Minicourse notes
Rearrangement groups of fractals and Thompson groups

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5th - 7th May 2025

What is this document?

These are notes for the minicourse *Rearrangement groups of fractals and Thompson groups* for the workshop "Random walks on groups, groups acting on fractals" at Montpellier (5th - 7th May 2025).

I originally wrote these notes for myself, as an attempt at organizing my thoughts and my lectures. In the end, I thought that they were decent enough that they could be shared, although they are still quite rough and sketchy. The notes probably include a little more than I manage to discuss in the lectures.

Why study these groups?

The importance of simple groups cannot be overstated. Classifying the infinite simple groups (even just the countable ones) is arguably impossible, so there have been attempts at subdividing countable groups into smaller families of groups, for example by distinguishing them by their finiteness properties [SWZ19]. Finitely presented simple groups are also important because of their involvement in the Boone-Higman conjecture, which states that a finitely generated group has solvable word problem if and only if it can be embedded into some finitely presented simple group, see [Bel+23].

Thompson groups T and V are the first examples of finitely presented infinite simple groups. Together with their smaller sibling F, they were introduced in the 60s by Richard Thompson in unpublished notes in the topic of logic and were later popularized by Higman [Hig74]. A popular and famously difficult question concerns the potential amenability of F [Gub23].

These three groups have countless equivalent definitions and generalizations that relate them to different topics in mathematics, among which we recall the following.

Topology: F, T and V are groups of piecewise-affine homeomorphisms.

Symbolic dynamics: V is the topological full group of the full shift on the alphabet of two digits [Mat15].

Fundamental groups: F, T and V are fundamental groups of certain Ore categories [Wit19].

Semigroup theory: F, T and V are planar, annular and symmetric diagram groups [GS97].

Logic: F is the group of associative laws [Bel04, Theorem 1.6.1].

Computer science: Lehnert's conjecture states that a group is co$\mathcal{CF}$ (i.e., its co-word problem is context-free) if and only if it embeds into V [BMN16].

Automata theory: F, T and V can be seen as groups of asynchronous automata.

Algebras: V is the automorphism group of a universal algebra [Hig74].

Links and knots: F encodes all knots and links in $\mathbb{R}^3$ [Jon17].

The diversity of settings in which Thompson groups make their appearance is what motivated Matthew Brin to call F *chameleon group* [Bri96].

The goal of these notes is to explore the first of these perspectives: Thompson groups as groups of "piecewise-canonical" homeomorphisms. Most generalizations of Thompson groups either have no "standard" actions on nice topological spaces or they act in some way on the interval, the circle or the Cantor space. Rearrangement groups instead provide a natural framework of "piecewise-canonical" homeomorphisms of a larger class of topological spaces (often fractals). This produces a rich class of groups with interesting behaviors regarding their finiteness properties [WZ19; PT24b], simplicity [BF15; Tar24b; Tar23b], generation properties [PT24a] and decision problems [Tar23a].

Also, rearrangement groups act on cool-looking fractals, which is never bad.

Lecture 1

Thompson groups

Let us introduce Thompson groups F, T and V as groups of piecewise-affine homeomorphisms of the unit interval $\mathbf{I} \coloneqq [0,1]$, the unit circle $\mathbf{S}^1 \coloneqq [0,1]/\{0,1\}$ and the Cantor space $\mathfrak{C}$.

We say that an element of $\mathbf{I}$ is dyadic if it belongs to $\mathbb{Z}[\frac{1}{2}]$. An element of $\mathbf{S}^1$ is **dyadic** if it has a dyadic representative in $\mathbf{I}$.

1.1 Thompson's group F

Definition 1.1. Thompson's group F is the group of the orientation-preserving piecewise-affine homeomorphisms g of $\mathbf{I}$ such that:

- g has finitely many breakpoints, all of which are dyadic;
- where g is affine, the slope is a power of 2.

Two examples are depicted in the left side of Figure 1.1.

Proposition 1.2. *F is torsion-free.*

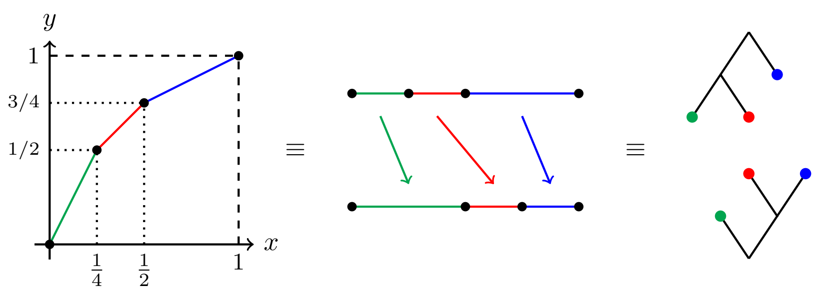
Proof. Given any $g \in F \setminus \{\mathrm{id}_I\}$, consider the point

$$x_0 = \inf\{x \in \mathbf{I} \mid g(x) \neq x\} \in \mathbf{I}.$$

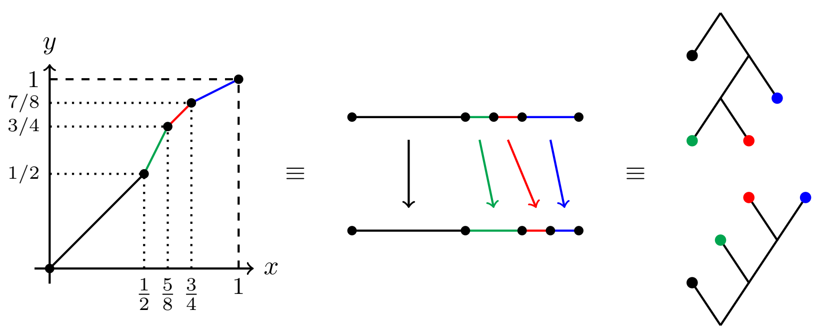
Note that $g(x_0) = x_0$ and $g'(x_0) = 2^k \neq 1$. Thus $(g^n)'(x_0) = 2^{kn}$, which are all distinct. So the powers g^n must all be distinct. □

Proposition 1.3. *For any $k \in \mathbb{N}$, given dyadics $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$ in $\mathbf{I}$, there exists some $g \in F$ such that $g(a_i) = b_i$ for all $i \in \{1, \ldots, k\}$. Actually, the commutator subgroup F' has this property too.*

We will omit this proof. This will be graphically evident (though a little boring to fully write out) using a description of F as the group of *dyadic rearrangements* of *tree pair diagrams*.



(a) The element X_0.



(b) The element X_1.

Figure 1.1: Two generators X_0 and X_1 for Thompson's group F.

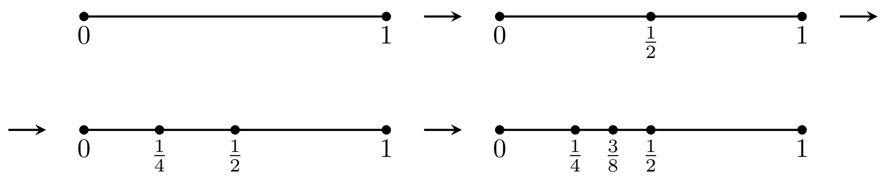


Figure 1.2: A dyadic subdivision built by a sequence of halvings.

Theorem 1.4. *Thompson's group F is generated by the elements X_0 and X_1 depicted in Figure 1.1.*

We will not show this here.

1.1.1 Diagrams for the elements of F

Let us call **dyadic subdivisions** those families of subintervals of $\mathbf{I}$ that are obtained by chopping the interval in half finitely many times, as in Figure 1.2. Dyadic subdivisions correspond to partitions of $\mathbf{I}\setminus\{1\} = [0,1)$ into finitely many **standard dyadic intervals**, which are intervals of the form

$$\left[\frac{a}{2^b}, \frac{a+1}{2^b}\right).$$

Elements of Thompson's group F can be described as **dyadic rearrangements**: pairs of dyadic subdivisions with the same number of standard dyadic intervals (the intervals are mapped affinely between the two subdivisions, in their order). See the central part of Figure 1.1 as examples.

The process of halving the interval can be represented by finite rooted binary trees: the root represents $[0,1)$ and each caret represents a halving, with the father corresponding to the interval being cut in half and the two children representing the two halves. (Note that only the leaves of such trees represent the standard dyadic intervals of the dyadic subdivision, whereas the inner vertices represent intervals that appear in the intermediate steps of the sequence of halvings.) Using such a description, elements of Thompson's group F can also be portrayed as **tree pair diagrams**, which are pairs of finite rooted binary trees with the same number of leaves. The right part of Figure 1.1 show two examples.

For the sake of brevity, we will refer to both these two ways of representing the elements of F as *diagrams*.

1.1.2 Reductions of diagrams

Sometimes a diagram features redundant pieces of data, for example in Figure 1.3. For the representation as a dyadic rearrangement, this happens when there are two standard dyadic intervals in the domain that are the halves of a larger standard dyadic interval and are mapped to two standard dyadic intervals with this same property. In a tree pair diagram, this is simply when

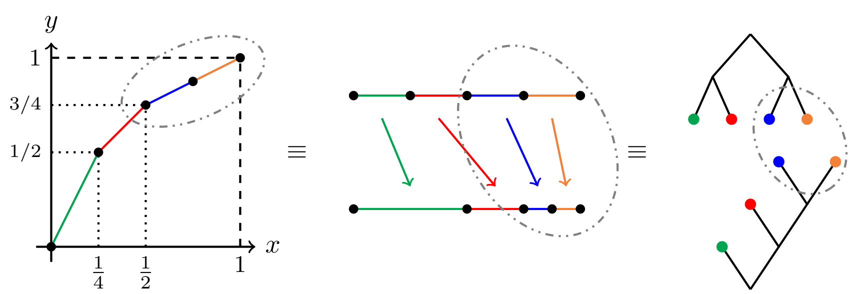


Figure 1.3: A non-reduced representative for $X_0 \in F$.

two leaves of a common caret in the domain tree are mapped to two leaves of a common caret in the range tree. When this happens, the diagram can be reduced, reverting the redundant halving for a dyadic rearrangement or removing the redundant carets in the tree pair diagram. For example, the diagrams portrayed in Figure 1.3 are reduced to those in Figure 1.1a.

It can be seen that two diagrams represent the same element of F if and only if the two diagrams differ by a sequence of reductions and expansions (the opposite of a reductions). Thus, F corresponds to the set of equivalence classes of diagrams under the equivalence relation of differing from a reduction or expansions.

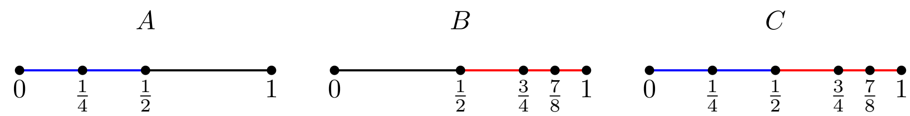
When no such reductions can be performed on a diagram, we say that the diagram is **reduced**. Informally, this means that such a diagram does not encode redundant pieces of data.

Lemma 1.5 (Existence and uniqueness of reduced diagrams for F)**.** *Each element of F is represented by a unique reduced diagram.*

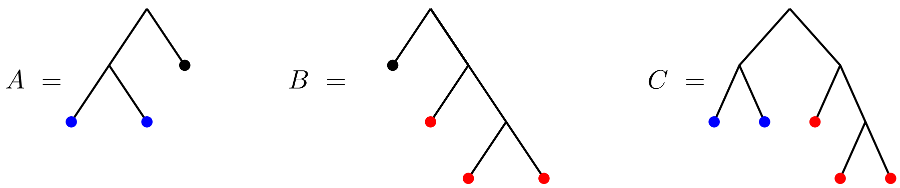
This can be shown using Newman's diamond Lemma [New42], which is also involved in the conjugacy problem for the third lecture and often appears when one wants to discuss uniqueness of reduced objects or normal forms. We will not explore the lemma here, but it informally states the following: if (1) whenever you reduce an object in two distinct way you can reduce them further to some common object and (2) there are no infinite chains of reductions, then there exists a unique reduced object (for each connected component of the rewriting system). It is easy to show that two distinct reductions of a diagram for an element of F can always be performed in whichever order one wishes, so Newman's diamond Lemma applies and thus there is a unique reduced diagram for each element.

1.1.3 Composition using diagrams

Let us see how the composition of two elements of F can be computed using either of the two diagrammatical approaches (dyadic rearrangements and tree pair diagrams). The procedure is similar to the sum of fractions: in order to



(a) Two dyadic subdivisions A and B and a common expansion C.



(b) Two finite rooted binary trees A and B and a common expansion C.

Figure 1.4: An example of common expansions of diagrams.

sum the numerators, one first needs to produce a common denominator E and "expand" both fractions so that their denominator is E.

Remark 1.6. Given two dyadic subdivisions or finite rooted binary trees A and B, there exist C that is a common expansion of them, as exemplified in Figure 1.4. (Actually, there exists a unique minimal such expansion.)

Say that you want to compute gh (which is $g \circ h$, so h is applied before g), given two diagrams $g = (A, B)$ and $h = (C, D)$, where A, B, C, D are dyadic subdivisions or finite rooted binary trees.

1. Compute a common expansion E of D and A.
2. Expand the diagram for g so that it becomes (E, B').
3. Expand the diagram for h so that it becomes (C', E).
4. The composition gh is represented by (C', B').

For example, Figure 1.5 depicts the composition of X_0 (from Figure 1.1a) with itself. The range of the top copy of X_0 and the domain of the bottom need to be expanded so that they match.

1.1.4 Abelianization and the commutator subgroup of F

Definition 1.7. The **support** of a homeomorphism f of a topological space X is the closure of the set

$$\{x \in X \mid f(x) \neq x\}.$$

Moreover, we say that f is **supported on** some $Y \subseteq X$ if the support of f is included in Y (or equivalently, if f is trivial outside of Y).

Proposition 1.8. *The commutator subgroup F' of F consists of the elements whose derivatives at* 0 *and* 1 *are trivial. Equivalently, these are the elements that are supported on some proper subinterval of* $\mathbf{I}$.

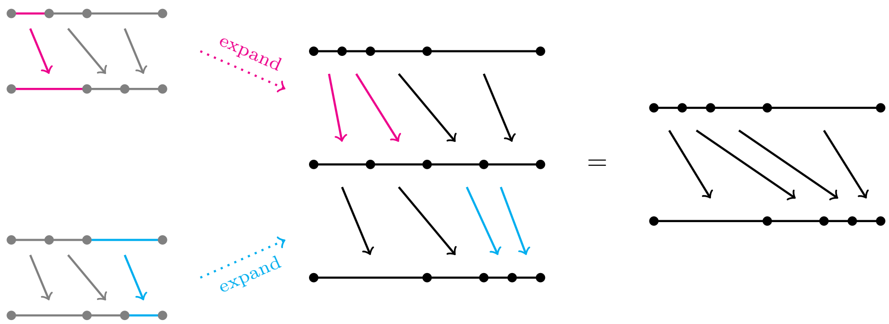


Figure 1.5: Computing the composition of X_0 with itself.

This can be formulated in terms of diagrams: F' consists of the dyadic rearrangements (A, B) such that the leftmost standard dyadic intervals of A and B are the same, as are the rightmost intervals; equivalently, F' consists of the tree pair diagrams (T_A, T_B) such that the leftmost leaves of T_A and T_B have the same height, as do the rightmost leaves.

Proof. Consider the map

$$\Phi \colon F \to \langle 2 \rangle_{\mathbb{Q}^*} \times \langle 2 \rangle_{\mathbb{Q}^*},\ g \mapsto (g'(0), g'(1)),$$

where $\langle 2 \rangle_{\mathbb{Q}^*}$ denotes the multiplicative subgroup of $\mathbb{Q}$ generated by 2 (i.e., the multiplicative group of the integer powers of 2). This is a group morphism, since 0 and 1 are fixed by every element of F. It is easy to create diagrams with arbitrary derivatives at 0 and at 1, so Φ is surjective. Since $\langle 2 \rangle_{\mathbb{Q}^*} \times \langle 2 \rangle_{\mathbb{Q}^*}$ is abelian, we have that $F' \leq K \coloneqq \mathrm{Ker}(\Phi)$.

The converse does not depend on F itself, but on a more general fact about 2-generated groups. Since F is 2-generated, the quotient F/F' is generated by two elements (namely, $X_0 F'$ and $X_1 F'$). It is abelian, so it is the direct product of two (finite, infinite or trivial) cyclic groups. But F/F' has (an isomorphic copy of) $\mathbb{Z} \times \mathbb{Z}$ as a quotient, so it must be $\mathbb{Z} \times \mathbb{Z}$ itself. □

Corollary 1.9. *The abelianization of F is $\mathbb{Z} \times \mathbb{Z}$.*

Corollary 1.10. *The commutator subgroup of F' is F' itself.*

Proof. Let $g \in F'$ and let $[a, b]$ be its support, which is a proper subset of $\mathbf{I}$ by Proposition 1.8. Let $c \in (0, a)$ and $d \in (b, 1)$ be dyadic numbers. Let us consider the subgroup $F[c, d]$ consisting of those elements of F that are supported on $[c, d]$, which can be shown to be isomorphic to F. Then g belongs to the commutator subgroup of $F[c, d]$ by Proposition 1.8. Since $F[c, d] \leq F'$, we have that $g \in F''$, as needed. □

1.1.5 Simplicity of the commutator subgroup of F

The following Lemma is inspired by strategies adopted by Epstein in [Eps70].

Lemma 1.11 (Double commutator trick)**.** *Let G be a group of homeomorphisms of X. Assume that $H \leq \mathrm{Homeo}(X)$ is normalized by G (i.e., $N_{\mathrm{Homeo}(X)}(H) \geq G$, which means $g^{-1}Hg = H,\ \forall g \in G$). If $J \cap h^{-1}(J) = \emptyset$ for some $h \in H$ and some proper nontrivial open subset J of X then, for all $f \in G$*

$$[g_1, g_2] \in H \text{ for all } g_1, g_2 \in G \text{ that are supported on } f(J).$$

Proof. Conjugating moves the support of an element, so if $g_1, g_2 \in G$ are supported on J then $h^{-1}g_1h$ is supported on $h^{-1}(J)$. Hence $[g_1, h] = g_1^{-1}(h^{-1}g_1h)$ agrees with g_1^{-1} on J. Then we have that

$$\Big[[g_1, h]^{-1}, g_2\Big] = [g_1, h]g_2^{-1}[g_1, h]^{-1}g_2 = g_1^{-1}g_2^{-1}g_1g_2 = [g_1, g_2].$$

Now, since H is normalized by G, the double commutator on the left of the equation belongs to H and thus $[g_1, g_2] \in H$. Since H is normalized by G, we can "conjugate" this statement by any $f \in G$ and obtain the desired fact. □

Theorem 1.12. *Any nontrivial subgroup H of F that is normalized by F' (i.e., $N_F(H) \geq F'$) must include F'.*

Proof. Assume $F' \leq N_F(H) = \{g \in G \mid H^g = H\}$. Consider any nontrivial $h \in H$. There exists some small enough dyadic subinterval $J \subseteq (0, 1)$ such that $J \cap h^{-1}(J) = \emptyset$. By Lemma 1.11, for any $f \in F'$ we have that

$$[g_1, g_2] \in H \text{ for all } g_1, g_2 \in F' \text{ that are supported on } f(J).$$

Now, by Propositions 1.3 and 1.8, this means that

$$[g_1, g_2] \in H \text{ for all } g_1, g_2 \in F'.$$

By Corollary 1.10 $F' = F''$, so F' is generated by $\{[g_1, g_2] \mid g_1, g_2 \in F'\}$. Hence, we ultimately have that H includes F'. □

Corollary 1.13. *The commutator subgroup F' of F is simple.*

1.1.6 Some additional properties of F

Theorem 1.14. *Thompson's group F admits the following infinite presentation*

$$F = \langle X_0, X_1, X_2 \cdots \mid X_k^{-1}X_nX_k = X_{n+1}, \forall k < n\rangle$$

and the following finite presentation

$$F = \langle X_0, X_1 \mid X_1^{-1}X_2X_1 = X_3,\ X_1^{-1}X_3X_1 = X_4\rangle,$$

where $X_i \coloneqq X_1^{X_0^{n-1}}$ (see Figure 1.6).

Proposition 1.15. *F has exponential growth (because the submonoid generated by X_0 and X_1^{-1} is free). However, F does not include free subgroups.*

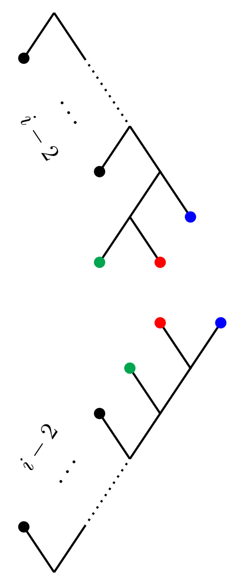


Figure 1.6: A schematic depiction of the element X_i (for $i \geq 2$).

1.2 Thompson's groups T and V

Multiple aspects of T and V are similar to those of their smaller sibling F, including their diagrammatic description and a dynamical proof of simplicity (in this case for the whole groups and not for the commutator subgroup). In particular, the proof of simplicity of T and V (which we will omit) use the same Lemma 1.11 that we employed for the simplicity of $[F, F]$.

1.2.1 Thompson's group T

Definition 1.16. Thompson's group T is the group of the orientation-preserving piecewise-affine homeomorphisms g of $\mathbf{S}^1$ such that:

- g has finitely many breakpoints, all of which are dyadic;
- where g is affine, the slope is a power of 2.

Diagrams are similar to those for F, with the addition of a cyclic permutation (see Figure 1.7). It is worth noting that such diagrams are no longer just *pairs*, but they are instead triples: one needs to specify two dyadic subdivisions of the circle along with a cyclic permutation of their intervals.

Proposition 1.17. *Thompson's group T has the following properties.*

- *T is finitely presented (actually F_∞).*
- *T is simple.*

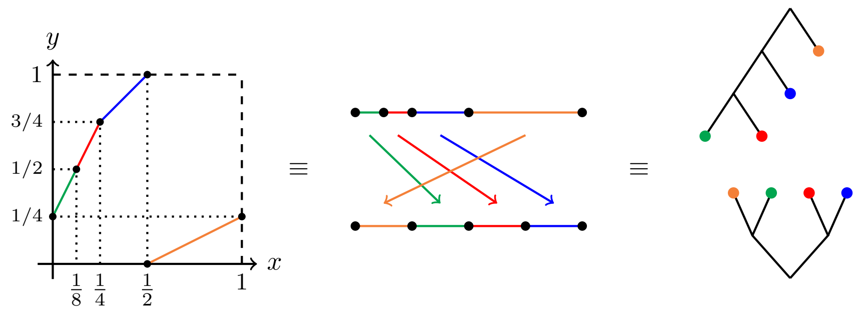


Figure 1.7: An element of Thompson's group T.

- *T includes any cyclic group.*
- *T can play ping-pong, so it contains free groups.*
- *T contains F.*

1.2.2 Thompson's group V

The description of V as a group of homeomorphism looks somewhat different from those of F and T, but it is going to be useful for understanding rearrangement groups.

Let us consider the binary Cantor space

$$\mathfrak{C}_2 \coloneqq \{0,1\}^\omega = \{x_1 x_2 x_3 \dots \mid x_i \in \{0,1\}\}.$$

We equip it with the product topology, which makes it a compact metrizable totally disconnected space without isolated points. It is actually the only such space up to homeomorphism, by Brouwer's Theorem.

A **cone** of $\mathfrak{C}_2$ is a subspace

$$C(w) = \{w\alpha \mid \alpha \in \mathfrak{C}_2\},$$

where w is any finite word in the alphabet $\{0,1\}$. Note that each cone $C(w)$ is homeomorphic to the whole $\mathfrak{C}_2$ itself via the map that removes the prefix w. In particular, every two cones are "canonically" homeomorphic by the map

$$C(u) \to C(v),\ u\alpha \mapsto v\alpha.$$

Definition 1.18. Thompson's group V is the group of the homeomorphisms g of the Cantor space $\mathfrak{C}_2$ that are given by finitely many prefix-exchanges, i.e., there exist partitions $\{C(u_1), \dots, C(u_k)\}$ and $\{C(v_1), \dots, C(v_k)\}$ of $\mathfrak{C}_2$ into cones such that

$$g(u_i\alpha) = v_i\alpha,\ \forall i \in \{1, \dots, k\}.$$

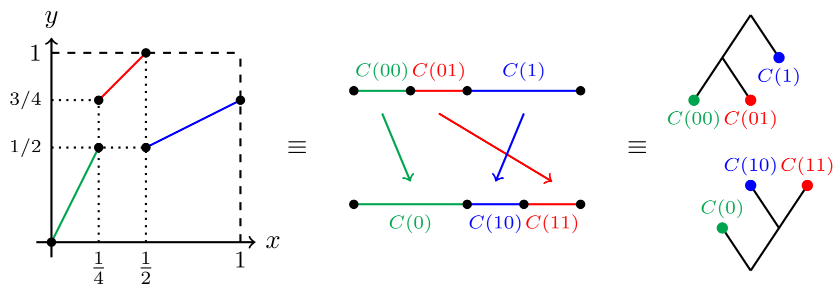


Figure 1.8: An element of Thompson's group V.

The link with Thompson's groups F and T is evident once we start depicting elements of V using diagrams. For example, Figure 1.8 depicts the element given by the three prefix-exchanges $00 \to 0$, $01 \to 11$ and $1 \to 10$. In general, the elements of V can be represented by a triple consisting of two dyadic subdivisions of the Cantor space (i.e., partitions into finitely many cones) along with a permutation between the two. The only difference with Thompson's group T is that, this time, the permutation need not be cyclic.

Proposition 1.19. *Thompson's group V has the following properties.*

- *V is finitely presented (actually F_∞).*
- *V is simple.*
- *V includes any finite group.*
- *V contains T (so it plays ping-pong and contains F).*

Lecture 2

Replacement systems and Rearrangement Groups

By (finite) **graph** we will mean the data (V, E, ι, τ) given by the following:

- V is a (finite) set of "vertices";
- E is a (finite) set of "edges";
- ι and τ are maps $E \to V$, mapping to "initial" and "terminal" vertices.

An **edge-colored graph** also comes equipped with

- a set C of "colors";
- a map $c\colon E \to C$.

In general, we will keep a fixed set of colors for every graph that we consider. If we need to work with monochromatic (i.e., not colored) graphs, we will simply assume that C is a singleton.

We say that an edge e is **incident** on a vertex v if $\iota(e) = v$ or $\tau(e) = v$ and that two edges are **adjacent** if they are incident on some common vertex.

Note that we allow the existence of **loops** (i.e., e such that $\iota(e) = \tau(e)$) and **parallel edges** (i.e., e_1, e_2 such that $\iota(e_1) = \iota(e)$ and $\tau(e_1) = \tau(e_2)$).

Graph isomorphisms are collections of bijections of the sets of vertices and edges that preserve ι, τ and c. More precisely, a **graph isomorphism** is

$$\phi = (\phi_V\colon V_1 \to V_2,\ \phi_E\colon E_1 \to E_2)$$

such that $\iota_2(\phi_E(e)) = \phi_V(\iota_1(e))$, $\tau_2(\phi_E(e)) = \phi_V(\tau_1(e))$ and $c(\phi_E(e)) = \phi_E(c(e))$ for all $e \in E_1$.

A **subgraph** of a graph (V, E, ι, τ) is a graph (V', E', ι', τ') such that $V' \subseteq V$, $E' \subseteq E$ and ι', τ' are the restrictions of ι, τ to E'.

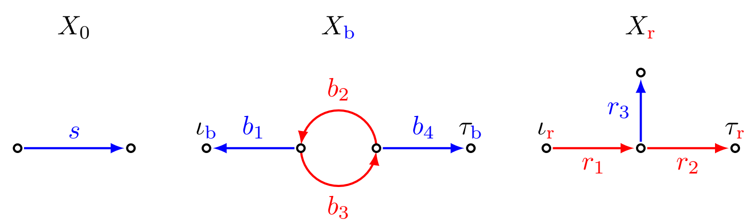


Figure 2.1: The airplane replacement system.

2.1 From graphs to fractals

Let us introduce the machinery that allows to build fractal topological spaces and rearrangement groups acting on them: the *replacement systems*.

2.1.1 Replacement systems

Definition 2.1. A set of **replacement rules** is a pair (R, C), with $R = \{X_c \mid c \in \mathrm{C}\}$, where

- C is a finite set of colors;
- for each color $c \in \mathrm{C}$, $X_c \in R$ is a finite graph edge-colored by C and equipped with distinct vertices ι_c and τ_c.

Each X_c is called c **replacement graph** (for example, *red replacement graph*) and the vertices ι_c and τ_c are called the **initial** and **terminal vertices** of X_c, respectively.

Definition 2.2. A **replacement system** (X_0, R, C) consists of a set of replacement rules (R, C) together with a finite graph X_0 that is edge-colored by C. The graph X_0 is called the **base graph** of the replacement system.

For example, Figure 2.1 depicts the **airplane replacement system**.

A c-colored edge e of a graph Γ can be **expanded** by replacing it with the graph X_c, identifying the initial and terminal vertices ι_c and τ_c of the replacement graph X_c with the initial and terminal vertices $\iota(e)$ and $\tau(e)$ of the edge e of Γ, respectively, producing a new graph $\Gamma \triangleleft e$. We will start expanding from the base graph X_0 and, say that we expand an edge e of X_0. For example, Figure 2.2 portrays two expansions of the airplane replacement system.

Note that we codify edges depending on what edges of previous expansions they descend on. The set of "codes" that correspond to edges is called the **language** of the replacement system. For example, sb_1r_2 belongs to the language while sb_1b_2 does not.

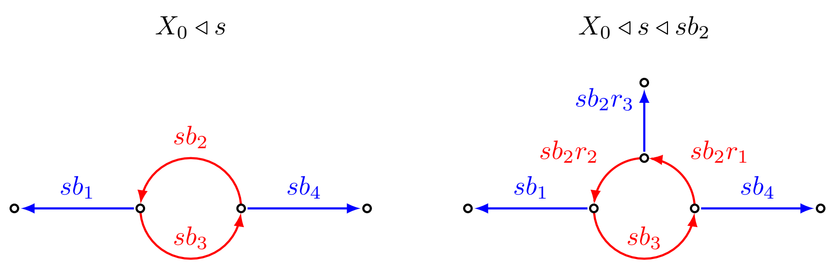


Figure 2.2: Two expansions of the airplane replacement system.

2.1.2 The gluing relation

The **symbol space** $\Omega_{\mathcal{R}}$ (where $\mathcal{R}$ is a replacement system) is the set of infinite codes whose every prefix belongs to the language of the replacement system. (For those who know about them, note that this is an edge shift.)

Definition 2.3. The **full expansion sequence** is the sequence of graph expansions $E_1, E_2, E_3, \dots$ obtained by expanding, at each step, every edge of the previous graph, starting from the base graph $E_1 = X_0$.

Note that E_k consists of the edges corresponding to words of length k.

Definition 2.4. The **gluing relation** of a replacement system is the binary relation on the symbol space $\Omega_{\mathcal{R}}$ defined by setting

$$\alpha \sim \beta \iff \forall n \in \mathbb{N},\ \alpha_1 \dots \alpha_n \text{ and } \beta_1 \dots \beta_n \text{ are incident in } E_n.$$

Definition 2.5. A replacement system (X_0, R, C) is **expanding** if

1. neither the base graph nor any replacement graph features isolated vertices;
2. the initial and terminal vertices of the replacement graphs are not connected by an edge;
3. the replacement graphs have at least three vertices and two edges.

Essentially all examples that we give are expanding, but there are interesting rearrangement groups arising from certain non-expanding (but still somehow well-behaved) replacement systems.

Proposition 2.6 (Proposition 1.9 of [BF19])**.** *If the replacement system is expanding, its gluing relation is an equivalence relation.*

Also note that the symbol space of an expanding replacement system is always a Cantor space (with the subspace topology coming from $\mathcal{E}^\omega$, where $\mathcal{E}$ here denotes the set of all edges of the base and replacement graphs).

Figure 2.3: The interval replacement system.

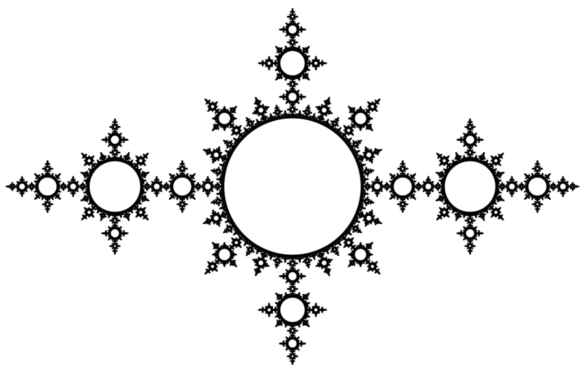


Figure 2.4: The airplane limit space.

2.1.3 The limit space

Definition 2.7. Let $\mathcal{R}$ be an expanding replacement system with gluing relation $\sim$ and symbol space $\Omega_{\mathcal{R}}$. The **limit space** of $\mathcal{R}$ is the quotient $\Omega_{\mathcal{R}}/\sim$.

Example 2.8 (The binary coding for the unit interval)**.** Consider the replacement system depicted in Figure 2.3. The symbol space is the binary Cantor space $\mathfrak{C}_2 = \{0,1\}^\omega$ and the gluing relation is

$$w1\bar{0} = w1000\ldots \sim w0\bar{1} = w0111\ldots$$

for all finite words w in the alphabet $\{0,1\}$. The limit space is then the unit interval, where each non-trivial equivalence class simply consists of the two binary expansions for that number. For instance, $1/2$ can be written as $0\bar{1} = 0111\ldots$ and as $1\bar{0} = 1000\ldots$.

The limit space for the airplane replacement system is a homeomorphic copy of the airplane Julia set and it is depicted in Figure 2.4. One can also build fractals such as the basilica and Douady rabbit Julia sets [DT25, Appendix C] and Ważewski dendrites [Tar23b].

Exercise 2.9. Find replacement systems for the circle S^1 and for the binary Cantor space $\mathcal{C}_2$.

Theorem 2.10 (Theorem 1.25 of [BF19])**.** *Limit spaces of expanding replacement systems are compact metrizable spaces.*

Theorem 2.11 ([PT25])**.** *The gluing relation is a rational relation (i.e., there exists a finite state automaton that read pairs of elements of the symbol space if and only if they are equivalent under the gluing relation).*

2.2 Rearrangement groups of limit spaces

Let us discuss a few other notions about limit spaces and then, finally, we will introduce rearrangements.

2.2.1 Cells and cellular partitions

Points of the limit space will be written as $[\![\alpha]\!]$, where $\alpha \in \Omega_{\mathcal{R}}$ is a representative of the gluing class.

Given a finite word w in the language of $\mathcal{R}$, a **cell** of the limit space is

$$[\![w]\!] \coloneqq \{[\![w\alpha]\!] \mid w\alpha \in \Omega_{\mathcal{R}}\},$$

which is the image of the cone $C(w)$ of $\Omega_{\mathcal{R}}$ (the set of elements of the symbol space with w as a prefix) via the quotient map. Note then that $[\![A]\!]$ has a different meaning depending on whether A is an infinite sequence in $\Omega_{\mathcal{R}}$ (then it is a point of the limit space) or a finite admissible word (then it is a cell, which is a subset of the limit space).

Every cell has one or two **boundary points**, depending on whether w is a loop or not. In Example 2.8, the boundary points of a cell $[\![w]\!]$ are the points $[\![w\bar{0}]\!]$ and $[\![w\bar{1}]\!]$. (Formally, a boundary point of $[\![w]\!]$ is defined as a point $[\![\alpha]\!]$ such that each prefix of α is an edge that is incident on a boundary vertex of the edge w.)

Definition 2.12. A **cellular partition** of the limit space is a collection of cells $\{[\![w_1]\!], \dots, [\![w_k]\!]\}$ such that $[\![w_1]\!] \cup \cdots \cup [\![w_k]\!]$ is the whole limit space and that any pairwise intersection $[\![w_i]\!] \cap [\![w_j]\!]$ is empty or consists of one or two boundary points.

In Example 2.8, cellular partitions correspond to dyadic subdivisions.

2.2.2 Rearrangements

Consider two cells $[\![a]\!]$ and $[\![b]\!]$ of the same color, either both loops or both non-loops (in this case we say that they are of the same *type*). Then there is a **canonical homeomorphism** $[\![a]\!] \to [\![b]\!]$ given by

$$[\![a\alpha]\!] \mapsto [\![b\alpha]\!].$$

In Example 2.8, a canonical homeomorphism is the unique affine map between two standard dyadic intervals.

Definition 2.13. A **rearrangement** is a self-homeomorphism of the limit space that restricts as a canonical homeomorphism on the cells on some cellular partition.

Following Example 2.8 once more, rearrangements are precisely the elements of Thompson's group F.

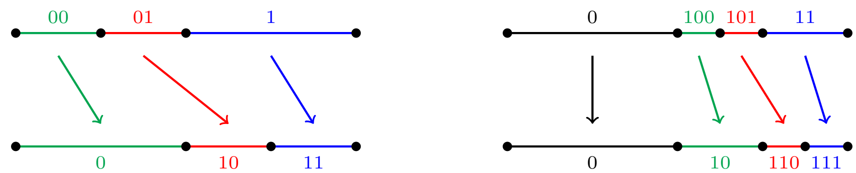


Figure 2.5: Graph pair diagrams for X_0 and X_1.

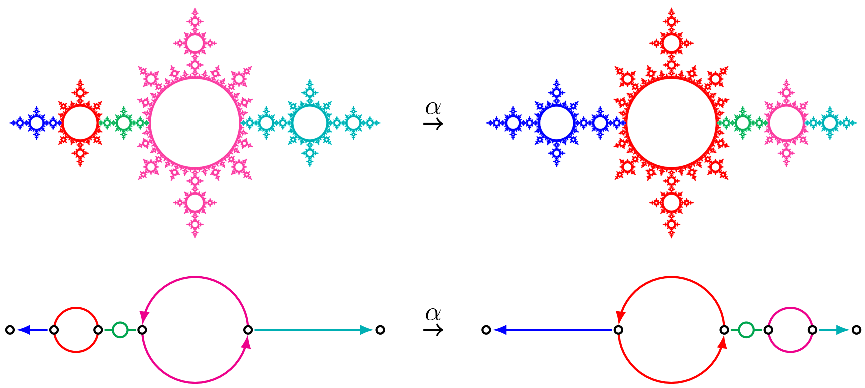


Figure 2.6: A rearrangement of the airplane limit space, along with a graph pair diagram that represents it.

Each rearrangement can be represented by a **graph pair diagram**, which is a triple (D, ϕ, R) where D and R are graph expansions and ϕ is a graph isomorphism. A graph pair diagram (D, ϕ, R) represents the rearrangement that restricts to canonical homeomorphisms $[\![e]\!] \to [\![\phi(e)]\!]$, for all edges e of D.

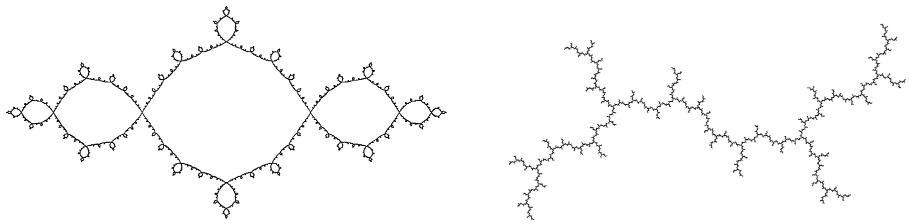
For our Example 2.8, a graph pair diagram naturally corresponds to a pair of dyadic subdivisions of the interval with the same number of standard dyadic intervals (the graph isomorphism is then uniquely determined). For instance, Figure 2.5 shows the graph pair diagrams for the generators X_0 and X_1 of F.

Going back to the airplane replacement system (Figure 2.1), a rearrangement and its graph pair diagram are depicted in Figure 2.6 (colors describe the graph isomorphism and each colored piece is only "rescaled" and rigidly moved by the canonical homeomorphisms).

Composition of graph pair diagrams works exactly like that of Thompson groups: to compute $(A, \phi, B) \circ (C, \psi, D)$ one needs to compute a common expansion E of A and D, then expand both diagrams to (E, ϕ', B') and (C', ψ', E), and finally the composition is $(C, \phi' \circ \psi', B')$.

The following groups (introduced prior to replacement systems) can be realized as rearrangement groups.

- Thompson groups F, T and V.
- The Higman-Thompson groups $F_{n,k}$, $T_{n,k}$, $V_{n,k}$.



(a) The basilica limit space. (b) A dendrite limit space.

Figure 2.7: More limit spaces.

- Topological full groups of edge shifts.
- The Thompson-like groups QF, QT and QV (non-expanding replacement systems).
- The Houghton groups H_n (non-expanding replacement systems).

2.2.3 The airplane rearrangement group

Let us write T_A for the rearrangement group associated to the airplane replacement system (Figure 2.1).

Theorem 2.14 ([Tar24b]). *The airplane rearrangement group T_A enjoys the following properties:*

- *T_A is generated by a copy of Thompson's group F together with a copy of Thompson's group T and thus it is finitely generated.*
- *T_A is finitely presented (actually F_∞) [PT24b].*
- *The abelianization T_A/T'_A is $\mathbb{Z}$.*
- *The commutator subgroup T'_A is simple.*
- *The commutator subgroup T'_A is finitely generated.*
- *T_A embeds into Thompson's group T.*

Other rearrangement groups of fractals include the basilica rearrangement group T_B from [BF15] acting on the basilica limit space depicted in Figure 2.7a (it is similar properties to T_A in many regards, but is virtually simple and is not finitely presented [WZ19]) and the dendrite rearrangement groups from [Tar23b] acting on Ważewski dendrites as the one depicted in Figure 2.7b (this is a countable family of groups that, differently from T_A and T_B, do not embed into T).

2.2.4 More results about rearrangement groups

Finiteness properties

Theorem 2.15 (Theorem 4.1 [BF19]). *Let $\mathcal{R}$ be a replacement system with finite branching (i.e., with bounded vertex degrees in its graph expansions). Let $\Gamma(\mathcal{R})$ be collection of all graphs that are obtained by finite sequences of edge expansions and reductions from the base graph of $\mathcal{R}$. Assume that, for every $m \geq 1$, all but finitely many graphs of $\Gamma(\mathcal{R})$ admit at least m distinct reductions. Then the rearrangement group of $\mathcal{R}$ is of type F_∞.*

This applies to F, T and V, but it does not apply to the airplane rearrangement group T_A nor to dendrite rearrangement groups. A similar but more flexible statement that works for such groups is included in [PT24b].

Lack of invariable generation

Definition 2.16. A group G is **invariably generated** if there exists a subset $S \subseteq G$ such that, for every choice $g_s \in G$ for $s \in S$, the group G is generated by $\{s^{g_s} \mid s \in S\}$ (with g^h we mean $h^{-1}gh$).

Theorem 2.17 (Main Theorem of [PT24a]). *Every CO-transitive subgroup G of a rearrangement group $G_\mathcal{X}$ is not invariably generated.*

A group acting on a topological space is **CO-transitive** (which stands for *compact-open transitive*) if it can map any "large" (proper) compact subset inside any "small" open subset. For a rearrangement group, this property can be fully characterized with cells. Except for Thompson's group F, the Houghton groups and the groups QF, QT and QV, every other known rearrangement group is CO-transitive and thus not invariably generated.

Finite subgroups

Theorem 2.18 (Theorem 2.9 [BF19]). *Every finite subgroup of a rearrangement group is a subgroup of the automorphism group of some graph expansion of the replacement system.*

Lecture 3

The conjugacy problem in rearrangement groups

The contents of this chapter are from [Tar23a] or [Tar24a, Chapter 5].

A problem is **solvable** if there exists an algorithm that infallibly (and correctly) answers *yes* or *no* in finite time. Given a fixed group, the **conjugacy problem** is the problem of deciding whether two given elements are conjugate or not in the given group.

3.1 An example: the conjugacy problem in free groups

The conjugacy problem is solvable in any finitely generated free group. Let us see an easy algorithm that solves the problem by computing, for each element, a unique "minimal" representative for each conjugacy class.

Given a word $A = a_1 \dots a_k$, we can consider the associated **cyclic word**, which is the equivalence class under cyclically permuting the digits. For instance, $a_2 a_3 \dots a_k a_1$ and $a_k a_1 \dots a_{k-2} a_{k-1}$ are elements of the free groups whose cyclic words are the same as that of $a_1 \dots a_k$. Free reductions can also be performed on cyclic words. See Figure 3.1 for an illustration of this.

1. Let $A = a_1 \dots a_n$ and $B = b_1 \dots b_m$ be elements of a free group.
2. Consider the cyclic words obtained from A and B.
3. Freely reduce the cyclic words until no reduction can be performed, finding words A' and B'.
4. If A' and B' correspond to the same cyclic word, then A and B are conjugate. Otherwise, A and B are not conjugate.

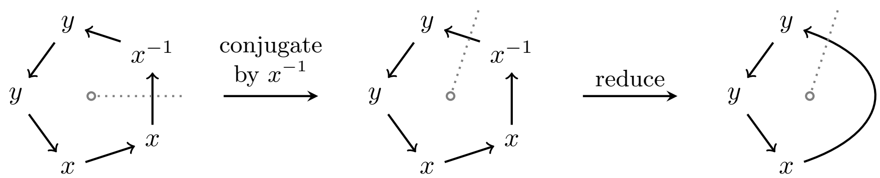


Figure 3.1: An illustration of why $x^{-1}y^2x^2$ and xy^2 are conjugate in the free group. The dotted lines allow to recover a word from a cyclic word and, keeping track of it, one can recover a conjugator.

This sort of algorithm applied to cyclic diagrams and transformation moves like that shown in Figure 3.1 will be used to solve the conjugacy problem in many rearrangement groups, with a key difference: we will not work with generators of the groups, but instead with generators of a groupoid.

3.2 Strand diagrams for rearrangements

In order to have some sort of *cyclic diagram* for the conjugacy classes, we need a representation of rearrangements with diagrams with a "beginning" and an "end" that can be glued together, as we did in Figure 3.1 for words (elements of a free group). This is why we introduce *strand diagrams*.

Throughout this section, let (X_0, R, C) be a fixed replacement system. We will use the airplane replacement system as a guiding example (Figure 2.1), but we modify it by replacing the base graph with its expansion (so the base graph is a copy of the blue replacement graph). This does not change the rearrangement group nor the limit space and it allows us to draw smaller pictures.

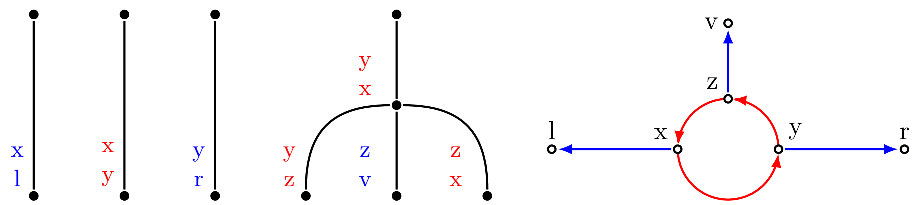
3.2.1 The forest of expansions

Expansions of the base graphs can be represented by labeled forests, as we explain here. We will implicitly equip each forest with an ordering of the roots and of the children of each node. Since there are multiple graphs involved in this construction, we will refer to the edges of forests with the term *strands*.

The base forest

Let $\mathcal{F}_0$ denote a forest consisting simply of a pair of vertices joined by one strand for each edge of the base graph X_0. If the edge of X_0 is $v \to w$, then we label the corresponding strand by a quadruple (v, w, c, i), where c is the color of the edge and i is an additional symbol for distinguishing parallel edges (i.e., if there are multiple edges $v \to w$, then the strands are labeled by $(v, w, 1), \dots, (v, w, k)$), which can be omitted when there are no parallel edges. We call such forest $\mathcal{F}_0$ the **base forest**. See for example Figure 3.2a. Note that, in pictures, we will color the label instead of displaying a symbol for the color, and we will often just write (v, w) for the label when the color is irrelevant.

(a) The forest expansion that corresponds to the base graph X_0.



(b) A forest expansion and the corresponding graph expansion $X_0 \triangleleft (y \to x)$.

Figure 3.2: Forest expansions of the airplane replacement system.

The replacement trees

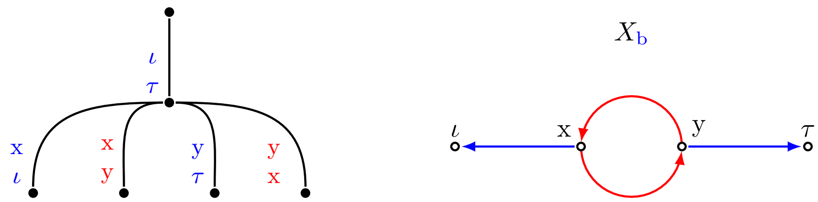
The base forest $\mathcal{F}_0$ can be expanded by appending to a leaf a copy of a **replacement tree**, which is a tree consisting of a strand labeled by (ι, τ) and colored by c from which multiple strands depart, one for each edge of the replacement graph X_c, each colored and labeled according to the edge of X_c. See Figure 3.3 for examples. When appending a replacement tree, the label (ι, τ) of the top strand needs to be changed to (v, w) according to the labels of the strand being expanded, changing each ι to v and each τ to w.

Forest expansions

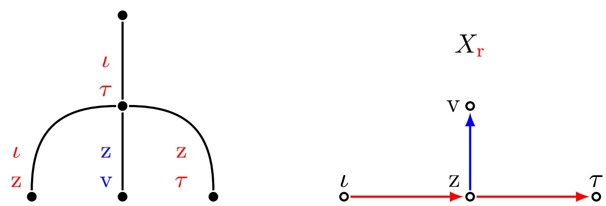
Each expansion of the replacement system can be represented by a forest: for instance, the expansion $X_0 \triangleleft e_1 \triangleleft e_2$ corresponds to the forest obtained by attaching the replacement tree to the strand corresponding to e_1, then attaching the replacement tree corresponding to e_2 to the corresponding edge. Figure 3.2b portrays an example. We will refer to such forests as **forest expansions**. Note that there is a bijective correspondence between forest expansions and graph expansions.

It is useful to note that the forest essentially consists of two pieces of information: the labels of the bottom strands describe the graph structure of a graph expansion (i.e., edge adjacencies), and the overall forest specifies which edges were expanded to obtain such graph (i.e., the coding of the edges in the alphabet of edges of the base and replacement graphs, see Section 2.1.1).

Moreover, keep in mind that the names symbols used to name vertices do not matter: two forest expansions are equivalent if one can be obtained from the other by simply renaming each symbol to something else.



(a) The blue replacement tree T_{b} beside the blue replacement graph.



(b) The red replacement tree T_{r} beside the red replacement graph.

Figure 3.3: The replacement trees of the airplane replacement system.

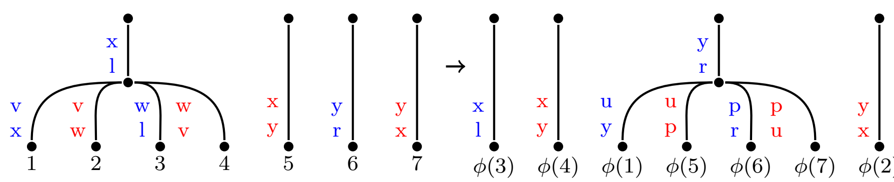


Figure 3.4: A forest pair diagram for the rearrangement portrayed in Figure 2.6.

3.2.2 Forest pair diagrams

A graph pair diagram (D, ϕ, R) can be equivalently represented by (T_D, ϕ_T, T_R), where T_D and T_R are the forest expansions corresponding to D and R and ϕ_T is a bijection between the leaves of the two forests corresponding to the graph isomorphism ϕ. Such diagrams are called **forest pair diagrams**. See Figure 3.4 for an example.

Remark 3.1. In this example we are implicitly using the fact that the orientation of blue edges does not matter in the airplane replacement system, so a blue label (v, w) is equivalent to a blue label (w, v). Essentially, this is because the expansion of a blue edge (v, w) is isomorphic to the expansion of a blue edge (w, v), so any rearrangement is allowed to swap the orientation of a blue edge up to an expansion of such edge.

Since the symbols used to refer to vertices of graph expansions do not matter, one can always rename those of the range forest expansion T_R so that the label corresponding to $\phi(e)$ in T_R is the same as the label corresponding to e in T_D, as done in Figure 3.5. This allows to express ϕ_T entirely via the labelings of the two forests, and in the remainder of these notes we will always assume that the

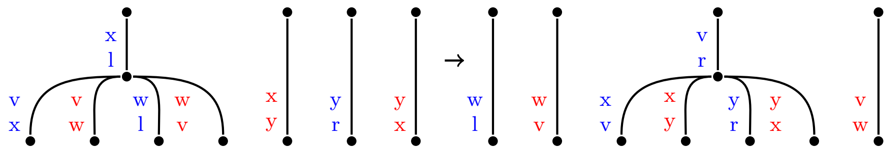


Figure 3.5: A forest pair diagram for the same element represented in Figure 3.4 after a renaming of symbols so that ϕ is determined by the labeling.

symbols of a forest pair diagrams are chosen in this way.

3.2.3 Strand diagrams and the replacement groupoid

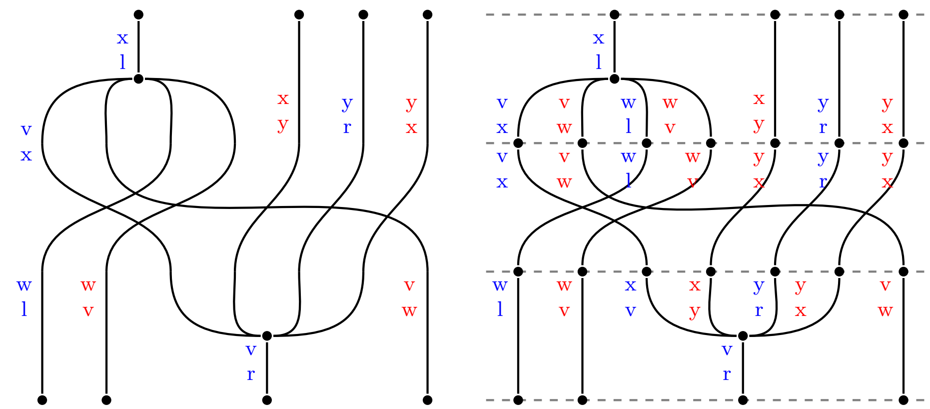
The leaves of the two forests of a forest pair diagrams can be glued as shown in Figure 3.6a. Strands that are glued have the same label, so they can be merged. Diagrams like this are called **strand diagrams**. These can be composed by gluing the top strands of a strand diagram with the bottom strands of another, after a renaming of the symbols of one (or both) strand diagrams so that the labels of the strands that are glued coincide.

Note that every strand diagram can be decomposed into a sequence of expansions, permutations and reductions, for example as in Figure 3.6b. Such small diagrams are called **expansion**, **permutation** and **reduction diagrams**, respectively.

Note that, in every diagram obtained by gluing the two forests of a forest pair diagram (as explained above), the top and the bottom strands are always labeled in such a way that represent the base graph X_0. If one lifts this assumption, they are left with a groupoid of generalized rearrangements whose elements are all strand diagrams that are generated by all of the (infinitely many) expansion, permutation and reduction diagrams. For instance, each of the three pieces into which the diagram in Figure 3.6b decomposes, as well as the composition of the first two and of the last two, are elements of this groupoid, which we will call **rearrangement groupoid**. A rearrangement groupoid depends on the replacement rules (i.e., the colors and on the replacement graphs R), but not on the base graph. In fact, these generalized rearrangements simply describe graph isomorphisms between graph expansions of all possible replacement systems based on a fixed set of replacement rules.

3.3 Closed strand diagrams and conjugacy classes

The advantage of using strand diagrams to describe the elements of a rearrangement group is that such diagrams have a "beginning" and an "end": the top and the bottom strands represent isomorphic graphs, so they can be glued together, as shown for example in Figure 3.7. We keep track of where the gluing happened using a *base line*, which is simply an ordering of the points that were glued, called the *base points*.



(a) The strand diagram for the forest pair diagram of Figure 3.5.

(b) The decomposition into generators of the rearrangement groupoid

Figure 3.6: Strand diagrams.

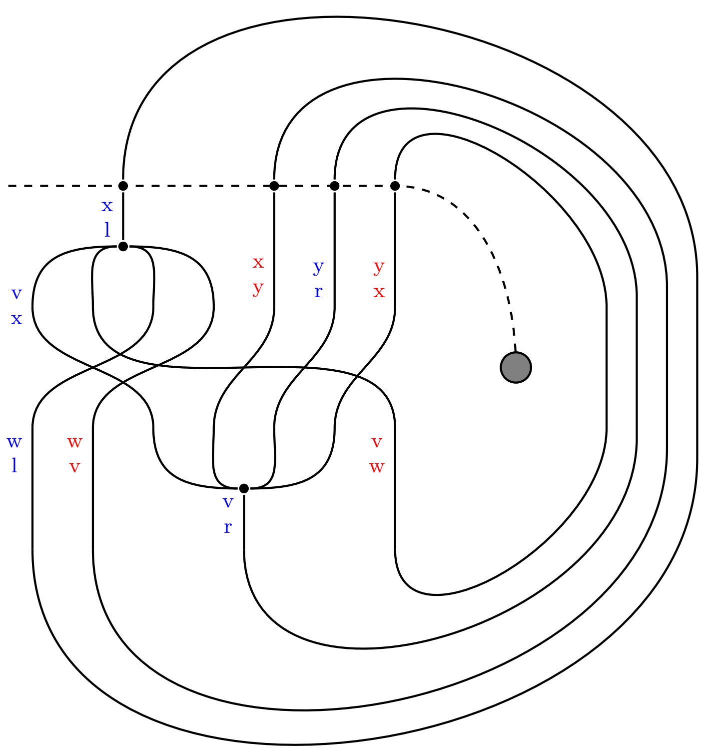


Figure 3.7: The closed strand diagram obtained from the strand diagram depicted in Figure 3.6. The base line is represented by the dashed line.

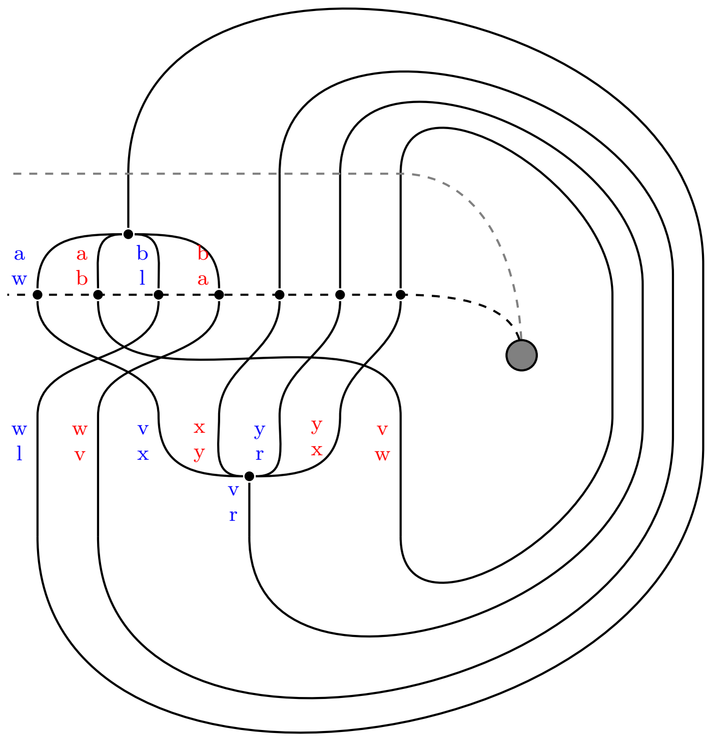


Figure 3.8: A shift of the diagram depicted in Figure 3.6 (the previous position of the base line is shown in gray).

3.3.1 Similarities of closed strand diagrams

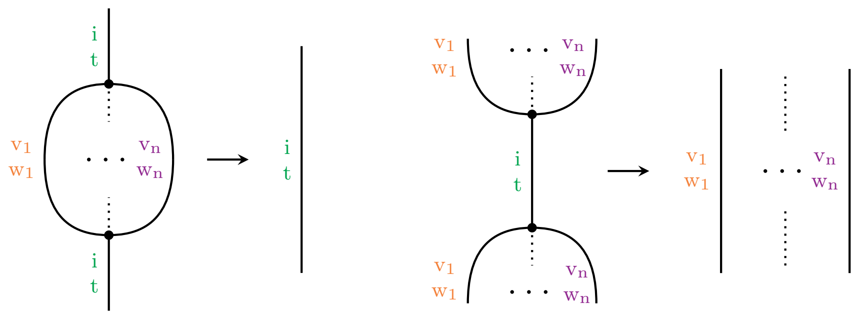
A **permutation** of a closed strand diagram consists of permuting the order of the base points.

Lemma 3.2. *A permutation move on the closed strand diagram corresponds to conjugacy by a permutation diagram in the groupoid of generalized rearrangements.*

A **shift** of a closed strand diagram consists of moving the base line through an expansion or a reduction, modifying symbols accordingly, as in Figure 3.8. There are rules for how symbols behave, which guarantee that cutting through the base line always results in a correctly labeled generalized rearrangement (an element of the rearrangement groupoid).

Lemma 3.3. *A shift move on the closed strand diagram corresponds to conjugacy by an expansion or reduction diagram in the groupoid of generalized rearrangements.*

Overall, we refer to permutations and shifts with the term **similarities**. The meaning of these transformations will be discussed later (Proposition 3.8).



(a) Type 1: an X-split on top of an X-merge produces a single strand.

(b) Type 2: an X-merge on top of an X-split produces multiple strands.

Figure 3.9: Types 1 and 2 reduction. Each strand should be labeled and colored according to the replacement rules.

Remark 3.4. It is possible to (algorithmically) determine whether two diagrams are similar: one needs to forget about the base line, which determines the ordering o the base points and the winding numbers (in other terms, it determines a cohomology class).

3.3.2 Reductions of closed strand diagrams

Closed strand diagrams can also be reduced with the following three types of moves, which we call **reductions**.

Types 1 and 2 reductions

Figure 3.9 schematically depicts reductions of types 1 and 2. These reductions can also be performed on non-closed strand diagrams. In closed strand diagrams, these reductions can only be performed on pieces of diagrams that do not cross the base line. In general, because of this, a type 1 of 2 reduction may need to be "unlocked" by a shift.

Type 1 reductions reflect the fact that expanding an edge and then reducing the resulting subgraph has no effect. Symmetrically, type 2 reductions reflect the fact that reducing a subgraph and then expanding the resulting edge has no effect. It is clear, thus, that such reductions do not modify the underlying rearrangement.

Type 3 reductions

Figure 3.10 schematically depicts reductions of type 3. These occur when there are parallel strands that do not expand nor reduce, such that they meet the base line in consecutive copies of a replacement graph. The number of such copies is what we refer to as *winding number*, as it is depicted as the amount of times a looping strand goes around the central "hole". In general, a type 3 reduction may need to be "unlocked" by a permutation.

Type 3 reductions reflect the fact that, sometimes, the base graphs of the domain and range graphs of a generalized rearrangement may be reduced. This transformation changes the element, as we will see later (Proposition 3.8).

Lemma 3.5. *A type 3 reduction on the closed strand diagram corresponds to conjugacy by one or more reduction diagrams in the groupoid of generalized rearrangements.*

3.3.3 Reduction-confluence of replacement rules

When talking about replacement systems, by the term *reduction* we will mean the inverse of an expansion. This is the transformation of a graph Γ given by replacing a subgraph Y isomorphic to some replacement graph X_c (up to an identification of the initial and terminal vertices of X_c) with an edge colored by c, provided this would not leave "dangling edges". (More formally, this means that, whenever some edge of $\Gamma \setminus Y$ is adjacent to a vertex of Y, that vertex must correspond to the initial or terminal vertex of X_c under the isomorphism between X_c and Y.)

Note that these reductions are not the same as reductions of closed strand diagrams. Moreover, these are not the same but correspond to reduction diagrams (one of the three types of generators of the rearrangement groupoid).

A set of replacement rules (R, C) is **reduction-confluent** if the rewriting system whose rewritings are the reductions is locally confluent. This means that, whenever $a \dashrightarrow b$ and $a \dashrightarrow c$ are two finite sequences of reductions of a, there exist a graph expansion d and two finite sequences of reductions $b \dashrightarrow d$ and $c \dashrightarrow d$.

For example, the airplane replacement rules are not reduction-confluent, as shown in Figure 3.11. It is not hard to show that, instead, essentially all other replacement rules considered in the literature (such as those for the basilica and dendrites, as well as those for Thompson groups F, T and V) are reduction-confluent.

Let us consider *similarity classes* (equivalence classes under similarities, which are permutations and shifts). We say that a reduction can be applied to a similarity class if it can be applied to some diagram in the class. As we mentioned, this allows us to "unlock" reductions by using permutations and shifts. We say that a similarity class is **reduced** if no reduction can be performed on it.

Lemma 3.6. *If the replacement rules are reduction-confluent, then each similarity class is reduced to a unique reduced similarity class.*

This lemma essentially tells us that, for reduction-confluent replacement rules, the order in which we reduce closed strand diagrams does not matter: we always reach the same reduced closed strand diagram (up to similarities).

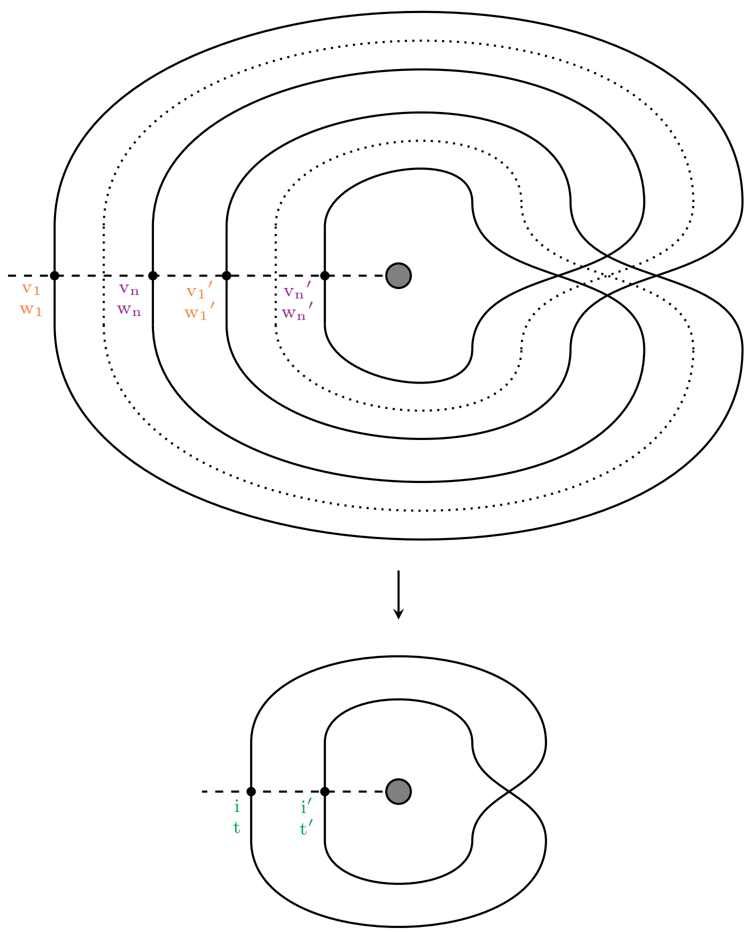
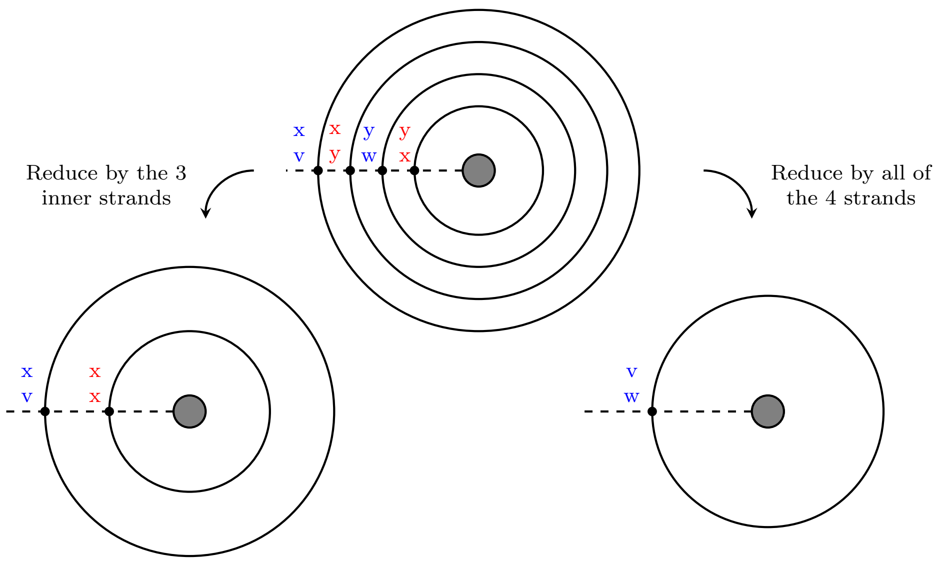
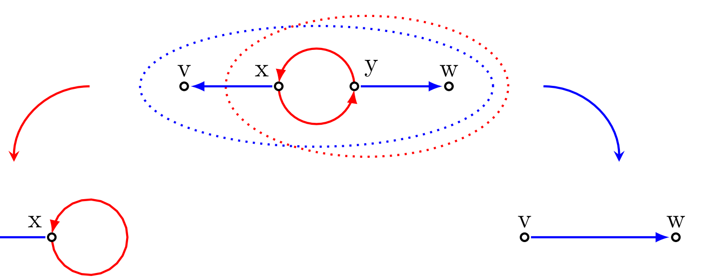


Figure 3.10: A schematic depiction of a Type 3 reduction with winding number equal to 2.



(a) Two reduced closed strand diagrams obtained from the identity of T_A.



(b) The graph reductions corresponding to the strand diagram reductions shown above.

Figure 3.11: The reason why the airplane replacement system is not reduction-confluent.

The proof of Lemma 3.6 relies on two facts: 1) that two reductions of the same diagram that “intersect” are either a type 1 and a type 2 (it can be seen that this is fixed by a type 3 reduction) or are both of type 3, and 2) that type 3 reductions correspond precisely to reductions of graphs, which are confluent by hypothesis.

3.3.4 Closed strand diagrams represent conjugacy classes

Remark 3.7. Understanding conjugacy in the groupoid allows to understand conjugacy in the groups. More precisely, If two elements of a rearrangement group G are conjugate in the groupoid, then any conjugator in the groupoid belongs G.

Proposition 3.8. *If two elements of a rearrangement group produce a common similarity class when reducing their closed diagrams, then they are conjugate in the rearrangement group.*

Proof. The is because similarities and reductions correspond to (possibly trivial) conjugacy, as we mentioned in previous lemmas. Let us list every case.

Types 1 and 2 reductions do not change the element represented by a diagram, so they correspond to trivial conjugacy.

Type 3 reductions correspond to conjugating by certain expansion diagrams (in the case in which the base graph has a periodic subgraph that can be reduced).

Permutations correspond to conjugating by permutation diagrams (the same permutation by which one permutes the base points).

Shifts correspond to conjugating by certain expansion or reduction diagrams (corresponding to the expansion or reduction the base line moved through when performing the shift). □

It is easy to check that the converse statement holds: if two elements are conjugate, then they produce a common similarity class (if $g = x^{-1}hx$, the element x can effectively be reduced in the closed strand diagram for g, leaving only h).

Note that the previous Proposition 3.8 holds without assuming reduction-confluence. When assuming reduction-confluence, by Lemma 3.6 and Proposition 3.8, one can solve the conjugacy problem using the following procedure.

1. Given two elements, compute their closed strand diagrams.
2. Compute their reduced closed strand diagrams.
   (a) To do this, first check whether reductions can be performed when forgetting about the base line.
   (b) If reductions can be performed, do that (possibly needing to first perform a shift or a permutation) until no more are available. By Lemma 3.6, the order in which reductions are performed does not matter.

3. Check whether the two reduced closed strand diagrams are similar (check isomorphism as graphs, cohomology and labeling).

4. If the reduced closed strand diagrams are similar, the elements are conjugate. Otherwise, they are not.

Theorem 3.9. *The conjugacy problem of any rearrangement group defined by reduction-confluent replacement rules is solvable.*

Corollary 3.10. *The conjugacy problem is solvable in the following groups.*

- *The basilica and rabbit rearrangement groups.*
- *The dendrite rearrangement groups.*
- *the Thompson-like groups QV, QT and QF.*

Also in the following groups, in which the conjugacy problem was previously shown to be solvable.

- *Thompson groups F, T and V (previously solved in multiple papers, including [BM14] which inspired this paper).*
- *The Higman-Thompson groups (previously solved in [BDR16]).*
- *The Houghton groups (previously solved in [ABM15]).*

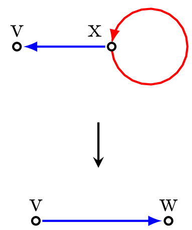
The conjugacy problem is also solvable in the airplane rearrangement group T_A with these methods, but some additional arguments are needed, as discussed in the next (and last) subsection.

3.3.5 Reduction-confluence is not necessary: the airplane

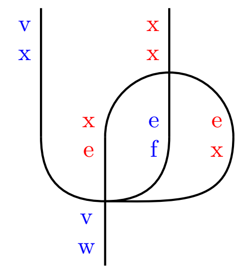
We mentioned that the airplane replacement rules are not reduction-confluent (see Figure 3.11), so Theorem 3.9 does not apply. However, the conjugacy problem can be solved simply by adding one graph reduction rule and the associated type 3 reduction of closed strand diagrams to the list of moves that we allow, the one portrayed in Figure 3.12a.

The additional graph reduction rule is a composition of expansions and reductions, so performing a type 3 reduction associated to it corresponds to conjugating by the diagram depicted in Figure 3.12b. This means that reductions still correspond to conjugacy. It can be checked that the issue portrayed in Figure 3.11b is essentially the only obstacle to reduction-confluence in the airplane and that the inclusion of this additional rule circumvents it. Ultimately, this solves the conjugacy problem in T_A too.

Sometimes (for instance, in the replacement system depicted in Figure 3.13) adding the graph reduction rule that seems to solve the issue ends up producing further problems (see Figure 3.14). Thus, it is still unclear when similar strategies can always be applied to circumvent the lack of reduction-confluence. Overall, these questions fit into the topic of *graph rewriting systems*, which is currently of interest in computer science.



(a) The additional graph reduction needed for solving the conjugacy problem in T_A.



(b) The conjugator corresponding to a Type 3 reduction associated with the graph reduction shown in Figure 3.12a.

Figure 3.12: The additional transformations needed to solve the conjugacy problem in the airplane rearrangement group T_A.

Figure 3.13: The red and blue replacement graphs for a non confluent-reduction set of replacement rules where closed strand diagrams do not seem to easily solve the conjugacy problem.

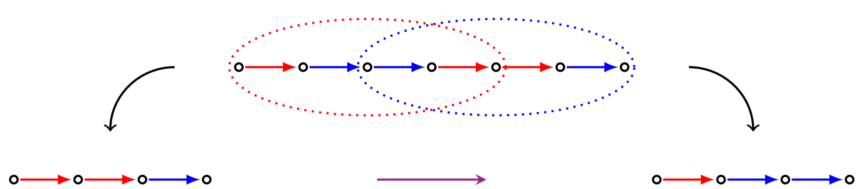


Figure 3.14: Adding this reduction to the reduction system associated to the replacement rules of Figure 3.13 does not make the reduction system confluent.

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