to \mathcal{E}(G^o, (s))^F$$

$$\chi^o \mapsto R^o_{\chi^o}(s)$$

is bijective.

**Corollary 2.2.3.** (a) If $\chi^o \in \text{Irr } W^o(s)$ and $\alpha \in A(s)$, then $R^o_{\chi^o}(s)$ is $R^o_{\chi^o}(s)$.

(b) If $\chi^o \in \text{Irr } W^o(s)$, then $A(R^o_{\chi^o}(s)) = A(s, \chi^o)$.

2.3. Canonical Extensions as Uniform Functions

Let $\chi^o$ be an irreducible character of $W^o(s)$. We define a function $\tilde{R}^o_{\chi^o}(s)$ on $G^o \rtimes A(s, \chi^o)$ by

$$\text{Res}_{G^o \rtimes A(s, \chi^o)}^{G^o \rtimes A(s, \chi^o)} \tilde{R}^o_{\chi^o}(s)$$

$$= \frac{\delta_{G^o}(s) \delta_{G^o}}{|W^o(s)|} \sum_{w \in W^o(s)} \tilde{\chi}_1(w\sigma) \text{ Res}_{G^o \rtimes A(s, \chi^o)}^{G^o \rtimes A(s, \chi^o)} R^{G^o \rtimes A(s, \chi^o)}_{T_{\chi^o}(s, \alpha)}(s).$$

(2.3.1)

for all $\alpha \in A(s, \chi^o)$.

**Proposition 2.3.2.** $\tilde{R}^o_{\chi^o}(s)$ is an irreducible character of $G^o \rtimes A(s, \chi^o)$ and is in fact the canonical extension of $R^o_{\chi^o}(s)$ (cf. Definition 1.6.3).

**Proof.** This follows immediately from Formula (1.6.2), from [B, Theorem 7.3.2], and from [DM2, Proposition 2.3].
2.4. Parameterization of \( E(G^F,(s)) \)

We denote by \( \mathcal{A}(s) \) the set of pairs \((\chi^*, \xi)\) where \( \chi^* \) is an irreducible character of \( W^o(s)^F \) and \( \xi \) is an irreducible character of \( \mathcal{A}(s, \chi^*) \). The group \( \mathcal{A}(s) \) acts by conjugation on \( \mathcal{A}(s) \), and we denote by \( \mathcal{F}(s) \) the set of orbits of \( \mathcal{A}(s) \) in \( \mathcal{A}(s) \). Moreover, if \((\chi^*, \xi) \in \mathcal{F}(s)\), we denote by \( \chi^* \otimes \xi \) its orbit under \( \mathcal{A}(s) \).

For all \( \chi^* \otimes \xi \in \mathcal{F}(s) \), we define

\[
R_{\chi^* \otimes \xi}^o(s) = R_{\chi^* \otimes \xi}^o(s) = \text{Ind}^G_{W(s)^F \rtimes \mathcal{A}(s, \chi^*)} \left( \tilde{R}_{\chi^*}(s) \otimes \xi \right).
\]

(2.4.1)

It follows from Corollary 2.2.2(a) that \( R_{\chi^* \otimes \xi}^o(s) \) only depends on the orbit of \((\chi^*, \xi)\) under \( \mathcal{A}(s) \). Moreover, it follows from Clifford theory and from Corollary 2.2.2(b) that we have

**Lemma 2.4.2.** The map

\[
\mathcal{F}(s) \to E(G^F,(s))
\]

\[
\chi^* \otimes \xi \mapsto R_{\chi^* \otimes \xi}^o(s)
\]

is bijective.

By [B, Proposition 2.3.1], \( \chi^* \) has a canonical extension \( \tilde{\chi} \) to the semidirect product \( W^o(s) \rtimes \mathcal{A}(s, \chi^*) \). By Clifford theory again we have

**Lemma 2.4.3.** The map

\[
\mathcal{F}(s) \to \text{Irr} W(s)^F
\]

\[
\chi^* \otimes \xi \mapsto \text{Ind}^{W(s)^F \rtimes \mathcal{A}(s, \chi^*)}_{W(s)^F \rtimes \mathcal{A}(s, \chi^*)} (\tilde{\chi} \otimes \xi)
\]

is bijective.

Lemmas 2.4.2 and 2.4.3 imply the following:

**Theorem 2.4.4.** There is a well-defined bijection

\[
\text{Irr} W(s)^F \to E(G^F,(s))
\]

\[
\chi \mapsto R_{\chi}(s).
\]

**Remark.** If necessary, we will write \( R^o_s(\chi) \) for the irreducible character \( R_{\chi}(s) \) of \( G^F \). By applying Theorem 2.4.4 in the case where \( G = G(s) \) and \( s = 1 \), we obtain a bijection,

\[
\text{Irr} W(s)^F \to E(G(s)^F,1)
\]

\[
\chi \mapsto R_{\chi}^{G(s)}(1).
\]
and it is easy to check that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Irr } W(s)^F & \sim & \text{Irr } W(s)^F
\\
\downarrow & & \downarrow
\\
\mathcal{E}(G^F, (s)) & \sim & \mathcal{E}(G(s)^F, 1).
\end{array}
\]

(2.4.5)

2.5. Induction from a Particular Subgroup of \( G \)

Let \( G' \) be a subgroup of \( G \) containing \( G^o \). It is \( F \)-stable because \( F \) acts trivially on \( A \). There exists a subgroup \( A' \) of \( A \) such that

\[ G' = G^o \rtimes A'. \]

If we construct \( G'^u \) in the way we construct \( G \), then \( G'^u \) may be identified with a subgroup of \( G^u \). We can also construct \( G'(s) \) so that it is contained in \( G(s) \), and we denote by \( W'(s) \) the Weyl group of \( G'(s) \) relative to \( T'_1(s) \) so that \( W'(s) \) is a subgroup of \( W(s) \).

**Proposition 2.5.1.** Let \( \chi' \) be an irreducible character of \( W'(s)^F \). Suppose

\[ \text{Ind}_{W'(s)^F}^{W(s)^F} \chi' = \sum_{\chi \in \text{Irr } W(s)^F} n_{\chi} \chi. \]

Then

\[ \text{Ind}_{G'}^{G} R_{\chi'}^G(s) = \sum_{\chi \in \text{Irr } W(s)^F} n_{\chi} R_{\chi}^G(s). \]

3. Lusztig Functors

**Hypothesis.** Throughout this section, and only in this section, \( A \) will be assumed abelian.

3.1. Notation

Let \( L \) be an \( F \)-stable Levi subgroup of a parabolic subgroup \( P \) of \( G \). Let \( A_L \) be the image of \( L \) through the composite morphism

\[ L \to G \to G/G^o \to A \]

(\( A_L \) is a subgroup of \( A \)). Because \( A \) is abelian, we can use the same argument as in [B, 7.6] to assume that \( L \) contains \( A_L \). Let \( A_L^u \) be the image of \( A_L \) under the anti-isomorphism \( A \to A^* \).
Let $L^*\circ F^*$ be an $F^*$-stable Levi subgroup of a parabolic subgroup of $G^{*\circ}$ that is a dual of $L^*$. We can choose $L^*\circ F^*$ to be $A^*_{L^*}$-stable, and we define $L^* = L^*\circ F^* \rtimes A^*_{L^*}$.

then $L^*$ is an $F^*$-stable Levi subgroup of a parabolic subgroup of $G^*$ and $L^{*\circ} = L^\circ$.

3.2. Jordan Decomposition and Lusztig Functors

Let $s$ be a semisimple element in $L^{*\circ} F^*$. We may assume that $s$ is nice in $G^*$. Then the subgroup $L(s)$ of $L$ following the construction of Section 1.4 can be chosen as a subgroup of $G(s)$. The linear character of $L(s)^F$ associated with $s$ as defined in Section 1.4 is then the restriction of $\hat{s}$ to $L(s)^F$. It results from this remark and from the transitivity of Lusztig induction functors (cf. [B, Proposition 6.3.3]) that the following diagram is commutative:

\[
\begin{array}{rcl}
\mathcal{E}(L^F, (s)L^{*\circ}) & \xrightarrow{\varphi_{L^*,(s)L^{*\circ}}} & \mathcal{E}(L(s)^F, 1) \\
\varepsilon_{L^*\circ F^*} R_{L^*} & & \varepsilon_{L^*\circ F^*} R_{L^*} \ \ \ \ (3.2.1)
\end{array}
\]

The description of the functor $R_{L(s)}^G$ in [B, Theorem 7.6.1] thus provides a description of the functor $R_{L}^G$ via the commutative diagram (3.2.1).

4. SHINTANI DESCENT IN THE GENERAL LINEAR GROUP

In this section, we explain the link between the theory of irreducible characters of $G^F$ and the theory of Shintani descent for the general linear group. For this purpose, we need to consider a particular case:

Hypothesis and Notations. Throughout this section, we assume that $r = 1$. We will denote $d = d_1$ and $n = n_1$ for simplicity. We also assume that $\sigma = (1, \ldots, d)$ and that $A$ is generated by $\sigma$.

4.1. The Group $G^F$

We denote by $G_1$ the general linear group $GL_n$, and we endow it with the split Frobenius endomorphism:

\[
F_0: G_1 \to G_1, \quad g \mapsto g^{(q)}.
\]
We denote by \( \phi_0 \) the automorphism of \( G_{1,F}^d \) induced by \( F_0 \). Then the map
\[
\theta: G_{1,F}^d \to G_{1}^{F_0}
\]
\[
g \mapsto (g, F_0(g), \ldots, F_0^{d-1}(g))
\]
is an isomorphism of groups and the following diagram is commutative:
\[
\begin{array}{ccc}
G_{1,F}^d & \xrightarrow{\theta} & G_{1}^{F_0} \\
\phi_0 \downarrow & & \downarrow \sigma^{-1} \\
G_{1,F}^d & \xrightarrow{\sigma} & G_{1}^{F_0}.
\end{array}
\]
This implies that \( \theta \) can be extended to an isomorphism denoted by
\[
\tilde{\theta}: G_{1,F}^d \times \langle \phi_0 \rangle \to G_{1}^{F}
\]
\[
g \phi_0^k \mapsto \theta(g) \sigma^{-k}
\]
for all \( g \in G_{1,F}^d \) and \( k \in \mathbb{Z} \).

4.2. Shintani Descent

Let \( g \in G_{1,F}^d \). By Lang's theorem, there exists \( x \in G_1 \) such that \( g = x^{-1}F_0^{d}(x) \). Then \( g' = F_0(x)x^{-1} \) belongs to \( G_{1,F}^d \), and the map that sends the conjugacy class of \( g \) in \( G_{1,F}^d \) to the \( \phi_0 \)-conjugacy class of \( g' \) in \( G_{1,F}^d \) is well-defined and is bijective. We denote it by
\[
N_{F_0/F_d}: \text{Cl}(G_{1,F}^d) \to H^1(\phi_0, G_{1,F}^d),
\]
where \( H^1(\phi_0, G_{1,F}^d) \) denotes the set of \( \phi_0 \)-conjugacy classes of \( G_{1,F}^d \) and \( \text{Cl}(G_{1,F}^d) \) denotes the set of conjugacy classes of \( G_{1,F}^d \). If we denote by \( \mathcal{E}(G_{1,F}^d, \phi_0) \) (respectively, \( \mathcal{E}(G_{1,F}^d) \)) the space of class functions on \( G_{1,F}^d \) obtained by restrictions from class functions on the group \( G_{1,F}^d \) (respectively, \( G_{1,F}^d \)), then \( N_{F_0/F_d} \) induces an isomorphism
\[
\text{Sh}_{F_0/F_d}: \mathcal{E}(G_{1,F}^d, \phi_0) \to \mathcal{E}(G_{1,F}^d),
\]
called the Shintani descent from \( F_0^d \) to \( F_0 \).

We recall the following theorem:

**Theorem 4.2.1 (Shintani).** Let \( \gamma_1 \) be an irreducible character of \( G_{1,F}^d \) stable under \( \phi_0 \). Then there exists an extension \( \tilde{\gamma}_1 \) of \( \gamma_1 \) to \( G_{1,F}^d \times \langle \phi_0 \rangle \) such that \( \text{Sh}_{F_0/F_d} \tilde{\gamma}_1 \) is, up to a sign, an irreducible character of \( G_{1,F}^d \).
4.3. Shintani Descent and Characters of $G^F$

We denote by $\theta^*$ and $\tilde{\theta}^*$ the isomorphisms of $\mathbb{Q}$-vector spaces:

$$\theta^*: \mathcal{C}(G^F) \to \mathcal{C}(G_1^F)$$

and

$$\tilde{\theta}^*: \mathcal{C}(G^F) \to \mathcal{C}(G_1^F \times \langle \phi_0 \rangle),$$

induced by $\theta$ and $\tilde{\theta}$, respectively.

Let $\gamma^F$ be an irreducible character of $G^F$, and let $\gamma_1 = \theta^*(\gamma^F)$. Then $\gamma_1$ is $\phi_0$-stable if and only if $\gamma^F$ is $\alpha$-stable.

**Hypothesis.** From now on, we assume that $\gamma_1$ is $\phi_0$-stable.

Let $s$ be a nice semisimple element of $G^{*F}$ such that $\gamma_1 \in \mathcal{Z}(G(F), (s)\mathcal{Z})$. Then $A(s) = A$ because $\gamma_1$ is $\phi_0$-stable. Let $\chi^*$ be the irreducible character of $W^0(s)$ (stable under $F$) such that $\gamma^F = R^*_x(s)$. Then $A(s, \chi^*) = A$.

**Theorem 4.3.1.** With the above notations, we have

(a) There exists a unique extension $\tilde{\gamma}_{1}$ of $\gamma_1$ to $G_1^F \times \langle \phi_0 \rangle$ such that $\text{Sh}_{F_0/F_{0},(s)}\tilde{\gamma}_{1}$ is an irreducible character of $G_1^F$. We call it the Shintani extension of $\gamma_1$.

(b) We have $\tilde{\gamma}_{1} = \tilde{\theta}^*(\tilde{R}_x^*(s))$.

(c) Let $e$ be a divisor of $d$, and let $\tilde{\gamma}_{1}^{(e)}$ be the Shintani extension of $\gamma_1$ to $G_1^F \times \langle \phi_0 \rangle$. Then $\tilde{\gamma}_{1}^{(e)}$ is the restriction of $\tilde{\gamma}_{1}$.

**Remark.** The result stated in (a) of Theorem 4.3.1 is slightly stronger than Shintani’s. It was already known for characters of the principal series [D M 3].

**Proof.** By Theorem 4.2.1, (a), (b), and (c) are immediate consequences of the following:

**Lemma 4.3.2.** $\tilde{R}_x^*(s)(\sigma^e)$ is a positive integer for all $e \in \mathbb{Z}$.

**Proof of Lemma 4.3.2.** Let $e \in \mathbb{Z}$. We first prove that

$$e_{G^*(s)^{\sigma^e}} = e_{G^*(s)}$$

and

$$e_{(G^*)^{\sigma^e}} = e_{(G^*)^{\sigma}}. \quad (\star)$$

Because $G^*(s)$ is a direct product of groups of the same type as $G^*$, it is sufficient to prove the result for $G^*$. But $(T^*)^{\sigma}$ is a maximal split subtorus of $G^*$, so it is a maximal split subtorus of $(G^*)^{\sigma}$. That proves $(\star)$.

Let $\chi'_x$ be the irreducible character of $W^0(s)^{\sigma^e} \times \langle \sigma \rangle$ associated with $\chi^*$ as in Section 2.1 (it was denoted $\chi_{\alpha^*,t}$, but we just want to have simpler notations).
Then, by formulas (2.3.1) and (★), we have
\[
\hat{R}^G_s(s)(\sigma^e) = \frac{\hat{E}^G_s(s)^{\sigma^e} \hat{E}^{\sigma^e}_s}{|W^*(s)|^{\sigma^e}} \sum_{w \in W^*(s)^{\sigma^e}} \tilde{\chi}_s(w \sigma) R^G_{(s, \sigma^e)}(\hat{s})(\sigma^e).
\]

Using [DM2, Theorem 4.13], we get
\[
\hat{R}^G_s(s)(\sigma^e) = \frac{\hat{E}^G_s(s)^{\sigma^e} \hat{E}^{\sigma^e}_s}{|W^*(s)|^{\sigma^e}} \sum_{w \in W^*(s)^{\sigma^e}} \tilde{\chi}_s(w \sigma) \dim R^G_{(s, \sigma^e)^{\sigma^e}}. \tag{1}
\]

But this last formula gives the degree of an irreducible character of \(((G^*)^{\sigma^e})^F\) (cf. [LS, Theorem 3.2]).

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**REFERENCES**


