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Jérémie Brieussel

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Croissance et moyennabilité de certains groupes d'automorphismes d'un arbre enraciné

Directeur :	Zuk Andrzej	Université Paris 7, France
Rapporteurs :	Bartholdi Laurent	ÉPF Lausanne, Suisse
	Wilson John	Université Oxford, GB

Thèse soutenue le 29 septembre 2008 devant le jury composé de :

Kato Tsuyoshi	Université Kyoto, Japon, Professeur
Skandalis Georges	Université Paris 7, France, Professeur
Valette Alain	Université Neuchâtel, Suisse, Professeur
Wilson John	Université Oxford, GB, Professeur
Zuk Andrzej	Université Paris 7, France, Professeur

“Prenez un bout de bois, cassez-le en deux bouts,
il y aura toujours deux bouts à chaque bout!”

R. Devos [Dev00]

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Introduction

0.1 Problèmes

Cette thèse a pour objet l'étude de la croissance des groupes et de propriétés proches comme la moyennabilité. La notion de croissance de groupe a été introduite par Svarc [Sva55] et Milnor [Mil68b] dans le cadre de l'étude des variétés riemaniennes.

La fonction de croissance notée $b_{\Gamma,S}(R)$ d'un groupe de type fini Γ relativement à un système fini donné de générateurs S est le nombre d'éléments du groupe qui peuvent s'écrire comme produit d'au plus R générateurs et de leurs inverses. Le comportement asymptotique de la fonction de croissance est indépendant du système générateur choisi. En particulier, si $\Gamma = \pi_1(X)$ est le groupe fondamental d'une variété riemanienne fermée X, sa croissance est équivalente à la croissance du volume des boules riemaniennes dans le revêtement universel $\tilde{X}$.

La notion de croissance des groupes s'est émancipée de la géométrie riemanienne dès les travaux de Wolf [Wol68] démontrant qu'un groupe résoluble est à croissance soit exponentielle, soit polynomiale, et que dans ce dernier cas le groupe est virtuellement nilpotent. Ce résultat a mené Milnor à poser (à titre d'exercice !) les deux questions suivantes :

Questions 0.1.1. *Milnor [Mil68a]*

1. *Existe-t-il un groupe dont la croissance ne serait ni polynomiale, ni exponentielle ?*
2. *Existe-t-il des groupes à croissance polynomiale qui ne soient pas virtuellement nilpotents ?*

Dans son célèbre article *Groups of polynomial growth and expanding maps*, Gromov a répondu négativement à la deuxième question, ce qui permet de décrire entièrement la croissance polynomiale :

Théorème 0.1.2 (Gromov [Gro81a], Guivarc'h [Gui73], Pansu [Pan83])**.** *Si Γ est un groupe dont la fonction de croissance est bornée par un polynome, alors le groupe Γ est virtuellement nilpotent. En particulier, sa fonction de croissance vérifie :*

$$b_{\Gamma,S}(R) \sim_{R\to\infty} C_{\Gamma,S} R^{d(\Gamma)},$$

où $C_{\Gamma,S}$ est une constante positive et $d(\Gamma)$ est un entier qui dépend explicitement de la structure du groupe nilpotent d'indice fini dans Γ.

Peu après, Grigorchuk [Gri83] a répondu par l'affirmative à la première question, montrant que des groupes d'automates introduits par Aleshin [Ale72] sont solutions. De tels groupes sont dits à *croissance intermédiaire*. Ce sont les groupes non virtuellement nilpotents dont le coefficient de croissance exponentiel $h_\Gamma(S) = \lim \sqrt[R]{b_{\Gamma,S}(R)}$ vaut 1 (indépendemment du système générateur S). A contrario, pour les groupes à croissance exponentielle, ce coefficient excède strictement 1 pour tous les systèmes générateurs. La question de la dépendance vis-à-vis du système générateur choisi a été formalisée par Gromov :

Question 0.1.3. *Gromov (Remarque 5.12 dans [Gro81b]) Existe-t-il des groupes à croissance exponentielle non uniforme ? C'est-à-dire des groupes à croissance exponentielle pour lesquels on ait* $\inf_S h_\Gamma(S) = 1$.

Wilson a répondu par l'affirmative à cette question dans deux articles récents [Wil04b], [Wil04a]. L'étude des groupes à croissance intermédiaire et des groupes à croissance non uniforme présente de nombreuses similarités. Voyons d'abord des exemples de classes de groupes dans lesquels ces propriétés sont prescrites.

Il est évident que les groupes libres n'ont pas de telles propriétés, ce qui se généralise aux produits libres avec amalgamations et aux extensions HNN [BdlH00], ainsi qu'aux groupes hyperboliques [Kou98] et aux groupes à un relateur ayant croissance exponentielle [GdlH01].

L'alternative de Tits [Tit72] stipulant qu'un sous-groupe de type fini de $GL_n(K)$ contient un groupe libre s'il n'est pas résoluble, jointe au résultat de Wolf susmentionné, assure que les sous-groupes de type fini des groupes linéaires ne peuvent avoir croissance intermédiaire. Ce résultat a été étendu par Eskin, Mozes et Oh [EMO02] pour montrer que des groupes à croissance non uniforme ne sauraient non plus être linéaires.

Le résultat de Wolf assurant que les groupes résolubles n'ont pas croissance intermédiaire admet une double généralisation. Il a d'abord été étendu aux groupes élémentairement moyennables par Chou [Cho80]. Ensuite, Osin a montré que des groupes résolubles ont croissance exponentielle uniforme (si non polynomiale) [Osi03], avant de montrer la même chose pour les groupes élementairement moyennables [Osi04].

0.2 Exemples archétypaux

Le bon contexte pour fabriquer des groupes à croissance intermédiaire ou non uniforme est celui de groupes agissant sur un arbre enraciné ou de manière équivalente par échange d'intervalles sur l'espace $[0, 1] \setminus \mathbb{Q}$. La classe des groupes d'automates constitue une large classe d'exemples de groupes agissant de manière naturelle sur un arbre enraciné de valence fixe. Les premiers exemples de tels groupes ont été construits par Aleshin [Ale72], afin d'obtenir des groupes infinis de torsions. Des constructions similaires de groupes de torsion sont dues à Sushchansky [Sus79], Grigorchuk [Gri80] et à Gupta et Sidki [GS83].

Ces groupes de torsions constituaient les premiers exemples de groupes à croissance intermédiaire comme l'a montré Grigorchuk dans [Gri83]. Plus généralement, il a construit dans l'article *Degrees of growth of finitely generated groups and the theory of invariant means* une famille de groupes G_ω, agissant fidèlement sur un arbre binaire enraciné, indexée par des suites ω à valeurs dans un ensemble à trois éléments $\{0, 1, 2\}$ vérifiant le :

Théorème 0.2.1. *Grigorchuk [Gri85]*

1. *Si ω n'est pas asymptotiquement constante, le groupe G_ω a croissance intermediaire.*
2. *Si ω est homogène* [1], *le groupe G_ω est un groupe de torsion et l'on dispose d'estimations explicites ($\beta < 1$) :*

$$e^{R^{\frac{1}{2}}} \leq b_{G_\omega}(R) \leq e^{R^\beta}.$$

   De plus dans ce cas, le groupe est moyennable mais pas élémentairement moyennable (Chou [Cho80]).
3. *Il est possible d'obtenir une variété indénombrable de taux de croissance intérmediaire deux à deux non comparables et en particulier on peut obtenir des taux de croissance arbitrairement proches de la croissance exponentielle.*

Le troisième point du théorème est à mettre en contraste avec la conjecture du même auteur selon laquelle un groupe à croissance bornée par $e^{\sqrt{R}}$ est automatiquement à croissance polynomiale et donc virtuellement nilpotent. Cette conjecture a été prouvée pour les groupes résiduellement p-groupes (voir [BG00]) ou résiduellement nilpotents (voir [LM91]).

Dans la même veine, Grigorchuk et Zuk ont introduit un groupe G engendré par un automate à trois états sur un alphabet à deux lettres, c'est-à-dire agissant fidèlement sur un arbre binaire enraciné, ayant les propriétés suivantes :

Théorème 0.2.2.

1. *(Grigorchuk Zuk [GZ02a]) Le groupe G est sans torsion, à croissance exponentielle et n'est pas sous-exponentiellement moyennable.*
2. *(Bartholdi Virag [BV05]) Le groupe G est moyennable.*

En particulier, il s'agit du premier exemple de groupe moyennable hors de la classe SG des groupes sous exponentiellement moyennables (plus petite classe de groupes contenant les groupes à croissance sous exponentielle et stable par extension,

[1] ω est dite r-homogène si toute sous-suite de longueur r prend les trois valeurs 0, 1 et 2.

quotient, sous-groupe et limite directe). Ce groupe est aussi le groupe de monodromie itérée du polynome $z^2 - 1$ et est appellé groupe du "Basilica" dans la littérature.

Toujours avec des méthodes similaires, Wilson a répondu par l'affirmative à la question de Gromov en construisant des groupes à croissance non uniforme :

Théorème 0.2.3 (Wilson [Wil04a]). *1. Si χ est une classe de groupes parfaits engendrés par des involutions, et dont tous les groupes G sont isomorphes à un produit en couronne permutationnel $G_1 \wr \mathcal{A}_d$ pour un autre G_1 dans χ et où $\mathcal{A}_d$ est le groupe des permutations alternées d'un ensemble à d (≥ 29) éléments, alors chaque groupe de la classe χ admet une famille (S_n) de générateurs pour lesquels $h_G(S_n) \to 1$.*

2. Le groupe d'automorphisme d'un arbre enraciné de valence non bornée contient une telle classe χ de groupes contenant un groupe libre (en particulier à croissance exponentielle et non moyennables).

Tous les exemples ci-dessus sont groupes d'automorphismes d'arbres enracinés. La structure de l'arbre y joue un rôle prépondérant. Un arbre enraciné (disons sphériquement homogène) se décompose comme d sous-arbres enracinés attachés aux sommets du premier niveau, c'est-à-dire aux sommets à distance 1 de la racine (notée $\emptyset$) :

$$T = T_1 \cup \cdots \cup T_d \cup \{\emptyset\}.$$

Cette structure entraîne une propriété essentielle du groupe d'automorphismes d'un tel arbre T (c'est-à-dire les automorphismes de graphe préservant la racine) :

$$Aut(T) \simeq (Aut(T_1) \times \cdots \times Aut(T_d)) \rtimes S_d = Aut(T) \wr S_d,$$

où le groupe S_d des permutations de d éléments agit sur les coordonnées du produit direct. En identifiant canoniquement les arbres T_i, on retrouve la définition du produit en couronne permutationnel $\wr$. Ainsi, se donner un automorphisme g de l'arbre revient à se donner d automorphismes g_i d'un (autre) arbre et une permutation σ. On notera cette identification :

$$g = (g_1, \ldots, g_d)\sigma.$$

Un exemple essentiel de sous-groupes de type fini de $Aut(T)$ est fourni par les groupes d'automates (pour des arbres à valence fixée). Ce sont les groupes engendrés par une partie finie A appelée ensemble des états, dont les éléments ont des coordonnées a_i dans l'écriture $a = (a_1, \ldots, a_d)\sigma$ qui sont elles-mêmes dans A. On peut leur associer un automate au sens de l'informatique théorique (voir par exemple [Zuk06]).

Les exemples des trois théorèmes énoncés ci-dessus disposent d'une propriété particulière de réduction de la longueur des mots dans le produit en couronne. C'est par récurrence sur la longueur des mots que l'on montre que certains groupes sont de torsion, ou qu'ils en sont libres. Le lien entre la norme d'un élément et les normes de ses images dans le produit en couronne est à l'origine de toutes ces estimations sur la croissance. On a les énoncés informels suivants :

1. Si la longueur des mots réduits diminue lors du passage au produit en couronne, alors le groupe a croissance sous-exponentielle, et l'on obtient une borne d'autant meilleure que la diminution est forte.

   Plus généralement, une hypothèse de diminution montre que la croissance du groupe G est plus petite que celle du groupe image $G_1 \wr S_d$.
2. Inversement si un d-uplet de mots réduits admet un antécédent par produit en couronne de courte longueur, alors la croissance du groupe est rapide. La construction de certains semi-groupes libres est basée sur la même idée (voir remarque 2.3.4).
3. Si la longueur des mots aléatoires décroît lors du passage en couronne, le groupe est moyennable.

0.3 Résultats de cette thèse

Cette thèse comporte quatre chapitres. Les trois premiers, rédigés en anglais, sont des articles écrits pendant ma thèse. Le premier chapitre décrit un travail effectué essentiellement au cours de ma première année sous le titre *Lower bound on the growth of some torsion 2-groups*, le deuxième présente un travail récent toujours en cours sous le titre *Groups with oscillating growth function.* Le troisième chapitre, le plus conséquent, présente un théorème de moyennabilité et divers résultats sur les groupes à croissance non uniforme de Wilson ; il a été soumis au journal *Mathematische Zeitschrift* le 12 février 2008 sous le titre *Amenability and non uniform growth of some directed automorphism groups of a rooted tree.* Le quatrième chapitre (en francais) présente deux généralisations du théorème de moyennabilité du chapitre 3, mises en place après la parution sur arXiv d'un article de Bartholdi, Kaimanovich et Nekrashevych sur la moyennabilité des groupes d'automates [BKN08].

0.3.1 Chapitre 1 : Étude de bornes inférieures

Le résultat principal de ce premier chapitre est une amélioration de la borne inférieure sur la croissance des groupes G_ω. On y montre le :

Théorème 0.3.1. *1. Si $\omega = 012012\ldots$, la fonction de croissance de G_ω est minorée par $e^{R^{\alpha_3}}$ pour $\alpha_3 = 0.5207\ldots$.*

2. Plus généralement, si ω est r-homogène, il existe un exposant $\alpha_r > \frac{1}{2}$ tel que la fonction $e^{R^{\alpha_r}}$ minore la croissance de G_ω.

Les résultats obtenus sont numériquement décevants, la meilleure borne connue auparavant (Bartholdi [Bar01], par ordinateur) valant $\alpha_3 = 0.5157\ldots$. La borne inférieure reste donc très éloignée de la borne supérieure $e^{R^{\beta_3}}$ où $\beta_3 = 0.7674\ldots$. Par ailleurs, le nombre α_r s'approche exponentiellement vite de $\frac{1}{2}$ alors que l'on voudrait penser que $\alpha_r \geq \alpha_3$.

La méthode utilisée consiste à construire un antécédent pour le produit en couronne d'un couple (g_0, g_1) d'éléments de longueur R. Si l'antécédent est de longueur $\leq 4R$, on obtient la borne classique de Grigorchuk. En m'inspirant du travail effectué par Muchnik et Pak ([MP01]) sur la borne supérieure, j'ai montré que l'on peut construire, en utilisant un algorithme élaboré par Leonov [Leo01], un antécédent de longueur $\leq (4 - c\delta)R$ où δ est le nombre d'apparitions d'un générateur (noté d) spécifique du groupe dans le couple (g_0, g_1) et $c = \frac{3}{2}$ est une constante (voir Lemme de réduction 1.3.7).

Ceci permet d'améliorer la borne de Grigorchuk si l'on peut s'assurer que le générateur d apparaît suffisament. Malheureusement il n'est pas possible de satisfaire une telle hypothèse. Il s'agit alors d'appliquer la méthode une deuxième fois, et de s'assurer que les générateurs b contribuent à fournir des générateurs d. Au pire des cas, il faut pratiquer une troisième itération.

Cette méthode est limitée en pratique, d'abord parce que les estimations de réduction de longueur n'ont lieu que sous un certain régime de l'algorithme. Je discute dans la section 1.4.2 les estimations que l'on obtiendrait si l'on connaissait la durée du régime. En particulier, si cette durée est maximale (hypothèse vraisemblable), on améliore $\alpha_3 = 0.5476\ldots$. D'autre part, pour des raisons techniques, la troisième itération ne peut se faire via l'algorithme de Leonov, entrainant une baisse de qualité des estimations. Enfin, il est vraisemblable que la constante $c = \frac{3}{2}$ ne soit pas optimale.

0.3.2 Chapitre 2 : Groupes à croissance oscillante

Portant sur la même famille de groupe G_ω, ce chapitre a pour but de construire des groupes dont la fonction de croissance soit oscillante (on note $f \prec g$ pour $\frac{f}{g} \to 0$) :

Théorème 0.3.2. *Soit $\rho(R)$ et $\tau(R)$ deux fonctions vérifiant $R^{\beta_3} \prec \tau(R) \prec \rho(R) \prec R$, il existe une suite ω à valeurs dans $\{0, 1, 2\}$ telle que le groupe G_ω ait une fonction de croissance $b_\omega(R)$ vérifiant :*

$$\begin{aligned} \log(b_\omega(R)) &\geq \rho(R) \quad \textit{pour une infinité de } R, \\ \log(b_\omega(R)) &\leq \tau(R) \quad \textit{pour une infinité de } R. \end{aligned}$$

On dit que $\log(b_\omega(R))$ oscille entre $\tau(R)$ et $\rho(R)$.

On peut de plus construire de tels groupes qui soient de torsion.

On peut également construire de tels groupes sans torsion, quitte à remplacer la fonction R^{β_3} par une fonction sous-linéaire non explicite.

L'idée de la démonstration est de munir l'espace $\mathcal{Y}$ des groupes G_ω de la métrique de coïncidence sur les boules, pour laquelle deux groupes (munis de leurs parties génératrices respectives) sont proches si leurs graphes de Cayley coïncident sur de grandes boules. Quitte à modifier la définition du groupe G_ω pour certains ω (en quantité dénombrable), l'application

$$\begin{array}{rcl}\Psi : (\Omega, d) & \to & (\mathcal{Y}, d) \\ \omega & \mapsto & G_\omega,\end{array}$$

est alors continue, où $\Omega = \{0,1,2\}^{\mathbb{N}}$ est muni de la distance de coïncidence sur les parties finies. De plus, l'espace $\mathcal{Y}$ contient une partie dense dont les groupes ont une log croissance majorée par CR^{β_3} et une autre dont les fonctions de log croissance sont minorées par cR (ces propriétés dépendent seulement de l'asymptotique de la suite ω). La construction de cet espace, sous-jacente à l'article [Ale72] d'Aleshin, explicitée par Grigorchuk [Gri85], est détaillée dans les sections 2.2 et 2.3. Le résultat est démontré dans la section 2.4, où l'on fournit aussi des estimations sur la fréquence des oscillations.

La section 2.5 concerne le cas des groupes sans torsion, où l'on considère un pendant $\tilde{G}_\omega$ sans torsion des groupes G_ω, également introduit par Grigorchuk [Gri86]. De tels groupes agissent fidèlement sur un arbre enraciné de valence infinie :

$$T = \bigcup_{i \in \mathbb{Z}} T_i \cup \{\emptyset\},$$

où l'action sur l'arbre T_i dépend seulement de la parité de i.

0.3.3 Chapitres 3 et 4 : Résultats de moyennabilité

Les sections 3.2, 3.3, 3.4 et 3.5 du chapitre 3 présentent un résultat de moyennabilité pour les groupes dirigés d'automorphismes d'un arbre enraciné sphériquement homogène. Un tel arbre $T_{\bar{d}}$ est décrit par la suite $\bar{d} = (d_i)_{i \in \mathbb{N}}$ des valences de niveau i.

Un automorphisme g est dit enraciné s'il agit seulement en permutant les sous-arbres du premier niveau, l'ensemble de tels automorphismes forme un sous-groupe isomorphe au groupe S_{d_0} de permutations. Un automorphisme g est dit dirigé par une géodésique $\mathcal{G}$ si son portrait est trivial sur cette géodésique et contenu dans un 1-voisinage de celle-ci. Plus explicitement, on peut supposer que cette géodésique est celle de gauche $\mathcal{G} = (111\dots)$ et que l'automorphisme est de la forme $g = g_0$ avec les identifications par produit en couronne :

$$g_k = (g_{k+1}, \sigma_2(k), \dots, \sigma_{d_k}(k)),$$

où chaque $\sigma_i(k)$ est enraciné. On note $\bar{H}$ l'ensemble de tels automorphismes, il est isomorphe à un produit direct infini de groupes finis. On appelle groupe des automorphismes dirigés et l'on note $G(S_{d_0}, \bar{H})$ le sous-groupe de $Aut(T_{\bar{d}})$ engendré par S_{d_0} et $\bar{H}$. Le résultat principal du chapitre 3 est le :

Théorème 0.3.3. *Le groupe $G(S_{d_0}, \bar{H})$ est moyennable si et seulement si l'arbre $T_{\bar{d}}$ est de valence bornée.*

La nécessité est facile, car en valence non bornée le groupe $\bar{H}$ lui-même n'est pas moyennable. Pour montrer la suffisance, on se restreint au cas des sous-groupes de type fini, donc aux groupes de la forme $G(S_{d_0}, H)$ où H est un sous-groupe fini de $\bar{H}$. On peut de plus le supposer saturé, c'est-à-dire que sa projection sur chaque facteur du produit direct infini $\bar{H}$ est équidistribuée.

Dans ce cadre, on adopte une démarche similaire à celle de Bartholdi et Virag pour le théorème 0.2.2, dite méthode du "baron de Münchhausen" (qui ne coulait pas car sa tête coulait moins vite que ses pieds). Il s'agit de construire une pseudo-norme ν (plus précisement une famille ν_k indexée par le niveau k) qui augmente par passage au produit en couronne :

$$\nu(g) \leq \nu(g_1) + \cdots + \nu(g_d),$$

quand on écrit $g = (g_1, \ldots, g_d)\sigma$. On construit aussi une marche aléatoire (non symétrique) Y_n ayant une propriété d'auto-similarité, c'est à dire que lorsqu'on écrit les coordonnées $Y_n = (Y_n^1, \ldots, Y_n^d)\sigma_n$ dans le produit en couronne, chaque coordonnée Y_n^t se comporte comme la marche aléatoire d'origine Y_n mais pour un temps (aléatoire) n_t moindre. On a :

$$Y_n^t \sim_{loi} Y_{n_t}, \text{ pour un temps } n_t \sim_{n\to\infty} \left(\frac{d-1}{d}\right)\frac{n}{d} \text{ presque sûrement.}$$

Ces propriétés permettent de montrer que la vitesse d'échappement (drift) $\lim \frac{\nu(Y_n)}{n}$ est nulle. On en déduit alors que cette vitesse est encore nulle pour la marche aléatoire simple Z_N pour le système générateur $S_d \cup H$.

La pseudo-norme ν construite ayant une croissance des boules exponentielle, on peut appliquer le critère de moyennabilité de Kesten sur les probabilités de retour [Kes59b].

Le chapitre 4 fait suite à la parution sur arXiv d'un article de Bartholdi, Kaimanovich et Nekrashevych [BKN08] montrant la moyennabilité des groupes d'automates bornés. Il présente deux généralisations du théorème 0.3.3.

La première est une légère extension de la notion d'automorphisme dirigé. On dit qu'un automorphisme g est δ-dirigé par une géodésique $\mathcal{G}$ s'il fixe la géodésique et son portrait est contenu dans un δ-voisinage de celle-ci. On note $\bar{M}(\delta)$ le sous-groupe formé par ces automorphismes. Comme précédemment, $\bar{M}(\delta)$ est produit direct infini de groupes finis :

$$\bar{M}(\delta) = F_0 \times F_1 \times F_2 \times \ldots .$$

Théorème 0.3.4. *Le groupe $G(S_{d_0}, \bar{M}(\delta))$ des automorphismes δ-dirigés est moyennable.*

Le nombre δ est ici peu significatif, on se ramène à $\delta = 1$ en restreignant la structure d'arbre aux niveaux multiples de δ (ce qui augmente la valence). Ce résultat constitue une extension du résultat intermédiaire crucial dans l'article [BKN08] sur le “Mother group” qui n’est autre que $G(S_d, M)$ où M est l’injection diagonale d’un facteur du produit direct des F_i (valence fixe). On en déduit un dernier résultat qui généralise légèrement le théorème sur les groupes d’automates bornés.

On conclut le chapitre 4 par un résultat noté par Erschler stipulant que le groupe $G(S_2, \bar{H}) < Aut(T_2)$ des automorphismes dirigés d’un arbre binaire est plus que moyennable, il a croissance intermédiaire.

0.3.4 Chapitre 3 : Liens entre la croissance intermédiaire et la croissance non uniforme

Les sections 3.6, 3.7, 3.8 sont consacrées aux groupes à croissance non uniforme introduits par Wilson. Pour construire de tels groupes, Wilson utilise des parties génératrices très particulières du groupe de permutation alterné $\mathcal{A}_d$ (pour $d \geq 29$).

Ces générateurs, l’un d’ordre 2, l’autre d’ordre 3, décrits explicitement dans [Wil04a], disposent de propriétés de point fixe, qui permettent d’obtenir des réductions lors du passage au produit en couronne. C’est un calcul basé sur cette propriété qui permet à Wilson de montrer une diminution de $h_G(S_n)$ en passant d’un système générateur à l’autre, dans le cadre de la classe de groupes χ du théorème 0.2.3. De plus, Wilson construit des exemples de sous-groupes de $G(S_{d_0}, \bar{H})$ qui sont dans la classe χ et qui sont à croissance exponentielle (car ils contiennent un groupe libre). On étend son résultat :

Théorème 0.3.5. *Soit $T_{\bar{d}}$ de valences $d_i \geq 29$, alors*

1. *Si la valence est non bornée, le groupe $G(S_{d_0}, \bar{H})$ intersecte la classe χ par des sous-groupes contenant un groupe libre, en particulier des groupes à croissance exponentielle non uniforme non moyennables (Wilson [Wil04a]).*
2. *Si la valence est bornée, le groupe $G(S_{d_0}, \bar{H})$ intersecte la classe χ par des sous-groupes contenant un semi groupe libre, en particulier des groupes à croissance exponentielle non uniforme moyennables.*

Le deuxième point est à mettre en contraste avec le résultat susmentionné de Osin [Osi04] assurant que des groupes à croissance exponentielle non uniforme ne peuvent être élémentairement moyennables. Notons que dans le cas de valence constante, on fabrique des groupes d’automates à croissance exponentielle non uniforme moyennables. Un tel exemple avait été fabriqué par Bartholdi [Bar03]. Sa moyennabilité (notée a posteriori dans [BKN08]) résulte de notre théorème 0.3.3.

De plus, le calcul de Wilson sur $h_G(S_n)$ peut s’interpreter géométriquement. En effet, les parties génératrices de $\mathcal{A}_d$ introduites par Wilson permettent de construire de nouveaux groupes $H_{\bar{d}}$ à croissance intermédiaire, engendrés par deux éléments d’ordre 2 et 3, agissant fidèlement sur un arbre enraciné de valences $d_i \geq 29$. Ces groupes, tout comme les groupes de la classe χ, ont aussi la propriété d’être denses (au sens de la topologie profinie) dans le groupe $Aut^e(T_{\bar{d}})$ des automorphismes alternés de l'arbre enraciné $T_{\bar{d}}$. On notera qu'a contrario, le groupe $Aut(T_{\bar{d}})$ n'admet aucun sous-groupe dense de type fini.

La non uniformité de la croissance des groupes de Wilson résulte de la :

Proposition 0.3.6. *Soit G un groupe appartenant à la classe χ, il existe un groupe $H_{\bar{d}}$ à croissance intermédiaire engendré par une partie finie T, et une suite S_n de parties génératrices de G avec :*

$$d((G,S_n),(H_{\bar{d}},T)) \underset{n\to\infty}{\longrightarrow} 0,$$

où d est la distance de coïncidence sur les boules définie au chapitre 2.

Cette proposition (sous-jacente au calcul de Wilson, explicite dans l'article [Bar03]) implique la non uniformité de la croissance. Notons que la proposition reste vraie même si G n'est pas moyennable, et même si l'action de G sur $T_{\bar{d}}$ n'est pas fidèle. Elle montre une proximité au sens géométrique entre les groupes à croissance intermédiaire et ces groupes à croissance non uniforme, proximité à opposer au résultat de dichotomie algébrique :

Proposition 0.3.7. *Soit G un groupe résiduellement fini appartenant à la classe χ, alors l'une seulement des possibilités suivantes est vérifiée :*

1. *soit G est à croissance sous-exponentielle,*
2. *soit G n'est pas dans la classe SG des groupes sous-exponentiellement moyennables.*

En particulier, aucun des groupes du théorème 0.3.5 à croissance exponentielle non uniforme n'est dans la classe SG.

Notons que ce résultat renforce dans ce contexte celui d'Osin et présente de nouveaux exemples de groupes moyennables hors de la classe SG. Il est basé sur un lemme dû à Neumann de classification des sous-groupes normaux de G. Je ne connais à ce jour pas de groupe dans χ qui ne soit pas résiduellement fini, ni de groupe parfait à croissance intermédiaire.

Chapitre 1

Lower bounds on growth

1.1 Introduction

Given a finitely generated group Γ endowed with a generating set S the growth function $b_{\Gamma,S}(n)$ is defined as the number of group elements which are products of less than a given number n of generators and their inverses. Given two different generating sets S and S' of Γ there exists a constant C such that $b_{\Gamma,S'}(n) \leq b_{\Gamma,S}(Cn)$. Defining the order relation $f \precsim g$ if there exists C such that $f(n) \leq g(Cn)$ and the equivalence relation $f \sim g$ if $f \precsim g$ and $g \precsim f$, the growth rate of the finitely generated group Γ is well defined up to equivalence. Milnor asked in 1968 the question of existence of groups of intermediate growth (see [Mil68a]). This means groups for which the growth rate would be faster than any polynomial and slower than any exponential.

Grigorchuk constructed a family of groups G_ω of intermediate growth in [Gri83] (see also [Gri85]). These groups are commensurable with those constructed by Aleshin in [Ale72] (see Theorem 2.3 in [Gri85]) and therefore they share properties of torsion and growth. The key example G associated to the sequence $\omega = 012012\ldots$ generated by an automaton satisfies the explicit estimates :

$$e^{n^\alpha} \precsim b_G(n) \precsim e^{n^\beta}.$$

It is a natural question whether if there exists γ such that $b_G(n) \sim e^{n^\gamma}$, which has motivated a series of papers to estimate more accurately the exponents α and β. In the paper [Gri83], Grigorchuk established $\alpha = \frac{1}{2}$ and $\beta = \log_{32}(31) = 0.9908\ldots$. The upper bound has been improved by Bartholdi to $\beta = \frac{\log(2)}{\log(2/\nu)} = 0.7674\ldots$ where ν is the real root of the polynomial X^3+X^2+X-2 (see [Bar98]). This result was obtained in a different way by Muchnik and Pak (see [MP01]) which generalizes to get $b_{G_\omega} \precsim e^{n^{\beta_r}}$ with $\beta_r = \frac{\log(2)}{\log(2/\nu_r)}$ with ν_r the real root of $X^r+X^{r-1}+X-2$ provided the sequence ω is r-homogeneous. The lower bound was improved successively by Leonov ([Leo01]) to get $\alpha = 0.5093\ldots$ and Bartholdi $\alpha = 0.5157\ldots$ ([Bar01]).

This paper improves the estimate on the lower bound :

Theorem 1.1.1. *The group $G = G_{012012\dots}$ has a growth function satisfying :*

$$e^{n^\alpha} \precsim b_G(n),$$

with $\alpha = 0.5207\dots$.

The proof is based on the algorithm used by Leonov in [Leo01], which is studied carefully (Section 1.3.2) to obtain induction relations on the growth function in Section 1.4 similar to that in [MP01]. In view of the proof the bound could be as good as $\alpha = 0.5476\dots$ as explained in Section 1.4.2.

In Section 1.5, the result of the previous sections is generalized to get a lower bound strictly better than $e^{\sqrt{n}}$ for the growth function of groups G_ω for homogeneous sequences ω.

1.2 Preliminary results

1.2.1 The group $G = G_{012012\dots}$ generated by an automaton

Let us denote by T a 2-regular rooted tree and T_0 and T_1 the subtrees under the two vertices of the first level. These three trees are canonically isomorphic. The group $Aut(T)$ of tree automorphism fixing the root satisfies :

$$Aut(T) \simeq Aut(T) \wr S_2 \simeq (Aut(T_0) \times Aut(T_1)) \rtimes S_2,$$

where S_2 is a permutation group with two elements acting on the two factors by permutations. Therefore an element f of the automorphism group $Aut(T)$ can be written $f = (f_0, f_1)\sigma$ where f_0 (respectively f_1) belongs to $Aut(T_0)$ (resp. $Aut(T_1)$) and σ is a permutation of $\{0, 1\}$ (denoted by ε if non trivial, nothing otherwise) representing the action on the first level. The computation rule is the following $ff' = (f_0, f_1)\sigma(f'_0, f'_1)\sigma' = (f_0 f'_{\sigma(0)}, f_1 f'_{\sigma(1)})\sigma\sigma'$. The group $G = G_{012012\dots}$ is a subgroup of $Aut(T)$ generated by the recursively defined elements :

$$a = (1, 1)\varepsilon, b = (a, c), c = (a, d), d = (1, b).$$

The elements of G of the form $g = (g_0, g_1)$ form a subgroup of G called the stabilizer of the first level, denoted by $St_1(G)$. The following properties are well-known and can be found in [dlH00] :

Property 1.2.1. *The group G is a quotient of the free product $S_2 * V$ between the group at two elements S_2 and a Klein group V. More precisely, the following relations hold :*

$$a^2 = b^2 = c^2 = d^2 = bcd = 1.$$

Lysionok ([Lys85]) has given the following presentation of this group :

$$G = \langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = bcd = W_n^4 = (W_n W_{n+1})^4 = 1\rangle$$

where $W_0 = ad$ and $W_{n+1} = \sigma(W_n)$ with σ defined by $\sigma(a) = aca, \sigma(b) = d, \sigma(c) = b, \sigma(d) = c$. Grigorchuk has shown that these relations are independent (see [Gri99]). Moreover, this group G is known (see [Ale72] or [Gri85]) to be a 2-group, which means any element g is of finite order a power of 2. In particular :

$$(ad)^4 = (ac)^8 = (ab)^{16} = 1.$$

Any element g of the group G is a product of elements in the generating set $S = \{a, b, c, d\}$ and thus is represented by a word $w = w(a, b, c, d)$ (remark that their inverses are not required as these generators are of order 2). A reduced representative w of an element g is a word of length $|g| = \min\{lg(w)\}$, where the minimum is taken among all representatives w of g, and the length $lg(w)$ of a word w is its number of letters. The number $|g|$ is called the norm of g relatively to the generating system S since it satisfies $|g^{-1}| = |g|$, $|gg'| \leq |g||g'|$ and $|g| = 0$ if and only if $g = 1$. The ball of radius n is the set $B_S(n) = \{g \in G||g| \leq n\}$. The growth function is its number of elements $b_S(n) = \#B_S(n)$.

The Lemmas of reduction in Section 1.3.3 will involve the number of each generator appearing in reduced representative words. So that for x in $\{a, b, c, d\}$, the number of x appearing in the word w is denoted by $|w|_x$. Note that an element g represented by w belongs to $St_1(G)$ if and only if $|w|_a$ is even.

By proposition 1.2.1, a reduced representative has form $w = a^\tau u_1 a u_2 a u_3 ... u_n a^{\tau'}$ with $u_k \in \{b, c, d\}$ and τ, τ' in $\{0, 1\}$, which justifies the :

Definition 1.2.2. As any word $w = a^\tau u_1 a u_2 a u_3 ... u_n a^{\tau'}$ is of length $2n - 1 \leq lg(w) \leq 2n + 1$, set $l(w) = n = |w|_b + |w|_c + |w|_d$ defining a new length function on reduced words. This new length induces on G a function $\|.\|$ which is a norm up to the fact that $l(a) = 0$; in particular it is a norm when restricted to $St_1(G)$. Setting $B(n) = \{g \in G|l(g) \leq n\}$, the function $b(n) = \#B(n)$ is a growth function equivalent to b_S.

The relation $(ad)^4 = 1$ implies that none of the pairs (u_k, u_{k+1}) is equal to (d, d), except for the unique case $g = adad = dada$. Indeed, the subword $au_k a u_{k+1} a$ would be equal to $adada = dad$, thus contradicting minimality. Let us be more precise concerning the sequence $\{u_k\}$.

A word w is said to be a d-word if it is of the form $w = x_0 ada x_1 ada x_2 .... ada x_k$, with $x_i \in \{b, c\}$. As $d^a = ada = (b, 1)$, such a word represents a d-element g of the stabilizer $St_1(G)$ of the first level such that $g = (a(ba)^k, \lambda)$ where $\lambda = x'_0 x'_1 \ldots x'_k$ with $x'_i = c$ if $x_i = b$ and $x'_i = d$ if $x_i = c$, so that λ belongs to $\{1, b\}$ if k is odd, and to $\{c, d\}$ if k is even (use Property 1.2.1).

Thus all the d-elements are described by the following family of representatives $\{x(d^a c)^k | k \in \{1, \ldots, 15\}, x \in \{b, c\}\}$ said to be of c-type. We can assume $k \leq 15$ because $(ab)^{16} = 1$ so that $(d^a c)^{16} = 1$. A representative w of an element g of G is said to satisfy the hypothesis (H_c) if all the maximal d-subwords of w are written with c-type representatives. Any element g has a representative (not necessarily unique) satisfying (H_c) which can be chosen to be of length $\|g\|$. This justifies the :

Assumption 1.2.3. Unless stated otherwise, it is assumed that reduced representative words $w = a^{\tau} u_1 a u_2 a u_3 ... u_n a^{\tau'}$ of elements of G satisfy the hypothesis (H_c), which means that whenever $u_k = d$ then $u_{k+1} = c$ (it is always possible except for $g = adad$).

1.2.2 A growth result on sequences of integers

The estimates on the growth function reduce to word combinatorics using the :

Proposition 1.2.4 (Lower bound on growth functions)**.** *If $b(n)$ is an unbounded sequence such that for each n large enough at least one of the following inequalities is true :*

$$b(n)^{p_1} \leq K b(\eta_1 n + K)$$
$$\dots$$
$$b(n)^{p_q} \leq K b(\eta_q n + K)$$

where $p_1, \dots, p_q \geq 2$ are fixed integers and $\eta_1, \dots, \eta_q, K$ are constants, then :

$$b(n) \gtrsim e^{n^{\alpha}},$$

where $\alpha = \min\{\frac{\log(p_1)}{\log(\eta_1)}, \dots, \frac{\log(p_q)}{\log(\eta_q)}\}$.

Proof. The proof is given for $q = 1$, the general case is similar. Iterating the inequality, we obtain for each integer k :

$$b(n)^{p^k} \leq K^{\frac{p^k-1}{p-1}} b(\eta^k n + K(\frac{\eta^k - 1}{\eta - 1}))$$

so that

$$\left(\frac{b(n)}{K}\right)^{p^k} \leq b(\eta^k n + K(\frac{\eta^k - 1}{\eta - 1})).$$

Let L be an integer such that $\lambda = \frac{b(L)}{K} > 1$. For an integer n large enough, choose k maximal such that : $N = \eta^{-k}(n - K(\eta^k - 1)/(\eta - 1)) \geq L$ and get :

$$b(n) \geq \left(\frac{b(N)}{K}\right)^{p^k} \geq \lambda^{p^k}$$

with $k \approx \log(n)/\log(\eta)$, which proves the proposition. □

1.3 Pull back methods

Given two arbitrary elements g_0, g_1 of the group G, each of the following two methods produces an element g such that $g = (g_0 z_0, g_1)$ where z_0 belongs to $\langle a, c\rangle$ which is a finite dihedral group $D_8 = \langle a, c\rangle = \langle a, c | a^2 = c^2 = (ac)^8 = 1\rangle$ of order 16.

1.3.1 The classical method

This method is described by Grigorchuk in [Gri85]. Fix two arbitrary minimal representatives w_0, w_1 of g_0, g_1. The word w_0 has the form $w_0 = a^{\tau} u_1 a u_2 a u_3 ... u_m a^{\tau'}$ where the u_i are elements of $\{b, c, d\}$. Set :

$$W_0 = b^{\varepsilon} \tilde{u_1}^a b \tilde{u_2}^a b \tilde{u_3}^a .... \tilde{u_m}^a b^{\varepsilon'}, \text{ where } \tilde{u} = \begin{cases} b & \text{if } u = c, \\ c & \text{if } u = d, \\ d & \text{if } u = b. \end{cases}$$

The following equalities :

$$\begin{array}{ll} b = (a, c), & b^a = aba = (c, a) \\ c = (a, d), & c^a = aca = (d, a) \\ d = (1, b), & d^a = ada = (b, 1) \end{array}$$

imply that $W_0 = (w_0, z_1)$ with $z_1 \in \langle a, c \rangle$. Then as $z_1^{-1} w_1 = a^{\tau} v_1 a v_2 a v_3 ... v_n a^{\tau'}$ for some v_i in $\{b, c, d\}$, set $W_1 = b^{\tau a} \tilde{v_1} b^a \tilde{v_2} b^a \tilde{v_3} .... b^a \tilde{v_n} b^{\tau' a}$ which ensures $W_1 = (z_0, z_1^{-1} w_1)$ with $z_0 \in \langle a, c \rangle$. The element g required is represented by :

$$W_{cl} = W_0 W_1 = (w_0 z_0, z_1 z_1^{-1} w_1) = (w_0 z_0, w_1).$$

Lemma 1.3.1. *The word W_{cl} satisfies :*

$$\begin{array}{rcl} l(W_{cl}) & = & 2(l(w_0) + l(w_1)) \pm 10, \\ |W_{cl}|_b & = & |w_0|_a + |w_1|_a + |w_0|_c + |w_1|_c \pm 8, \\ |W_{cl}|_c & = & |w_0|_d + |w_1|_d, \\ |W_{cl}|_d & = & |w_0|_b + |w_1|_b. \end{array}$$

Proof. The construction implies that $l(W_0) = 2m \pm 1$, $|W_0|_b = |w_0|_a + |w_0|_c$, $|W_0|_c = |w_0|_d$ and $|W_0|_d = |w_0|_b$. Moreover, as $z_1 \in \langle a, c \rangle = D_8$, we have $l(z_1^{-1}) \leq 4$ and $0 \leq |z_1^{-1}|_a + |z_1^{-1}|_c \leq 8$. Thus $l(z_1^{-1} w_1) \leq l(w_1) + 4$, so that $l(W_1) = 2l(w_1) \pm 9$, and $|W_1|_b = |w_1|_a + |w_1|_c \pm 8$. The lemma follows. □

This lemma and the classical pull back method give the :

Proposition 1.3.2 (The classical lower bound [Gri85]). *The growth of the Grigorchuk group G satisfies :*

$$b(n) \gtrsim e^{\sqrt{n}}.$$

Proof. To any pair of elements g_0, g_1 of $B(n)$ is associated an element g of length $l(g) = l(W_{cl}) \leq 4n + 10$. Moreover, such an element g can be obtained through this method from at most $\#\langle a, c \rangle = 16$ different couples, which gives the estimate :

$$b(n)^2 \leq 16 b(4n + 10) \tag{1.3.1}$$

and Proposition 1.2.4 allows to conclude. □

1.3.2 An algorithm due to Leonov

Instead of pulling back successively the two elements g_0, g_1, the following method pulls back both at the same time. This algorithm was given by Leonov in [Leo01]. Its presentation is simplified by Assumption 1.2.3. Depending on the first letters of w_0, w_1, each step of the algorithm provides $W = (v_0, v_1)$ such that

$$l(w_0') + l(w_1') < l(w_0) + l(w_1),$$

where $w_0' = v_0^{-1}w_0$ and $w_1' = v_1^{-1}w_1$. The next step is run on w_0', w_1'. The number of letters, d, c and b pulled back at a step of the algorithm are respectively :

$$\begin{aligned} n &= (l(w_0) + l(w_1)) - (l(w_0') + l(w_1')) > 0, \\ n_d &= (|w_0|_d + |w_1|_d) - (|w_0'|_d + |w_1'|_d), \\ n_c &= (|w_0|_c + |w_1|_c) - (|w_0'|_c + |w_1'|_c), \\ n_b &= (|w_0|_b + |w_1|_b) - (|w_0'|_b + |w_1'|_b). \end{aligned}$$

The algorithm is described for pairs of words (type I) of the form :

$$\begin{aligned} w_0 &= x_0ax_1ax_2a\dots \\ w_1 &= ay_1ay_2ay_3\dots \end{aligned} \tag{1.3.2}$$

The case of pairs of words (type II) of the form :

$$\begin{aligned} w_0 &= ax_1ax_2ax_3\dots \\ w_1 &= y_0ay_1ay_2a\dots \end{aligned}$$

reduces to the previous type replacing W by W^a and remarking that $l(W^a) = l(W)$ and $|W^a|_x = |W|_x$ for x in $\{b, c, d\}$. The pair of words (w_0', w_1') will always belong to these first types (I and II).

In the other types, it is sufficient to initiate the algorithm : if w_0 and w_1 both start with a (type III), set $W = d^a = (b, 1)$ which reduces to type I. If none of w_0 and w_1 starts with a (type IV), discuss on x_0 : if $x_0 = b$, set $W = d^a$ to get type I, if $x_0 = c$, set $W = b^a$ to get to type III, and if $x_0 = d$, set $W = c^a$ to get to type III. So that up to increasing $l(W), l(w_0), l(w_1)$ by at most 2, the algorithm is initiated.

The possible steps of the algorithm are described in Figure 1.1. The algorithm is run under Assumption 1.2.3 that the representatives chosen for g_0, g_1 satisfy the hypothesis (H_c), that is whenever $x_k = d$, then $x_{k+1} = c$ and the same holds for the y_k. Cases from 0d to 5 exhaust all possibilities, more precisely :

Fact 1.3.3. *Given two elements g_0, g_1 and fixed c-type representatives w_0, w_1 of the form (1.3.2), at least one case from 0d to 5 of Figure 1.1 can be applied. Moreover, such a case is unique unless $x_0 = x_1 = y_1 = b$, in which situation both cases 1 and 5 can apply. Priority is given to case 1.*

case	$w_0(a,b,c,d)$ $w_1(a,b,c,d)$	$W(a,b,c,d)$	$v_0(a,b,c,d)$ $v_1(a,b,c,d)$	w'_0 w'_1
0d	$dx_1^a\dots$ $y_1^a\dots$	c^a	d a	$x_1^a\dots$ $y_1y_2^a\dots$
0c	$cx_1^a\dots$ $y_1^a\dots$	b^a	c a	$x_1^a\dots$ $y_1y_2^a\dots$
1	$bx_1^a\dots$ $b^ay_2\dots$	b^adc^a	b b^a	$x_1^a\dots$ $y_2y_3^a\dots$
2d	$bd^acx_3^a\dots$ $d^acy_3^a\dots$	b^acc^ab	cd^a d^ac	$bx_3^a\dots$ $y_3^a\dots$
2c	$bd^acx_3^a\dots$ $c^ay_2\dots$	b^abc^a	cad c^a	$b^ax_3\dots$ $y_2y_3^a\dots$
3	$bc^ax_2\dots$ $d^acy_3^a\dots$	d^acb^ac	bc^a dad	$x_2x_3^a\dots$ $b^ay_3\dots$
41d	$bc^ax_2\dots$ $c^adc^ay_4\dots$	$b^abc^acc^ab$	cd^ada c^adac	$bx_2^a\dots$ $y_4^a\dots$
41c	$bc^ax_2\dots$ $c^acy_3^a\dots$	$b^abc^abc^a$	cd^ad c^aca	$b^ax_2\dots$ $y_3y_4^a\dots$
42d	$bc^adc^ax_4\dots$ $c^aby_3^a\dots$	$d^acd^acc^abc^adb^a$	bc^adac c^aba	$x_4^a\dots$ $y_3y_4^a\dots$
42c	$bc^acx_3^a\dots$ $c^aby_3^a\dots$	$d^acb^acb^acc^adc^ad$	bc^aca c^ab	$x_3x_4^a\dots$ $y_3^a\dots$
43d	$bc^abd^acx_5^a\dots$ $c^aby_3^a\dots$	$d^acb^acb^acc^acd^ab$	bc^acd^aba ac	$x_5x_6^a\dots$ $b^ay_3\dots$
43c	$bc^abc^ax_4\dots$ $c^aby_3^a\dots$	$d^acb^acd^acb^ac$	bc^abc^a 1	$x_4x_5^a\dots$ $c^aby_3^a\dots$
44d	$bc^a(bb^a)^qdc^ax_6\dots$ $c^aby_3^a\dots$	$d^acb^ac(d^acd^ac)^{q-1}$ $d^acb^acc^abc^adb^a$	$bc^a(bb^a)^{q-1}bc^adab$ c^aba	$x_6^a\dots$ $y_3y_4^a\dots$
44c	$bc^a(bb^a)^qcx_5^a\dots$ $c^aby_3^a\dots$	$d^acb^ac(d^acd^ac)^q$ $b^acc^adc^ad$	$bc^a(bb^a)^qca$ c^ab	$x_5x_6^a\dots$ $y_3^a\dots$
45d	$bc^a(bb^a)^qbd^acx_7^a\dots$ $c^aby_3^a\dots$	$d^acb^ac(d^acd^ac)^q$ $d^acc^acb^ab$	$bc^a(bb^a)^qbd^aca$ ac	$x_7x_8^a\dots$ $b^ay_3\dots$
45c	$bc^a(bb^a)^qbc^ax_6\dots$ $c^aby_3^a\dots$	$d^acb^ac(d^acd^ac)^q$ d^acb^ac	$bc^a(bb^a)^qbc^a$ 1	$x_6x_7^a\dots$ $c^aby_3^a\dots$
5	$bb^ax_2\dots$ $y_1^a\dots$	d^acd^ac	bb^a 1	$x_2x_3^a\dots$ $y_1^a\dots$
6	$c^a(bb^a)^{q+1}cx_7^a\dots$ $bc^ay_3\dots$	$bb^ac(d^acd^ac)^{q+1}b^ac$	$c^a(bb^a)^{q+1}ca$ cd^ad	$x_7\dots$ $b^ay_3\dots$

FIG. 1.1 – The possible steps of the Leonov algorithm

Proof of Fact 1.3.3. When $x_0 = d$, apply case 0d, when $x_0 = c$ apply case 0c. Otherwise $x_0 = b$.

When $y_1 = b$, apply case 1. Otherwise $y_1 \in \{c, d\}$.

When $x_1 = d$, and $y_1 = x \in \{c, d\}$, apply case 2x. Otherwise $x_1 \in \{b, c\}$.

When $x_1 = b$, apply case 5. Otherwise $x_1 = c$.

When $y_1 = d$, apply case 3. Otherwise $y_1 = c$.

When $y_2 = x \in \{c, d\}$, apply case 41x. Otherwise $y_2 = b$.

At this point, $x_0 = b$, $x_1 = c$, $y_1 = c$, $y_2 = b$. Discuss on $x_2 a x_3 a \dots$ which has necessarily the form $(ba)^k xa \dots$ for an integer $k \leq 8$ (because $(ab)^{16} = 1$) and $x \in \{c, d\}$.

When $k = 0$, apply case 42x.

When $k = 1$, apply case 43x.

When $k \geq 2$ is even, apply case 44x.

When $k \geq 3$ is odd, apply case 45x. □

This Fact allows to define a first algorithm to pull back w_0, w_1. Once the algorithm is initiated, apply the appropriate case S_1 from 0d to 5. Then apply the appropriate case S_2 to the remaining w'_0, w'_1, etc. This defines (uniquely) a sequence of cases $(S_1, S_2, S_3, \dots)$.

After k steps, the algorithm provides a word $W = W_{initial} W(S_1) \dots W(S_k) = (v_0, v_1)$ and there remains to pull back $(w'_0, w'_1) = (v_0^{-1} w_0, v_1^{-1} w_1)$.

Remarks 1.3.4.

1. To compute (w'_0, w'_1), the relation $(ad)^4 = 1$ is sometimes used such as in case 2d ($w'_0 = (adacbada)cax_3 \dots = dcax_3 \dots$). This implies that after stepping the algorithm, w'_0, w'_1 are not necessarily subwords of w_0, w_1. However the only possible modification is to turn a c into a b (cases 2d, 2c, 3, 41d and 41c) which would be the first letter of w'_0, w'_1, so that hypothesis (H_c) remains.
2. In cases 44, 45 and 6, q is an integer ≤ 3. Indeed, since $(ab)^{16} = 1$ the subword $a(ba)^8 = b(ab)^7$ cannot appear in a reduced word.
3. This algorithm, though better than the classical algorithm, is not sufficient to obtain the good estimates of Lemma 1.3.7 of d-reduction. It will be slightly modified (see Remark 1.3.8 for the improved version of the algorithm). Case 6 will be used only in the modified algorithm.

Definition 1.3.5. Given a word $w = a^{\tau} u_1 a u_2 a u_3 \dots u_n a^{\tau'}$ and $\lambda \in [0, 1]$, we define $\lambda w = a^{\tau} u_1 a u_2 a u_3 \dots u_k a$, for $k = [\lambda n]$, where $[x]$ denotes the integer part of x.

Remark 1.3.6. [End of the algorithm] The discussion of the algorithm can be applied provided both $l(w_0)$ and $l(w_1)$ are ≥ 11 (case 45d with $q = 3$), so that we get a word W_L such that $W_L = (v_0, v_1)$ and the words left to pull back are $w'_0 = v_0^{-1} w_0$ and $w'_1 = v_1^{-1} w_1$, with either $l(w'_0) \leq 10$ or $l(w'_1) \leq 10$. Up to reindex, assume $l(w'_0) \leq 10$

and then by construction, the word v_1 is of the form λw_1 for some λ in $[0,1]$ (but its three last letters if the algorithm ends with case 2, 3 or 41). The pair (w_0', w_1') is pulled back with the classical method, so that we get $W_{cl} = (w_0' z_0, w_1')$. All in all, we constructed $W = W_{initial} W_L W_{cl} = (w_0 z_0, w_1)$.

1.3.3 Lemmas of reduction

In this section is shown that the Leonov algorithm provides shorter pulled back than the classical algorithm. In particular, Lemma 1.3.7 of d-reduction shows that whenever two generators d are pulled back, the length of W_L is reduced by three comparing with the classical algorithm. A number N_{red} will denote the number of other reductions occurring.

By lack of information on the number of generators d occurring in reduced words, the process will be applied again to the outputs of the Leonov method. Lemma 1.3.9 of b-reduction asserts that generators b contribute to give generators d on these outputs unless they raise other reductions. Another Lemma 1.4.6 (of c^*-reduction) will also be used in the proof of Proposition 1.4.1. The numbers N_{red}, N_e, N'_{red} and N'_e are defined at each step of the algorithm in this purpose as (the number n_{c^*} will be defined before Remark 1.4.5) :

$$\begin{aligned} N_{red} &= 2n - \tfrac{3}{2} n_d - l(W), \\ N'_{red} &= 2n - n_d - l(W), \\ N_e &= |W|_d + \tfrac{4}{3} N_{red} - n_b, \\ N'_e &= |W|_d + \tfrac{4}{3} N'_{red} - n_b. \end{aligned}$$

They are given in Figure 1.2 as well as the number of generators pulled back. Concerning cases 44 and 45, we assume $q = 1$ since they are equivalent to the succession of $q-1$ cases 5 and the concerned case with $q = 1$. Concerning case 6, we assume $q = 0$ for the same reason.

Lemma 1.3.7 (of d-reduction)**.** *After stepping the algorithm k times, we have constructed $W_L = W_{initial} W_1 W_2 \dots W_k = (v_0, v_1)$ which satisfies :*

$$l(W_L) \leq 2(l(v_0) + l(v_1)) - \frac{3}{2}(|v_0|_d + |v_1|_d) + 10.$$

Proof. Check in the tab that at each step, the following inequality is true $l(W) \leq 2n - \frac{3}{2} n_d$, or equivalently $N_{red} \geq 0$. This is always true but for case 0d. Note also that for all cases but 0d, a better bound $l(W) \leq 2n - 2n_d$ (or equivalently $N'_{red} \geq n_d$) is obtained.

There remains to treat case 0d, which is paired with the next case(s) S, setting $W = W(0d)W(S)$, in order to compensate for the lack of reduction. (Note that the numbers N_{red}, N_e, N'_{red} and N'_e satisfy : $N_{red}(W_1 W_2) = N_{red}(W_1) + N_{red}(W_2)$).

If S is the case 0d, then the two following cases will be cases 0c, the result holds with $l(W) \leq 2n - 2n_d$ (for $W = c^a c b^a b$). If S belongs to the list $L_1 = \{0c, 1, 2d, 2c\}$

case	n_b	n_c	n_d	n	$l(W)$	$\|W\|_d$	N_{red}	N'_{red}	N_e	N'_e	$n_{c^*} \leq$
0d	0	0	1	1	1	0	$-\frac{1}{2}$	0	$\frac{-2}{3}$	0	0
0c	0	1	0	1	1	0	1	1	$\frac{4}{3}$	$\frac{4}{3}$	1
1	2	0	0	2	3	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	0
2d	0	2	2	4	4	0	1	2	$\frac{4}{3}$	$\frac{8}{3}$	2
2c	0	2	1	3	3	0	$\frac{3}{2}$	2	2	$\frac{8}{3}$	2
3	0	2	1	3	4	1	$\frac{1}{2}$	1	$\frac{5}{3}$	$\frac{7}{3}$	2
41d	0	3	1	4	6	0	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{4}{3}$	3
41c	0	3	0	3	5	0	1	1	$\frac{4}{3}$	$\frac{4}{3}$	2
42d	2	3	1	6	9	3	$\frac{3}{2}$	2	3	$\frac{11}{3}$	3
42c	2	3	0	5	10	3	0	0	1	1	2
43d	2	3	1	6	10	2	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{4}{3}$	2
43c	2	2	0	4	8	2	0	0	0	0	0
44d	4	3	1	8	13	3	$\frac{3}{2}$	2	1	$\frac{5}{3}$	2
44c	4	3	0	7	14	5	0	0	1	1	2
45d	4	3	1	8	14	4	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{4}{3}$	2
45c	4	2	0	6	12	4	0	0	0	0	0
5	2	0	0	2	4	2	0	0	0	0	0
6	2	3	0	5	9	2	1	1	$\frac{4}{3}$	$\frac{4}{3}$	2

FIG. 1.2 – The numbers of letters pulled back in the Leonov algorithm

then $l(W) \leq 2n - 2n_d$, and if S belongs to $L'_1 = \{3, 41d, 41c, 42d, 43d, 44d, 45d\}$ then $l(W) \leq 2n - \frac{3}{2}n_d$.

Otherwise, case 0d is followed by a succession of cases in the list $L_2 = \{42c, 43c, 44c, 45c, 5\}$. If this succession has no end, case 0d was the last letter d to pull back. Otherwise there is a next case S', which can belong to L_1, then $l(W) \leq 2n - 2n_d$, or belong to L'_1, then $l(W) \leq 2n - \frac{3}{2}n_d$.

The last unchecked situation is 0d followed by a succession of cases in the list L_2, then case 0d again, which is followed by a succession of l_1 (possibly 0) cases 5 and then a case $S'' \neq 5$.

If S'' is in the list L_1, then $l(W) \leq 2n - \frac{3}{2}n_d$. If S'' equals 0d, then the two following cases are cases 0c and $l(W) \leq 2n - \frac{3}{2}n_d$ holds. Otherwise,that is if S'' is case 3 or one of the cases 4, the algorithm has to be slightly modified (see Remark 1.3.8 below for a complete description of the modified algorithm).

The situation has the form :

$Case$	$0d$	$42c, 44c$	$43c, 45c$	5	$0d$	5^{l_1}	$S'' \in \{3, 4\}$
$w_0 =$	d	$acab$			a	$(baba)^{l_1}$	$bacaxa\ldots$
$w_1 =$	a	$baca(baba)^q ca$	$baca(baba)^q baca$	$baba$	d		$aya\ldots$

where at least one of the cases 42c or 44c occurs within the terms of the list L_2, otherwise w_0 would not satisfy (H_c). This can be rewritten :

$$\begin{aligned} w_0 &= d(acab)^l a(baba)^{l_1} bacaxa \dots \\ w_1 &= av_1' daya \dots \end{aligned}$$

where v_1' is a product of $baca(baba)^q ca$ (case 42c and 44c), $baca(baba)^q baca$ (case 43c and 45c) and $baba$ (case 5). Discuss on the parity of l (which is the number of occurrence of cases 42c and 44c between the two cases 0d) how to modify the algorithm.

If the number $l = 2l'$ is even, set :

$$W = c^a(cb^a cd^a cb^a cd^a)^{l'} bd^a c = \begin{pmatrix} d(acabacab)^{l'} aba \\ ab \end{pmatrix}$$

which is equivalent to applying case 0d, then l' time a modified version of case 43c (conjugate by a and inverse) and finally case 1. There comes $l(W) \leq 2n - 2n_d$.

If $l = 2l' + 1$ is odd, set :

$$W = c^a(dc^a dc^a)^{l''}(cb^a cd^a)^{2l'} bb^a c(d^a cd^a c)^{l_1+1} b^a c = \begin{pmatrix} d(acab)^{2l'} acaba(baba)^{l_1} baca \\ a(baba)^{l''} cadad \end{pmatrix}$$

where w_1 was written in the form $w_1 = a(baba)^{l''} bacay \dots$. This is equivalent to apply successively case 0d, l'' cases 5 (modified by a-conjugacy), l' cases 43c (modified) and case 6. There comes $l(W) \leq 2n - 2n_d$. □

Remark 1.3.8. [The algorithm as should be run by a computer] Assume given two elements g_0, g_1 and their fixed c-type representatives w_0, w_1. The algorithm is run as follows :

1. First initiate the algorithm to get to type I as (1.3.2).
2. While not encountering case 0d, run the first algorithm (following Fact 1.3.3), using cases 0c to 5.
3. When case $S_k = 0d$ occurs, look at the sequence of next cases $(S_{k+1}, S_{k+2}, \dots)$.
   (a) If the sequence is not of the following form : $S_{k+i} \in L_2 = \{42c, 43c, 44c, 45c, 5\}$ for $1 \leq i < j$, $S_{k+j} = 0d$, $S_{k+j+1} = \dots = S_{k+j+l_1} = 5$, $5 \neq S_{k+j+l_1+1} \in \{3, 4\}$ for some $j \geq 2$, $l_1 \geq 0$, then step case $S_k = 0d$ and go to 2.
   (b) If the sequence has the above form, and the number $l = 2l'$ of cases 42c and 44c occuring between S_{k+1} and S_{k+j} is even, then step case $S_k = 0d$, then step l' cases 43c modified (conjugate by a and inverse), then step case 1 and go to 2.
   (c) If the sequence has the above form, and the number $l = 2l'+1$ of cases 42c and 44c is odd, then step case $S_k = 0d$, then step l'' case 5 modified by a conjugacy (the number l'' is defined at the end of the proof of Lemma 1.3.7), then step l' cases 43c (modified by a conjugacy and inverse), then step case 6 and go to 2.

4. When one of the remaining words w_0', w_1' is too short to apply the first algorithm, finish the pull back with the classical method.

Lemma 1.3.9 (of b-reduction)**.** *In the setting of lemma 1.3.7, denote* $N_{red}(W_L) = 2(l(v_0)+l(v_1)) - \frac{3}{2}(|v_0|_d + |v_1|_d) - l(W_L)$*, then :*

$$|W_L|_d + \frac{4}{3}N_{red}(W_L) \geq |v_0|_b + |v_1|_b.$$

Note that this inequality holds for the classical algorithm (Lemma 1.3.1), so that for the final pull back $W = W_L W_{cl} = (w_0 z_0, w_1)$*, we have :*

$$|W|_d + \frac{4}{3}N_{red}(W_L) \geq |w_0|_b + |w_1|_b.$$

Proof. It is sufficient to check that at each step of the algorithm $n_b \leq |W|_d + \frac{4}{3}N_{red}$, that is $N_e \geq 0$. This is true for all cases but case 0d. For the case 0d, pair it with the next case S. The result holds for any S but case 1 (which does not happen since w_0 and w_1 are c-type representatives) and cases 43c, 45c and 5. These three last cases contribute to modify only the word w_1. Apply them until another case S' is obtained. If S' (which can neither be 1, 43c, 45c, 5) is case 0d, then the two next ones are 0c, and the lemma holds. Otherwise it is an appropriate pairing. □

Another weaker version of these lemmas of reduction is useful :

Lemma 1.3.10. *In the setting of lemma 1.3.7 we have :*

$$l(W_L) \leq 2(l(v_0)+l(v_1)) - (|v_0|_d + |v_1|_d) + 10.$$

Moreover, if $N'_{red}(W_L) = 2(l(v_0)+l(v_1)) - (|v_0|_d + |v_1|_d) - l(W_L)$*, then :*

$$|W_L|_d + \frac{4}{3}N'_{red}(W_L) \geq |v_0|_b + |v_1|_b.$$

Proof. Just check in Figure 1.2 that $N'_{red} \geq 0$ and $N'_e \geq 0$ for all cases. □

1.4 The growth of G

1.4.1 A new lower bound

The algorithm above allows to replace estimate (1.3.1) in the proof of Proposition 1.3.2 by the following :

Proposition 1.4.1. *For any positive* ε*, for any integer* n*, there exist* β*,* γ *and* δ *in* $[0,1]$ *such that* $1-3\varepsilon \leq \beta+\gamma+\delta \leq 1$ *and the three following inequalities are true :*

$$\begin{aligned} b(n)^2 &\leq Kb((4-\frac{3}{2}\delta)n+K), \\ b(n)^4 &\leq Kb((16-6\delta-3\beta)n+K), \\ b(n)^8 &\leq Kb((56+2\delta-4\gamma+\varepsilon)n+K), \end{aligned}$$

where K is a constant (depending on ε).

Corollary 1.4.2. *The growth of G satisfies :*

$$b(n) \gtrsim e^{n^\alpha}$$

where $\alpha = 0.5207\ldots$.

Proof of the corollary. Applying Proposition 1.2.4, we get

$$\alpha = \min_{\delta+\gamma+\beta=1}\{\max\{\frac{\log(2)}{\log(4-\frac{3}{2}\delta)}, \frac{\log(4)}{\log(16-6\delta-3\beta)}, \frac{\log(8)}{\log(56+2\delta-4\gamma)}\}\} = 0.5207\ldots$$

□

Before proving the proposition, first define the numbers β, γ and δ involved, a positive ε being fixed. Given a word $w = a^\tau u_1 a u_2 a u_3 ... u_m a^{\tau'}$ with $u_i \in \{b, c, d\}$ and w in $B(n)$, set :

$$\delta_w = \frac{|w|_d}{n}, \gamma_w = \frac{|w|_c}{n}, \beta_w = \frac{|w|_b}{n}.$$

It implies $\delta_w, \gamma_w, \beta_w \in [0,1]$ and $\delta_w + \gamma_w + \beta_w = \frac{m}{n} \leq 1$. Choose for each g in $B(n)$ a representative word w_g of minimal length satisfying (H_c) (remind $l(w_g) = |w_g|_d + |w_g|c + |w_g|_b$).

Fact 1.4.3. *For any $\varepsilon > 0$, and for any integer n, there exist β, γ, δ such that $\beta + \gamma + \delta \leq 1$, and a subset $B'(n) \subset B(n)$ such that $\#B'(n) \geq \varepsilon^3 \#B(n)$ and for all g in $B'(n)$:*

$$\beta \leq \beta_{w_g} \leq \beta + \varepsilon,$$
$$\gamma \leq \gamma_{w_g} \leq \gamma + \varepsilon,$$
$$\delta \leq \delta_{w_g} \leq \delta + \varepsilon.$$

Moreover, setting $n' = \max\{l(w_g) | g \in B'(n)\} \leq n$ ensures that for all $g \in B'(n)$, we have :

$$(1-3\varepsilon)n' \leq l(w_g) \leq n'.$$

This Fact ensures that, given an error ε, there exists a substantial subset $B'(n)$ of $B(n)$ in which all representatives have nearly the same length and proportion of generators d, c, b.

Proof of Fact 1.4.3. For any $\varepsilon > 0$, the cube $[0,1]^3$ splits in $(1/\varepsilon)^3$ cubes of the form $C_{i,j,k} = [i\varepsilon, (i+1)\varepsilon] \times [j\varepsilon, (j+1)\varepsilon] \times [k\varepsilon, (k+1)\varepsilon]$ for i, j, k integers in $[0, 1/\varepsilon]$. There necessarily exists i_0, j_0, k_0 such that :

$$\#\{g \in B(n) | (\delta_{w_g}, \gamma_{w_g}, \beta_{w_g}) \in C_{i_0,j_0,k_0}\} \geq \varepsilon^3 \#B(n).$$

For such i_0, j_0, k_0, set $\delta = i_0\varepsilon$, $\gamma = j_0\varepsilon$, $\beta = k_0\varepsilon$ and denote $B'(n) = \{g \in B(n) | (\delta_{w_g}, \gamma_{w_g}, \beta_{w_g}) \in C_{i_0,j_0,k_0}\}$. □

Remark 1.4.4. In the following proof, we will assume for ease of notations that $n' = n$. If this were not the case, it would be sufficient to rescale, replacing n, δ, γ, β by $n', \delta', \gamma', \beta'$ with :

$$\delta' = \frac{\delta}{\delta+\gamma+\beta}, \gamma' = \frac{\gamma}{\delta+\gamma+\beta}, \beta' = \frac{\beta}{\delta+\gamma+\beta}.$$

This would provide Proposition 1.4.1 with n' instead of n in the right side of the inequalities, which would improve the estimates.

Proof of Proposition 1.4.1. To get the first inequality, consider all pairs of elements (g_0, g_1) in $B'(n)$ (defined in Fact 1.4.3) and their chosen c-type representatives (w_0, w_1). Applying the Leonov method to pull back this pair, the resulting W satisfies (Lemma 1.3.7) :

$$l(W) \leq 2(l(w_0)+l(w_1)) - \frac{3}{2}|w_0|_d - \frac{3}{2}|\lambda w_1|_d + K,$$

and so in particular $l(W) \leq (4-\frac{3}{2}\delta)n+K$, where we use $|w_0|_d \geq \delta n$ and $l(w_0), l(w_1) \leq n$ (Fact 1.4.3). Note that the constant K includes the difference between $(w_0, \lambda w_1)$ and $W_L = (v_0, v_1)$, as well as the initiation of the algorithm. Moreover, by lack of information on the number $\lambda \in [0,1]$, the term $|\lambda w_1|_d$ cannot be estimated (see next section 1.4.2). We obtain :

$$b(n)^2 \leq \frac{\#B'(n)^2}{\varepsilon^6} \leq \frac{16}{\varepsilon^6} b((4-\frac{3}{2}\delta)n+K),$$

since at most 16 different pairs can give the same element g represented by W.

To get the second inequality, consider a 4-tuple $(g_{00}, g_{01}, g_{10}, g_{11})$ in $B'(n)$, with their c-type representatives. Applying the Leonov method to the pair (w_{00}, w_{01}) raises a word w_0 such that (Lemma 1.3.7) :

$$\begin{aligned} l(w_0) &\leq 2(l(w_{00})+l(w_{01})) - \frac{3}{2}|w_{00}|_d - \frac{3}{2}|\lambda_0 w_{01}|_d - N_{red,0} + K, \\ &\leq (4-\frac{3}{2}\delta)n - \frac{3}{2}|\lambda_0 w_{01}|_d - N_{red,0} + K. \end{aligned}$$

Moreover Lemma 1.3.9 gives also :

$$|w_0|_d + \frac{4}{3}N_{red,0} \geq |w_{00}|_b + |w_{01}|_b = 2\beta n.$$

The word w_1 is built in a similar way, hence satisfies the same estimates. We intend to apply the Leonov algorithm to (w_0, w_1), however those words might not satisfy the hypothesis (H_c), therefore we build other words $w_i' =_G w_i$ as follows.

First if a subword of the form $uadadav$ with $u, v \in \{b, c\}$ appears, replace it as $u(adada)v = u(dad)v = (ud)a(dv)$. This lowers $|w_i|_d$ by 1 but increases $N_{red,i}$ by 1 too, so that estimates above remain. Secondly we modify all d-subwords as c-type representatives, which modifies the numbers of c and b but not the number of d. (Note that these words w_i' are not minimal length representatives of $g_i =_G w_i$).

Now the Leonov method can be applied to (w_0', w_1') providing a word W such that :

$$\begin{aligned}
l(W) &\leq 2(l(w_0') + l(w_1')) - \frac{3}{2}|w_0'|_d - \frac{3}{2}|\lambda w_1'|_d + K \\
&\leq 4(4 - \frac{3}{2}\delta)n - 3|\lambda_0 w_{01}|_d - 3|\lambda_1 w_{11}|_d - 2N_{red,0} - 2N_{red,1} \\
&\qquad -\frac{3}{2}|w_0'|_d - \frac{3}{2}|\lambda w_1'|_d + K \\
&\leq (16 - 6\delta)n - \frac{3}{2}(|w_0'|_d + \frac{4}{3}N_{red,0}) - \frac{3}{2}(|\lambda w_1'|_d + \frac{4}{3}N_{red,1}) \\
&\qquad -3|\lambda_0 w_{01}|_d - 3|\lambda_1 w_{11}|_d + K \\
&\leq (16 - 6\delta - 3\beta)n - \frac{3}{2}(|\lambda w_1'|_d + \frac{4}{3}N_{red,1}) - 3|\lambda_0 w_{01}|_d - 3|\lambda_1 w_{11}|_d + K, \\
&\leq (16 - 6\delta - 3\beta)n + K,
\end{aligned}$$

allowing to conclude :

$$b(n)^4 \leq \frac{\#B'(n)^4}{\varepsilon^{12}} \leq \frac{(16)3}{\varepsilon^{12}} b((16 - 6\delta - 3\beta)n + K).$$

To get the third inequality, consider an 8-tuple $(g_{000}, g_{001}, ..., g_{111})$ in the set $B'(n)$ with the c-type representatives. We first pull back the pairs $(w_{i_1 i_2 0}, w_{i_1 i_2 1})$ using the classical method and get $w_{i_1 i_2}$ such that (Lemma 1.3.1) :

$$\begin{aligned}
l(w_{i_1 i_2}) &= 4n \pm 10, \\
|w_{i_1 i_2}|_d &= |w_{i_1 i_2 0}|_b + |w_{i_1 i_2 1}|_b = 2\beta n, \\
|w_{i_1 i_2}|_b &= |w_{i_1 i_2 0}|_a + |w_{i_1 i_2 1}|_a + |w_{i_1 i_2 0}|_c + |w_{i_1 i_2 1}|_c \geq 2(1 + \gamma)n - 2.
\end{aligned}$$

Then, pull back the pairs $(w_{i_1 0}, w_{i_1 1})$ using the Leonov method. This requires to choose c-type representatives. Thus some of the b appearing in $w_{i_1 0}, w_{i_1 1}$ (namely most of those adjacent to d) are changed into c as long as the Leonov algorithm is run (to finish the pull back with the classical method, they remain b). This provides two new representatives $w_{i_1 0}', w_{i_1 1}'$, and we denote by $|w_{i_1 0}'|_{c^*}, |w_{i_1 1}'|_{c^*}$ the number of $c = c^*$ which come from b in $w_{i_1 0}, w_{i_1 1}$. At each step of the Leonov algorithm, denote by n_{c^*} the number of c^* pulled back (see Figure 1.2).

Remark 1.4.5. [on the numbers n_{c^*}] The outputs of the classical algorithm are of the form $u_1 u_2^a u_3 u_4^a \ldots$ with $u_{2k} = b$ so that c^* are obtained only at even places, contrary to d which occur only on odd places. This explains why $n_{c^*} = 2 < 3 = n_c$ in case 43 for instance. Moreover, since a letter c^* is adjacent to a generator d it is clear that $|w'|_{c^*} \leq 2|w|_d$.

Lemma 1.4.6. *[of c^* reduction] If w_0 and w_1 are obtained as outputs of the classical algorithm, and if W is their pulled back using the Leonov method, then :*

$$|v_0|_b + |v_1|_b + \frac{1}{2}(|v_0|_{c^*} + |v_1|_{c^*}) \leq |W|_d + \frac{4}{3}N_{red}'.$$

Proof. It is sufficient to check that at each step $n_{c^*} \leq 2N'_e$. This is always true but for case 41d. Then, if $n_{c^*} \leq 2$ the result holds, and if $n_{c^*} = 3$ the next case is case 1 or case 2 and the result holds. □

At this point, we have constructed words $w'_{i_1 i_2}$ satisfying $l(w'_{i_1 i_2}) = 4n \pm 10$ and $|w'_{i_1 i_2}|_d = 2\beta n$. Moreover $|w'_{i_1 i_2}|_{c^*} = \gamma^* n \leq 2|w'_{i_1 i_2}|_d = 4\beta n$, thus $|w'_{i_1 i_2}|_b = (2 + 2\gamma - \gamma^*)n$. Applying the Leonov algorithm raises w_0 and w_1 such that (Lemma 1.3.10) :

$$\begin{aligned} l(w_i) &\leq 2(l(w'_{i0}) + l(w'_{i1})) - |w'_{i0}|_d - |\lambda_i w'_{i1}|_d - N'_{red,i} + K \\ &\leq (16 - 2\beta)n - |\lambda_i w'_{i1}|_d - N'_{red,i} + K \end{aligned}$$

where the number of d is bounded by (Lemma 1.4.6) :

$$\begin{aligned} |w_i|_d + \frac{4}{3}N'_{red,i} &\geq |w'_{i0}|_b + |w'_{i1}|_b + \frac{1}{2}(|w'_{i0}|_{c^*} + |w'_{i1}|_{c^*}) \\ &\geq 2(2 + 2\gamma - \gamma^*)n + \frac{1}{2}(2\gamma^*)n \\ &\geq (4 + 4\gamma - \gamma^*)n. \end{aligned}$$

Finally apply the Leonov algorithm to w_0 and w_1 (as for the second inequality, they should be modified first as c-type representatives, which does not change the estimates above and below). We get W which satisfies :

$$\begin{aligned} l(W) &\leq 2(l(w_0) + l(w_1)) - \frac{3}{2}|w_0|_d - \frac{3}{2}|\lambda w_1|_d + K \\ &\leq 4(16 - 2\beta)n - 2|\lambda_0 w_{01}|_d - 2|\lambda_1 w_{11}|_d - 2N'_{red,0} - 2N'_{red,1} \\ &\qquad - \frac{3}{2}|w_0|_d - \frac{3}{2}|\lambda w_1|_d + K \\ &\leq (64 - 8\beta)n - \frac{3}{2}(|w_0|_d + \frac{4}{3}N'_{red,0}) + K \\ &\leq (64 - 8\beta - \frac{3}{2}(4 + 4\gamma - \gamma^*))n + K \end{aligned}$$

and as $1 - 3\varepsilon \leq \beta + \gamma + \delta \leq 1$ (Remark 1.4.4), there comes $64 - 8\beta - \frac{3}{2}(4 + 4\gamma - \gamma^*) \leq 58 - 8\beta - 6\gamma + 6\beta = 58 - 2\beta - 6\gamma \leq 58 - 2(\beta + \gamma) - 4\gamma \leq 56 + 2\delta - 4\gamma + 6\varepsilon$. It is now possible to conclude :

$$b(n) \leq \frac{\#B'(n)^8}{\varepsilon^{24}} \leq \frac{16^7}{\varepsilon^{24}} b((56 + 2\delta - 4\gamma + 6\varepsilon)n + K).$$

□

1.4.2 Expectable other lower bounds

Improving the lower bound reduces to minimizing the length of W knowing $l(w_i) \leq n$. The lemmas of reduction give inequalities of the form :

$$l(W) \leq 2(l(w_0) + l(w_1)) - \frac{3}{2}|w_0|_d - \frac{3}{2}|\lambda w_1|_d + K.$$

λ_0	0	0.2	0.4	0.6	0.8	1
α	0.5207	0.5255	0.5306	0.5359	0.5416	0.5476

FIG. 1.3 – Expectable lower bounds

However, in the proof of Proposition 1.4.1 the term $|\lambda w_1|_d$ was not estimated. In this section we give the bounds that would be obtained if we could prove $\lambda \geq \lambda_0$ for a fixed λ_0 in $[0,1]$ (recall that this λ depends on the pair (w_0, w_1)).

Definition 1.4.7. A representative word w is said to be λ_0-homogeneous if it satisfies $|\lambda_0 w|_x = \lambda_0 |w|_x$ for $x = b, c, d$.

Proposition 1.4.8. *If we assume that $\lambda \geq \lambda_0 \in [0,1]$ and that the c-type representatives of the elements of the ball $B(n)$ are λ_0-homogeneous for all n, then the inequalities obtained in Proposition 1.4.1 would be :*

$$\begin{aligned} b(n)^2 &\leq Kb(\eta_d(\lambda_0)n + K), \\ b(n)^4 &\leq Kb(\eta_b(\lambda_0)n + K), \\ b(n)^8 &\leq Kb((\eta_c(\lambda_0) + \varepsilon)n + K), \end{aligned}$$

where $\eta_d(\lambda_0) = 4 - (1+\lambda_0)\frac{3}{2}\delta$, $\eta_b(\lambda_0) = 16 - (1+\lambda_0)(6\delta + 3\beta)$ and $\eta_c(\lambda_0) = 64 - (1+\lambda_0)(8 - 2\delta + 4\gamma)$.

Thus the exponent of the lower bound would become :

$$\alpha = \min_{\delta+\gamma+\beta=1}\{\max\{\frac{\log(2)}{\log(\eta_d(\lambda_0))}, \frac{\log(4)}{\log(\eta_b(\lambda_0))}, \frac{\log(8)}{\log(\eta_c(\lambda_0))}\}\}.$$

The numerical values of α such that $b(n) \gtrsim e^{n^\alpha}$ given λ_0 are listed in Figure 1.3.

1.5 Generalization to the groups G_ω

1.5.1 The family of groups G_ω

The group G is a particular case of an uncountable family of groups defined in a similar way in [Gri85]. Given a sequence $\omega = \omega_0\omega_1\omega_2\omega_3\ldots$ in $\Omega = \{0,1,2\}^{\mathbb{N}}$, and denoting the shift application by σ (so that $\sigma\omega = \omega_1\omega_2\omega_3\ldots$), four automorphisms $a, b_\omega, c_\omega, d_\omega$ of the binary rooted tree T are defined recursively by :

$$a = (1,1)\varepsilon, b_\omega = (u^b_\omega, b_{\sigma\omega}), c_\omega = (u^c_\omega, c_{\sigma\omega}), d_\omega = (u^d_\omega, d_{\sigma\omega}),$$

where :

$$(u^b_\omega, u^c_\omega, u^d_\omega) = \begin{cases} (a,a,1) & \text{if } \omega_0 = 0, \\ (a,1,a) & \text{if } \omega_0 = 1, \\ (1,a,a) & \text{if } \omega_0 = 2. \end{cases}$$

The group $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$ is the subgroup of $Aut(T)$ generated by these automorphisms. Property 1.2.1 still holds :

Property 1.5.1. *The group G_ω is a quotient of the free product $S_2 * V$ between the group at two elements S_2 and a Klein group V. Indeed, the following relations hold :*

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = 1.$$

Definition 1.5.2. A sequence ω is said to be flat if there exist integers k such that $\omega_{k+1} = \omega_{k+2} = \cdots = \omega_{k+r}$ for arbitrarily large r. It is said to be r-homogeneous if for any k in $\mathbb{N}$ the subsequence $\omega_{k+1}\omega_{k+2}\ldots\omega_{k+r}$ contains the three numbers 0, 1 and 2. The set of r-homogeneous sequences is denoted by Ω^r.

Remark 1.5.3. It is an obvious but crucial fact that Ω^r is stable under the shift σ.

The group G considered in the previous sections is canonically isomorphic to each of the three groups $G_{012012\ldots}$, $G_{120120\ldots}$ and $G_{201201\ldots}$. It is known ([Gri85]) that the growth function $b_\omega(n)$ of G_ω is bounded below by $e^{\sqrt{n}}$ provided ω is not flat. Muchnik and Pak have shown in [MP01] that the growth is bounded above by $e^{n^{\beta_r}}$ provided ω is r-homogeneous, where $\beta_r = \frac{\log(2)}{\log(2)-\log(\nu_r)}$ with ν_r a positive real root of the equation $x^r + x^{r-1} + x = 2$. The lower bound is improved by the :

Theorem 1.5.4. *If ω is an r-homogeneous sequence, then the growth function of G_ω satisfies :*

$$b_\omega(n) \gtrsim e^{n^{\alpha_r}}$$

where $\alpha_r = \frac{\log 2^{r-1}}{\log \eta_r} > \frac{1}{2}$, with $\eta_r = 4^{r-1} - \frac{2}{2^{r+1}+1}$.

1.5.2 Generalization of the pull back methods

The relations $(ad)^4 = (ac)^8 = (ab)^{16} = 1$ were crucial to apply Leonov method. In the generalized case of G_ω, they are replaced by the :

Proposition 1.5.5. *Let ω be an r-homogeneous sequence, then $(ax)^{2^{r+2}} = 1$ for all x in $\{b_\omega, c_\omega, d_\omega\}$. Moreover :*

$$(ad_\omega)^4 = 1 \text{ if } \omega_0 = 0,$$
$$(ac_\omega)^4 = 1 \text{ if } \omega_0 = 1,$$
$$(ab_\omega)^4 = 1 \text{ if } \omega_0 = 2.$$

Proof. If $\omega_0 = 0$, then $(ad_\omega)^4 = (ad_\omega ad_\omega)^2 = (d_{\sigma\omega}, d_{\sigma\omega})^2 = 1$. The same computation gives cases $\omega_0 = 1$ and $\omega_0 = 2$. More generally, compute $(ax_\omega)^{2^s} = (ax_\omega ax_\omega)^{2^{s-1}} = (x_{\sigma\omega}a, ax_{\sigma\omega})^{2^{s-1}} = \cdots = (ax_{\sigma^j\omega}, x_{\sigma^j\omega}a, \ldots, ax_{\sigma^j\omega})^{2^{s-j}}$. For $x_\omega = d_\omega$ (respectively c_ω, b_ω), choose j minimal such that $(\sigma^j\omega)_0 = \omega_{j-1} = 0$ (respectively 1, 2), which ensures $(ax_{\sigma^j\omega})^4 = (x_{\sigma^j\omega}a)^4 = 1$, thus $(ax_\omega)^{2^s} = 1$ for $s = j+2$ with $j \leq r$. □

These relations allow us to consider subwords similar to the d-subwords introduced in section 1.2. Indeed, define d_ω-subwords if $\omega_0 = 0$, c_ω-subwords if $\omega_0 = 1$ and b_ω-subwords if $\omega_0 = 2$, with the appropriate d_ω, c_ω or b_ω-type representatives.

ω_0	$\omega_1 = 0$	$\omega_1 = 1$	$\omega_1 = 2$
0	$W(a, b_\omega, c_\omega, d_\omega)$ $w_0, w_1(a, b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega})$	$W(a, b_\omega, c_\omega, d_\omega)$ $w_0, w_1(a, d_{\sigma\omega}, b_{\sigma\omega}, c_{\sigma\omega})$	$W(a, c_\omega, b_\omega, d_\omega)$ $w_0, w_1(a, d_{\sigma\omega}, c_{\sigma\omega}, b_{\sigma\omega})$
1	$W(a, b_\omega, d_\omega, c_\omega)$ $w_0, w_1(a, c_{\sigma\omega}, b_{\sigma\omega}, d_{\sigma\omega})$	$W(a, d_\omega, b_\omega, c_\omega)$ $w_0, w_1(a, d_{\sigma\omega}, b_{\sigma\omega}, c_{\sigma\omega})$	$W(a, d_\omega, b_\omega, c_\omega)$ $w_0, w_1(a, c_{\sigma\omega}, d_{\sigma\omega}, b_{\sigma\omega})$
2	$W(a, c_\omega, d_\omega, b_\omega)$ $w_0, w_1(a, b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega})$	$W(a, d_\omega, c_\omega, b_\omega)$ $w_0, w_1(a, b_{\sigma\omega}, d_{\sigma\omega}, c_{\sigma\omega})$	$W(a, c_\omega, d_\omega, b_\omega)$ $w_0, w_1(a, c_{\sigma\omega}, d_{\sigma\omega}, b_{\sigma\omega})$

FIG. 1.4 – Appropriate permutations in the pull back methods

The pull back methods produced a word $W(a, b, c, d)$ given two words $w_0(a, b, c, d)$ and $w_1(a, b, c, d)$. They can be generalized to produce a word $W(a, e^1_\omega, e^2_\omega, e^3_\omega)$ given two words $w_0, w_1(a, e^1_{\sigma\omega}, e^2_{\sigma\omega}, e^3_{\sigma\omega})$ where $(e^1_\omega, e^2_\omega, e^3_\omega)$ and $(e^1_{\sigma\omega}, e^2_{\sigma\omega}, e^3_{\sigma\omega})$ are appropriate permutations described in Figure 1.4 of $(b_\omega, c_\omega, d_\omega)$ and $(b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega})$ depending on (ω_0, ω_1). The classical method can be used for any values of (ω_0, ω_1), the Leonov method can be used provided $\omega_0 \neq \omega_1$. The only properties required are $(ae^3_\omega)^4 = (ae^3_{\sigma\omega})^4 = (ae^1_{\sigma\omega})^{2^{r+2}} = 1$, $e^1_\omega = (a, e^2_{\sigma\omega})$, $e^2_\omega = (a, e^3_{\sigma\omega})$ and $e^3_\omega = (1, e^1_{\sigma\omega})$. So that the Leonov algorithm can be applied to $e^2_{\sigma\omega}$-type representatives and Lemma 1.3.10 generalizes in :

Lemma 1.5.6 (of reduction)**.** *Let W be the pulled back of w_0, w_1 with the generalized Leonov method, then either :*

$$l(W) \leq 2(l(w_0) + l(w_1)) - |w_0|_{e^3_{\sigma\omega}} + K,$$

either the same inequality holds interchanging w_0 and w_1. Moreover :

$$|W|_{e^3_\omega} + 2N'_{red} \geq |w_0|_{e^1_{\sigma\omega}} + |w_1|_{e^1_{\sigma\omega}}.$$

Proof. The proof of Lemma 1.3.10 of reduction applies here, the constant K depends on the order of (ae^1_ω) which is bounded by 2^{r+2} provided ω is r-homogeneous. The second inequality follows from checking that $|W|_d + 2N'_{red} \geq n_b$ in Figure 1.2. □

1.5.3 Proof of Theorem 1.5.4

Proposition 1.2.4 is used to prove Theorem 1.5.4 once the following lemma is known :

Lemma 1.5.7. *Let ω belong to Ω^r, then for any positive ε, there exists constants K_1, K_2 such that for any n :*

$$b_{\sigma^{r-1}\omega}(n)^{2^{r-1}} \leq K_1 b_\omega((\eta_r + \varepsilon)n + K_2)$$

with $\eta_r = 4^{r-1} - \frac{2}{2^{r+1}+1}$.

Define $b_r(N) = \min\{b_\omega(N) | \omega \in \Omega^r\}$. The minimum is reached for some ω_N since the set takes values in the positive integers. Set $n = \left[\frac{N-K_2}{\eta_r+\varepsilon}\right]$, there comes :

$$b_r(n)^{2^{r-1}} \leq b_{\sigma^{r-1}\omega_N}(n)^{2^{r-1}} \leq K_1 b_{\omega_N}(N) = K_1 b_r(N) = K_1 b_r((\eta_r+\varepsilon)n + K_2).$$

As G_ω is infinite for all ω non flat, $b_\omega(n) \geq n+1$, thus $b_r(n) \geq n+1$ is an unbounded sequence. Apply Proposition 1.2.4 to get $b_r(n) \gtrsim e^{n^{\alpha_r}}$ which ensures Theorem 1.5.4 with the extra property that the lower bound function is uniform on $\omega \in \Omega^r$.

Proof of Lemma 1.5.7. Given a positive ε, Fact 1.4.3 allos to define a subset $B'_{\sigma^{r-1}\omega}(n)$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in $[0,1]$ such that for any $e^2_{\sigma\omega}$-type representative w of an element g in $B'_{\sigma^{r-1}\omega}(n)$ we have $\varepsilon_j n \leq |w|_{e^j_{\sigma\omega}} \leq (\varepsilon_j+\varepsilon)n$. Moreover, we can assume $\varepsilon_1+\varepsilon_2+\varepsilon_3 \geq 1-3\varepsilon$ and $\#B'_{\sigma^{r-1}\omega}(n) \geq \varepsilon^3 b_{\sigma^{r-1}\omega}(n)$. Thus the proof reduces to building a pull back W of length $l(W) \leq \eta_r n + K_2$ given a 2^{r-1}-tuple $(w_{0\ldots0}, \ldots, w_{1\ldots1})$ in $B'_{\sigma^{r-1}\omega}(n)$. The procedure will depend on ω.

We say that ω changes at place k if $\omega_k \neq \omega_{k+1}$ and consider the two last changes in the sequence $\omega_0 \ldots \omega_{r-1}$, say at places t and s. First use the classical method $r-s-2$ times to get a 2^{s+1}-tuple $(w_{0\ldots0}, \ldots, w_{1\ldots1})$ in $B_{\sigma^{s+1}\omega}(n)$ such that :

$$\begin{array}{l} l(w_{i_1\ldots i_{s+1}}) \leq 4^{r-s-2}n + K, \\ |w_{i_1\ldots i_{s+1}}|_{e^j_{\sigma^{s+1}\omega}} \geq \varepsilon_j n, \end{array}$$

for all $j = 1,2,3$. Then at place s, use the Leonov method to get $w_{i_1\ldots i_s}$ satisfying :

$$\begin{array}{rcl} l(w_{i_1\ldots i_s}) & \leq & 2(l(w_{i_1\ldots i_s 0}) + l(w_{i_1\ldots i_s 1})) - N'_{red,i_1\ldots i_s} - |w_{i_1\ldots i_s 0}|_{e^3_{\sigma^{s+1}\omega}} + K \\ & \leq & (4^{r-s-1} - \varepsilon_3)n - N'_{red,i_1\ldots i_s} + K, \end{array}$$

and $|w_{i_1\ldots i_s}|_{e^3_{\sigma^s\omega}} + 2N'_{red,i_1\ldots i_s} \geq |w_{i_1\ldots i_s 0}|_{e^1_{\sigma^{s+1}\omega}} + |w_{i_1\ldots i_s 1}|_{e^1_{\sigma^{s+1}\omega}} \geq 2\varepsilon_1 n$ (Lemma 1.5.6). Now the classical method applied $s-t-1$ times raises a 2^{t+1}-tuple such that :

$$\begin{array}{l} l(w_{i_1\ldots i_{t+1}}) \leq 4^{s-t-1} l(w_{i_1\ldots i_s}) + K, \\ |w_{i_1\ldots i_{t+1}}|_{e^j_{\sigma^{t+1}\omega}} \geq |w_{i_1\ldots i_s}|_{e^j_{\sigma^s\omega}}. \end{array}$$

The Leonov algorithm applied at place t gives $w_{i_1\ldots i_t}$ such that :

$$\begin{array}{rcl} l(w_{i_1\ldots i_t}) & \leq & 2(l(w_{i_1\ldots i_t 0}) + l(w_{i_1\ldots i_t 1})) - |w_{i_1\ldots i_t 0}|_{e^3_{\sigma^{t+1}\omega}} + K \\ & \leq & 4^{s-t} l(w_{i_1\ldots i_s}) - |w_{i_1\ldots i_s}|_{e^3_{\sigma^s\omega}} + K \\ & \leq & 4^{s-t}(4^{r-s-1} - \varepsilon_3)n - N'_{red,i_1\ldots i_s}) - |w_{i_1\ldots i_s}|_{e^3_{\sigma^s\omega}} + K \\ & \leq & (4^{r-t-1} - 4^{s-t}\varepsilon_3 - 2\varepsilon_1)n + K. \end{array}$$

The classical method applied t times finally gives W such that :

$$l(W) \leq 4^t(4^{r-t-1} - 4^{s-t}\varepsilon_3 - 2\varepsilon_1)n + K \leq (4^{r-1} - 4\varepsilon_3 - 2\varepsilon_1)n + K.$$

As $(ae^2_{\sigma^{r-1}\omega})^{2^{r+2}} = 1$, there can be at most 2^{r+1} successive $e^2_{\sigma^{r-1}\omega}$ in $w_{i_1\ldots i_{r-1}}$, so that $\varepsilon_2 \leq \frac{2^{r+1}}{2^{r+1}+1}$ and $\varepsilon_1 + \varepsilon_3 \geq \frac{1}{2^{r+1}+1} - 3\varepsilon$. Finally $(4^{r-1} - 4\varepsilon_3 - 2\varepsilon_1) \leq 4^{r-1} - \frac{2}{2^{r+1}+1} + 3\varepsilon$, and the Lemma follows. □

Chapitre 2

Groups with oscillating growth

2.1 Introduction

Given a group Γ and a finite generating set $S = S^{-1}$, the ball $B_{\Gamma,S}(R)$ of radius R is defined as $B_{\Gamma,S}(R) = \{\gamma \in \Gamma | |\gamma|_S \leq R\}$, where $|\gamma|_S = \min\{r | \gamma = s_1 \dots s_r$ for $s_i \in S \cup \{id_\Gamma\}\}$ defines a norm on Γ (this means $|\gamma\gamma'|_S \leq |\gamma|_s + |\gamma|_S$, $|\gamma^{-1}|_S = |\gamma|_S$ and $|\gamma|_S = 0$ if and only if $\gamma = id_\Gamma$). Note that it coincides with the ball of center id_Γ and radius R in the Cayley graph $Cay(\Gamma, S)$ of Γ with respect to S. The growth function is $b_{\Gamma,S}(R) = \#B_{\Gamma,S}(R)$. The following two properties are well known :

Property 2.1.1. *Let Γ be a finitely generated group. Let $S = S^{-1}$ and $S' = S'^{-1}$ two finite generating sets, then :*

1. *(submultiplicativity) for all R, R' one has $b_{\Gamma,S}(R + R') \leq b_{\Gamma,S}(R)b_{\Gamma,S}(R')$.*
2. *(dependance on generating set) let us denote $C_{S,S'} = \max\{|s|_{S'} | s \in S\}$, then :*

$$b_{\Gamma,S'}(\frac{1}{C_{S',S}}R) \leq b_{\Gamma,S}(R) \leq b_{\Gamma,S'}(C_{S,S'}R).$$

The second property implies that though the growth function of a group Γ depends on the generating set S, its asymptotic behavior does not. The growth function modulo the relation $\approx$ below is a group invariant.

Notation 2.1.2. Let us introduce a few relations between functions $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

1. $f \precsim g$ if there exists a constant C such that $f(R) \leq g(CR)$ for all R.
2. $f \approx g$ if $f \precsim g$ and $g \precsim f$.
3. $f = o(g)$ if for every $\varepsilon > 0$ there exists R_ε such that $f(R) \leq \varepsilon g(R)$ for all $R \geq R_\varepsilon$.

Submultiplicativity implies that the function $\log(b_{\Gamma,S}(R))$ is subadditive, hence :

$$\frac{\log(b_{\Gamma,S}(R))}{R} \longrightarrow \lambda_{\Gamma,S} \geq 0,$$

where the property $\lambda_{\Gamma,S} > 0$ does not depend on the generating set S. When this is the case, the group is said to have exponential growth. The log growth function is then linear :

$$\log(b_{\Gamma,S}(R)) = \lambda_{\Gamma,S} R + o(R).$$

The growth of the group Γ is qualified subexponential when $\lambda_{\Gamma,S} = 0$ for one (hence for all) generating set S. It is the case for instance when Γ is abelien and more generally nilpotent. In this case the growth rate of $b_{\Gamma,S}$ is polynomial and more precisely :

$$\log(b_{\Gamma,S}(R)) = d(\Gamma)\log(R) + C + o(1),$$

where $d(\Gamma)$ is an integer (see [Gui73]) and the second term is due to Pansu (see [Pan83]). The famous theorem of Gromov ([Gro81a]) states that these groups are the only ones (up to finite extension) for which the growth function is bounded by a polynomial.

It was a question of Milnor whether if there exists groups of subexponential growth which are not virtually nilpotent (see [Mil68a]). Such groups are said to have intermediate growth. A large family of exemples has been constructed by Grigrochuk in [Gri85] (see also [Gri86]). These groups are recursively defined as 3-generated automorphism groups G_ω of a binary tree and are indexed by a Cantor space Ω (section 2.2). The dependence on the index is continuous with respect to the distance of coincidence on large balls, for which both subgroups of exponential growth and of smaller growth are dense (section 2.3). Classical estimates due to Grigorchuk (and others) are of the form :

$$R^\alpha \precsim \log(b_\Gamma(R)) \precsim R^\beta,$$

for some $\alpha < \beta$ in $[\frac{1}{2}, 1[$.

The main Theorem 2.4.2 of this paper stating the existence of groups the growth function of which is oscillating (section 2.4) implies in particular that there are groups the growth function of which is infinitely often less than such a lower bound and infinitely often more than such an upper bound. Some estimates on the frequency of oscillation are also given.

Oscillating groups can moreover be chosen to be torsion, or torsion free which requires to construct another similar space of groups as studied in section 2.5.

2.2 Preliminary

2.2.1 Automorphism of the binary rooted tree

Let $T = T_2$ denote the binary rooted tree, that is the graph with vertices finite sequences $(i_1 i_2 \dots i_k)$ of $i_j \in \{0,1\}$ including the empty sequence (denoted $\emptyset$) and non oriented edges linking $(i_1 \dots i_k)$ to $(i_1 \dots i_k i_{k+1})$. Endowed with the graph metric (every edge has length 1), the sphere of center $\emptyset$ and radius k is called the level (layer)

k and consists of sequences in $\{0,1\}$ of length k. To each vertex $v=(i_1\dots i_k)$ is associated the hanging subtree T_v which is the restriction of the graph to vertices (and links associated) of the form $(vi_{k+1}\dots i_{k+m})$. Note that each tree T_v is canonically isomorphic to T.

The group $Aut(T)$ of automorphisms of $T=T_2$ is the subgroup of graph automorphisms fixing the root. This implies that the layers are preserved and that an automorphism $g\in Aut(T)$ acts on each layer by permutation. The identification between T, T_0 and T_1 raises the following isomorphism :

$$Aut(T)\simeq (Aut(T_0)\times Aut(T_1))\rtimes S_2\simeq Aut(T)\wr S_2,$$

where S_2 is the group with two elements acting by permutation of coordinates on the direct product $Aut(T_0)\times Aut(T_1)\simeq Aut(T)\times Aut(T)$. This isomorphism allows to decompose automorphisms in the wreath product as $g=(g_0,g_1)\sigma$, which means $g(i_1i_2\dots i_k)=\sigma(i_1)g_{i_1}(i_2\dots i_k)$. In particular there is a projection $p: Aut(T)\rightarrow S_2$ such that $p(g)=\sigma$. The kernel of p is called the stabilizer of the first level denoted $St_1(Aut(T))$. More generally there is another isomorphism :

$$Aut(T)\simeq \underbrace{(Aut(T)\times\cdots\times Aut(T))}_{2^n times}\rtimes(S_2\wr\cdots\wr S_2)\simeq Aut(T)\wr Aut(T^n),$$

where $Aut(T^n)\simeq(S_2\wr\cdots\wr S_2)$ is the automorphism group of the rooted finite tree T^n which is the subtree of T consisting of vertices in the n firsts levels. This allows to decompose automorphisms in the iterated wreath product as $g=(g_{0\dots0},\dots,g_{1\dots1})_n\sigma_n$ where σ_n is the image of the projection $p_n: Aut(T)\rightarrow Aut(T^n)$, the kernel of which is denoted $St_n(Aut(T))$. Note that $St_{n+1}(Aut(T))\lhd St_n(Aut(T))$ and that their intersection is the trivial group :

$$\bigcap_{n\in\mathbb{N}} St_n(Aut(T))=\{id_{Aut(T)}\}.$$

In particular, the group of automorphism of the rooted tree T is profinite via :

$$Aut(T)=\varprojlim_{n\rightarrow\infty} Aut(T^n).$$

Another description of an automorphism $g\in Aut(T_2)$ is given by its portrait, which is the function :

$$\begin{array}{rcl} p(g): T_2 & \rightarrow & S_2 \\ v & \mapsto & p_v(g), \end{array}$$

defined by the recursive relations $g_v=(g_{v0},g_{v1})p_v(g)$ for any vertex v of T_2 (initiation by $g_\emptyset=g=(g_0,g_1)p_\emptyset(g)$). The following formula shows that the restriction of $p(g)$ to the first n layers describes the action of g on the n first layers :

$$g(i_1,i_2,\dots,i_k)=(p_\emptyset(g)i_1,p_{i_1}(g)i_2,\dots,p_{i_1\dots i_{k-1}}(g)i_k).$$

2.2.2 Groups recursively defined

Let $\Omega = \{0,1,2\}^{\mathbb{N}}$ be the set of infinite sequences $\omega = \omega_0\omega_1\omega_2\omega_3\dots$ taking values in $\{0,1,2\}$. This set can be endowed with the following distance :

$$d(\omega,\omega') = \min\{\frac{1}{n+1} | \omega_0 = \omega'_0, \dots, \omega_n = \omega'_n\},$$

which turns it into a Cantor set. The shift on sequences denoted by $\sigma : \Omega \to \Omega$ where $\sigma(\omega_0\omega_1\omega_2\dots) = \omega_1\omega_2\omega_3\dots$ is a continuous application.

To each sequence ω in Ω is associated a subgroup G_ω of $Aut(T)$ the action of which is recursively defined as follows (see [Gri85]). Define via the wreath product $a = (1,1)\varepsilon$ with ε the non trivial element of S_2 and 1 the identity of $Aut(T)$ (acting on the vertices as $a(i_1 \dots i_k) = (\varepsilon(i_1)i_2\dots i_k)$).

Three automorphisms of $Aut(T)$ are recursively defined via the formulas :

$$b_\omega = (u^b_\omega, b_{\sigma\omega}), c_\omega = (u^c_\omega, c_{\sigma\omega}), d_\omega = (u^d_\omega, d_{\sigma\omega}),$$

where :

$$(u^b_\omega, u^c_\omega, u^d_\omega) = \begin{cases} (a,a,1) & \text{if } \omega_0 = 0, \\ (a,1,a) & \text{if } \omega_0 = 1, \\ (1,a,a) & \text{if } \omega_0 = 2. \end{cases}$$

It follows from this definition that it is sufficient to know the n first values of the sequence ω to know the action of these associated automorphisms on the subtree T^n of the n firsts levels.

Property 2.2.1. *The three automorphisms $b_\omega, c_\omega, d_\omega \in Aut(T)$ satisfy :*

$$b^2_\omega = c^2_\omega = d^2_\omega = b_\omega c_\omega d_\omega = 1.$$

In particular they generate a Klein group (non cyclic with four elements) provided the sequence ω is not constant.

Proof. Independently of the sequence ω, the automorphism b_ω (hence its square) belongs to $St_1(Aut(T))$. Show by joint (on ω) induction on n that $b^2_\omega \in St_n(Aut(T))$ for any integer n. Indeed, provided $b^2_{\sigma\omega}$ belongs to $St_n(Aut(T))$ equalities :

$$b^2_\omega = (u^b_\omega, b_{\sigma\omega})^2 = ((u^b_\omega)^2, b^2_{\sigma\omega}) = (1, b^2_{\sigma\omega})$$

ensure that b^2_ω belongs to $St_{n+1}(Aut(T))$. This proves (and similarly for c_ω and d_ω) that b^2_ω belongs to the intersection of stabilizers of all levels hence is trivial. The same process and the calculation :

$$b_\omega c_\omega d_\omega = (u^b_\omega, b_{\sigma\omega})(u^c_\omega, c_{\sigma\omega})(u^d_\omega, d_{\sigma\omega}) = (u^b_\omega u^c_\omega u^d_\omega, b_{\sigma\omega}c_{\sigma\omega}d_{\sigma\omega}) = (1, b_{\sigma\omega}c_{\sigma\omega}d_{\sigma\omega})$$

provide the last equality. This shows that they generate a quotient of a Klein group. There remains to show that when ω is not constant, none of $b_\omega, c_\omega, d_\omega$ is trivial.

Show for instance that $d_\omega \neq 1$ when ω is not taking constant value 0. Let n be the minimum integer such that $\omega_n \neq 0$, then :

$$d_\omega(\underbrace{1\dots1}_{n}0i) = 1d_{\sigma\omega}(\underbrace{1\dots1}_{n-1}0i) = \dots = \underbrace{1\dots1}_{n}d_{\sigma^n\omega}(0i) = \underbrace{1\dots1}_{n}0a(i),$$

which shows d_ω is not the trivial automorphism. □

Remark 2.2.2. Note that $d_{000\dots} = c_{111\dots} = b_{222\dots} = 1$, so that if ω is constant, $b_\omega, c_\omega, d_\omega$ only generate the group S_2 with two elements.

Definition 2.2.3 (Grigorchuk [Gri85]). The group G_ω associated to the sequence ω is the subgroup of $Aut(T)$ generated by the 4-tuple $S_\omega = (a, b_\omega, c_\omega, d_\omega)$. It follows from Property 2.2.1 that G_ω is a quotient of the free product $S_2 * V$, where V is a Klein group. The growth function of the group G_ω relatively to the generating set S_ω will be denoted $b_{G_\omega,S_\omega}(R) = b_\omega(R)$.

2.2.3 A distance between colored graphs

Let us consider $\mathcal{G} = (V, E_1, \dots, E_k)$ and $\mathcal{H} = (W, F_1, \dots, F_{k'})$ two colored graphs, where V, W denote the set of vertices and E_i, F_j the set of edges of color e_i, f_j. The colored graphs are said to be isomorphic (denoted $\mathcal{G} \simeq \mathcal{H}$) if there exist two bijections $\varphi : V \to W$ and $\sigma : \{e_1, \dots, e_k\} \to \{f_1, \dots, f_{k'}\}$ such that for any edge (v, v') in E_i of color e_i one has $\varphi((v, v')) = (\varphi(v), \varphi(v')) \in F_{\sigma(i)}$ and conversely for any edge (w, w') in F_j there exists (v, v') in $E_{\sigma^{-1}(j)}$ such that $\varphi^{-1}((w, w')) = (v, v')$. Moreover if v_0 and w_0 are marked points respectively in V and W, and if $\varphi(v_0) = w_0$, the bijection φ will be called an isomorphism of marked colored graphs.

When $\mathcal{G} = (V, v_0, E_1, \dots, E_k)$ is a marked colored graph, call $B_{\mathcal{G}}(R)$ the ball of center v_0 and radius R for the graph distance (every edge has length 1 independently of its color). This allows to define the following distance (as in [Gro81a] or [Gri85]) :

Definition 2.2.4. Let $\mathcal{G} = (V, v_0, E_1, \dots, E_k)$ and $\mathcal{H} = (W, w_0, F_1, \dots, F_{k'})$ be two marked colored connected graphs, set :

$$d(\mathcal{G}, \mathcal{H}) = \inf\{\frac{1}{R} | B_{\mathcal{G}}(R) \simeq B_{\mathcal{H}}(R)\}.$$

The following Proposition is essentially contained in both [Gro81a] and [Gri85] :

Proposition 2.2.5. *Let $X_{m,k}$ be the space of marked connected colored graphs of valency bounded by m and number of colors bounded by k. Then the space $(X_{m,k}, d)$ is a compact metric space.*

Proof. Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of graphs of the space $(X_{m,k}, d)$. It is sufficient to prove that there exists a graph $\mathcal{G}_\infty$ with less than k colors and valency bounded by m, and an infinite subsequence $(n_j)_j$ such that $d(\mathcal{G}_{n_j}, \mathcal{G}_\infty) \to 0$, which follows from the :

Fact 2.2.6. *Given such a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ in $X_{m,k}$, an integer R and an infinite subset $I \subset \mathbb{N}$, there exists an infinite subset $I' \subset I$ such that for all i, i' in I' :*

$$B_{\mathcal{G}_i}(R) \simeq B_{\mathcal{G}_{i'}}(R).$$

Proof of Fact 2.2.6. A ball of radius R in a colored graph is determined by the number N_v of vertices (bounded by m^{R+1} as the valency is bounded by m) and the colored edges between them, which is described by a function $f : N_v \times N_v \to \{\text{colors}\} \cup \{\emptyset\}$, allowing less than $(k+1)^{N_v^2}$ possibilities. In particular the number of isomorphism classes is finite, so that one occurs infinitely many (I') times in I. □

Indeed, this Fact allows to construct inductively infinite subsets $I_{R+1} \subset I_R \subset I_0 = \mathbb{N}$ for every integer R such that $B_{\mathcal{G}_i}(R) \simeq B_{\mathcal{G}_{i'}}(R)$ for all i, i' in I_R. Set $n_j = \min(I_j \setminus \{0, 1, \ldots, n_{j-1}\})$. The graph $\mathcal{G}_\infty$ is defined by the isomorphisms $B_{\mathcal{G}_\infty}(R) \simeq B_{\mathcal{G}_i}(R)$ for all $i \in I_R$ and thus belongs to $X_{m,k}$. The construction implies $d(\mathcal{G}_\infty, \mathcal{G}_{n_j}) \leq \frac{1}{j}$. □

2.2.4 The particular case of Cayley graphs

Let Γ be a group and $S \subset \Gamma$ a finite generating subset, and assume that $S = S^{-1}$. The Cayley graph $Cay(\Gamma, S)$ of the group Γ relatively to the generating set S is the colored graph with set of vertices Γ and edges $(\gamma, \gamma s)$ of color $s \in S$. Moreover the identity of the group is a canonical marked vertex.

Note also that in the case of Cayley graph, edges of a given color are in bijection with the set of vertices via : $\Gamma \ni \gamma \mapsto (\gamma, \gamma s) \in E_s$. This implies that to a word $w = s_1 \ldots s_k$ in S is associated a unique path starting from the identity in the Cayley graph $c_w = (1, s_1, s_1 s_2, \ldots, s_1 \ldots s_k)$. The path is said freely reduced if w is reduced in the free group $\mathbb{F}_S$ of basis S.

Fact 2.2.7. *Let $Y_m \subset X_{m,m}$ be the subspace of graphs which are Cayley graphs of m-generated groups (ie groups with generating set $S = S^{-1}$ of size less than m), then Y_m is a closed subspace of $(X_{m,m}, d)$.*

Proof of Fact 2.2.7. Let $\mathcal{G}_n = Cay(\Gamma_n, S_n)$ where S_n is a generating set of size less than m for Γ_n for each n. Assume that $\mathcal{G}_n \to \mathcal{G}$ in the space $X_{m,m}$, which is possible by compactness. Aim to show existence of a group Γ generated by a subset S of size less than m such that $\mathcal{G} = Cay(\Gamma, S)$.

Note that as soon as $d(\mathcal{G}_n, \mathcal{G}) \leq \frac{1}{2}$ the number $\#S_n = \#B_{\mathcal{G}_n}(1) - 1$ is constant (assumed equal to m). Let S denote the vertices of $\mathcal{G}$ at distance exactly one from the marked point. From each vertex of $\mathcal{G}$ there is exactly one edge of a given color $s \in S$ (indeed, this is locally true for all $\mathcal{G}_n$ for n large enough). This permits to define given an element $w = s_{i_1} \ldots s_{i_k}$ of the free group $\mathbb{F}_S$ of basis S, a path c_w in the graph $\mathcal{G}$ starting from the marked point $1_{\mathcal{G}}$ and following successively edges of color s_{i_j}. Denote $c_w(1)$ the endpoint of such a path. This provides an application :

$$\begin{aligned} \varphi : \mathbb{F}(S) &\to Vertex(\mathcal{G}) \\ w &\mapsto c_w(1). \end{aligned}$$

Aim to show this application induces on $Vertex(\mathcal{G})$ a structure of quotient group of $\mathbb{F}_S$ the Cayley graph of which (relatively to the generating set $\varphi(S)$) is $\mathcal{G}$ itself.

This is the case as soon as the operation defined by $c_{w_1}(1).c_{w_2}(1) = c_{w_1 w_2}(1)$ is well defined, which is the case provided that if $c_{w_1}(1) = c_{w_1'}(1)$ and $c_{w_2}(1) = c_{w_2'}(1)$, then $c_{w_1 w_2}(1) = c_{w_1' w_2'}(1)$. This is indeed the case since this holds as soon as n is big enough so that $d(\mathcal{G}_n, \mathcal{G}) \leq \frac{1}{|w_1|_S + |w_1'|_S + |w_2|_S + |w_2'|_S}$. □

Definition 2.2.8. Let $w(s_1, \ldots, s_k)$ a reduced representative word of the free group $\mathbb{F}_S = \mathbb{F}_{(s_1,\ldots,s_k)}$ on the basis S, we call oracle on w for (Γ, S) the knowledge of wether if $w(s_1, \ldots, s_k) = id_\Gamma$ or $w(s_1, \ldots, s_k) \neq id_\Gamma$, equivalently wether if c_w is a loop or not.

More precisely, the oracle on (Γ, S) is the function $\mathcal{O} : \mathbb{F}_S \to \{0, 1\}$ taking value $\mathcal{O}(w) = 1$ if c_w is a loop in $Cay(\Gamma, S)$ and $\mathcal{O}(w) = 0$ otherwise.

Say that the groups Γ and Δ with ordered generating set $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ have same l-oracle if (the generating sets S and T have same size and) their oracles coincide for w word of length less than l. This will be denoted :

$$(\Gamma, S) \sim_l (\Delta, T).$$

The following Lemma relating oracles and Cayley graphs will be helpfull :

Lemma 2.2.9. *The colored ball $B_{Cay(\Gamma,S)}(R)$ depends only on the class of $(2R+1)$-oracle of (Γ, S). More precisely, if $(\Gamma, S) \sim_{2R+1} (\Delta, T)$ then :*

$$d(Cay(\Gamma, S), Cay(\Delta, T)) \leq \frac{1}{R}.$$

Proof. Consider reduced words w_1, w_2 on the free group $\mathbb{F}_S$ of word length less than R. The equivalence classes modulo the relation $\mathcal{O}(w_1(S)w_2(S)^{-1}) = 1$ provide the representatives of length less than R of the elements $\gamma \in B_{Cay(\Gamma,S)}(R)$, in particular, the $2R$-oracle class gives the size of the ball $B_{Cay(\Gamma,S)}(R)$.

Once is fixed an arbitrary representative word w_γ (say of minimal length) of every element of the R-ball in Γ. There remains to draw edges (γ_1, γ_2) where $\gamma_2 = \gamma_1 s$, for what the values of the $(2R+1)$-oracle $\mathcal{O}(s^{-1} w_{\gamma_1}(S)^{-1} w_{\gamma_2}(S))$ are sufficient. □

2.3 Space of groups G_ω

2.3.1 Metric space of Cayley graphs

Denote $\mathcal{Y}$ the space of Cayley graphs $\mathcal{G}_\omega = Cay(G_\omega, S_\omega)$ of the Grigorchuk groups. The metric d from Definition 2.2.4 turns it into a metric space. The following Lemma shows that the application :

$$\begin{array}{rcl} \Psi : (\Omega_0, d) & \to & (\mathcal{Y}, d) \\ \omega & \mapsto & \mathcal{G}_\omega \end{array}$$

is continuous where Ω_0 is the subset of Ω consisting of non asymptotically constante sequences.

Lemma 2.3.1. *Let ω and ω' sequences in Ω which are not asymptotically constant and coincide on their n first values (that is $d(\omega,\omega') \leq \frac{1}{n+1}$), then :*

$$d(Cay(G_\omega, S_\omega), Cay(G_{\omega'}, S_{\omega'})) \leq \frac{1}{R_n},$$

where $R_n \geq 2^{n-2}$.

Remark 2.3.2. The assumption of non asymptotic constance ensures that for every integer k the balls $B_{\sigma^k\omega}(1)$ and $B_{\sigma^k\omega'}(1)$ are isomorphic, or equivalently that the Klein groups of Property 2.2.1 are non degenerate. This allows to initiate induction in the following proof.

Proof. Proceed by joint (on sequences ω in Ω non asymptotically constant) induction on n to show that the r_n-oracle class of (G_ω, S_ω) depends only on the n first values of ω, for $r_n = 2^{n-1}+1$, which implies the Lemma in accordance with Lemma 2.2.9.

Let $w(S_\omega)$ be a free reduced word in $\mathbb{F}_4 = \mathbb{F}_{S_\omega}$ of length less than $r_{n+1} = 2^n + 1$. If $w'(S_\omega)$ is a reduced representative of w in the quotient $S_2 * V$ of $\mathbb{F}_4$, then $w(S_\omega) = id_{G_\omega}$ if and only if $w'(S_\omega) = id_{G_\omega}$. Now w' has the specific form :

$$w'(a, b_\omega, c_\omega, d_\omega) = a^\tau x_1 a x_2 a x_3 ... x_n a^{\tau'},$$

with τ, τ' in $\{0,1\}$ and x_i in $\{b_\omega, c_\omega, d_\omega\}$. Use the following relations :

$$\begin{array}{lll} b_\omega = (u^b_\omega, b_{\sigma\omega}), & c_\omega = (u^c_\omega, c_{\sigma\omega}), & d_\omega = (u^d_\omega, d_{\sigma\omega}), \\ ab_\omega a = (b_{\sigma\omega}, u^b_\omega), & ac_\omega a = (c_{\sigma\omega}, u^c_\omega), & ad_\omega a = (d_{\sigma\omega}, u^d_\omega). \end{array} \quad (2.3.1)$$

to obtain a result of the form :

$$w'(a, b_\omega, c_\omega, d_\omega) = (w_0(a, b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega}), w_1(a, b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega}))\sigma_{w'},$$

where $\sigma_{w'}$ is independent of the sequence ω and the words $w_0(S_{\sigma\omega})$ and $w_1(S_{\sigma\omega})$ in $\mathbb{F}_4 = \mathbb{F}_{S_{\sigma\omega}}$ depend only on the value of $(u^b_\omega, u^c_\omega, u^d_\omega)$, that is on ω_0. The wreath product relations above imply that the word length satisfy $|w_0|, |w_1| \leq \frac{|w'|+1}{2} \leq \frac{r_{n+1}+1}{2} = r_n$, and as $w(S_\omega) = id_{G_\omega}$ if and only if $\sigma_{w'} = 1$, $w_0(S_{\sigma\omega}) = id_{G_{\sigma\omega}}$ and $w_1(S_{\sigma\omega}) = id_{G_{\sigma\omega}}$, the induction hypothesis can be applied to show the r_{n+1}-oracle of (G_ω, S_ω) is determined by the $n+1$ first values of ω. □

Following section 6 in [Gri85], another group $\bar{G}_\omega$ generated by a finite set $\bar{S}_\omega$ is defined for $\omega \in \Omega \setminus \Omega_0$ (asymptotically constant), such that if $\bar{\mathcal{Y}}$ is the space of Cayley graphs $\bar{\mathcal{G}}_\omega = Cay(\bar{G}_\omega, \bar{S}_\omega)$ (with $\bar{\mathcal{G}}_\omega = \mathcal{G}_\omega$ if $\omega \in \Omega_0$), then the application Ψ extends to a continuous application from Ω to $\bar{\mathcal{Y}}$.

Proposition 2.3.3. *If ω_u is a constant sequence of Ω, there exists a group $\bar{G}_{\omega_u}$ generated by a family $\bar{S}_{\omega_u} = (\bar{a}, \bar{b}_{\omega_u}, \bar{c}_{\omega_u}, \bar{d}_{\omega_u})$ such that (for $\omega^{(n)} \in \Omega_0$) :*

$$\mathcal{G}_{\omega^{(n)}} \underset{\omega^{(n)} \to \omega_u}{\longrightarrow} \bar{\mathcal{G}}_{\omega_u}.$$

Moreover, the group $\bar{G}_{\omega_u}$ is virtually metabelien of exponential growth :

$$b_{\omega_u}(R) \geq \theta^R, \text{ for some } \theta > 1.$$

Proof. The sequence of colored graphs $\mathcal{G}_n$ is a sequence in $Y_4 \subset X_{4,4}$ hence admits an accumulation point in $X_{4,4}$ (Proposition 2.2.5), which is in fact a limit (Lemma 2.3.1) and the Cayley graph of some group $\bar{G}_{\omega_u}$ (Fact 2.2.7). Moreover, all groups $G_{\omega^{(n)}}$ for $n \geq 2$ have the same ball of radius 3 so that all relations of Property 2.2.1 are satisfied, as well as non degeneracy of the Klein group $V = \langle b_{\omega^{(n)}}, c_{\omega^{(n)}}, d_{\omega^{(n)}} \rangle$. In particular, the group $\bar{G}_{\omega_u}$ is generated by four elements $\bar{S}_{\omega_u} = (\bar{a}, \bar{b}_{\omega_u}, \bar{c}_{\omega_u}, \bar{d}_{\omega_u})$ and is a quotient of $S_2 * V$.

Assume for definiteness that ω_u is the constant sequence taking value 0. Let us denote $\langle \bar{d}_{\omega_u} \rangle_N$ the normal subgroup of $\bar{G}_{\omega_u}$ generated by $\bar{d}_{\omega_u}$. Note that as $V/\langle \bar{d}_{\omega_u} \rangle_N \simeq S_2 \simeq \langle \bar{b}_{\omega_u} \rangle$, the quotient group $\bar{G}_{\omega_u}/\langle \bar{d}_{\omega_u} \rangle_N = D$ is a quotient of the infinite dihedral group $S_2 * S_2 = D_\infty$, hence is virtually cyclic. There remains to show $\langle \bar{d}_{\omega_u} \rangle_N$ is abelian to prove virtual metabelianity of $\bar{G}_{\omega_u}$.

Proceed by induction on $r = \max\{|g|_{\bar{S}_{\omega_u}}, |h|_{\bar{S}_{\omega_u}}\}$ to show that if $n \geq N_r$ is large enough, then $[g d_{\omega^{(n)}} g^{-1}, h d_{\omega^{(n)}} h^{-1}] = 1$ where $g = w_g(S_{\omega^{(n)}})$ and $h = w_h(S_{\omega^{(n)}})$ (initiated by $N_1 = 2$). Use the wreath product image $g = (g_0, g_1)\sigma$ with some σ in S_2 and g_i in $G_{\sigma\omega^{(n)}}$ of $S_{\sigma\omega^{(n)}}$-word length less or equal to $\frac{r+1}{2}$. Compute :

$$g d_{\omega^{(n)}} g^{-1} = \begin{cases} (g_0 d_{\omega^{(n)}} g_0^{-1}, 1) & \text{if } \sigma = \varepsilon, \\ (1, g_1 d_{\omega^{(n)}} g_1^{-1}) & \text{if } \sigma = 1, \end{cases}$$

so that induction applies as soon as $N_{r+1} \geq N_{\frac{r+1}{2}} + 1$.

Assume the virtually solvable group $\bar{G}_{\omega_u}$ does not have exponential growth, then by results of Milnor [Mil68b] and Wolf [Wol68] it would be virtually nilpotent, hence finitely presented. In particular, for n large enough the relations would be satisfied by the group $G_{\omega^{(n)}}$, which would be a quotient of $\bar{G}_{\omega_u}$ hence virtually nilpotent and of polynomial growth. This is absurd, because $G_{\omega^{(n)}}$ can be chosen to have intermediate growth (see section 2.3.2). □

Remark 2.3.4. Note the difference between $G_{000\ldots}$ and $\bar{G}_{000\ldots}$. In the second case, the group can still be thought generated by $a = (1,1)\varepsilon$, $b_\omega = (a, b_{\sigma\omega})$, $c_\omega = (a, c_{\sigma\omega})$ and $d_\omega = (1, d_{\sigma\omega})$, but $d_{000\ldots}$ should not be considered trivial when computing in the group, because this would require to know the whole infinite sequence ω.

This remark provides another (constructive) proof of exponential growth of $\bar{G}_{000\ldots}$, indeed that the semigroup generated by ab and ac is free, (which implies $\theta \geq \sqrt{2}$).

Assume the contrary and take $w, w' \in \mathbb{S}_2 = \langle ab, ac\rangle$ such that $w(ab,ac) =_{\bar{G}_{000\dots}} w'(ab,ac)$ and such words of minimal length l, which is assumed even up to multiplying both sides by ab. Then there images in the wreath product $w = (w_0, w_1)$ and $w' = (w'_0, w'_1)$ have the same property but have length $\leq \frac{l+1}{2}$, which is absurd. The same argument holds for such words of length less than R provided $\omega_0 = \dots = \omega_{\log_2 R+1} = 0$.

Now assume that $\omega \in \Omega \setminus \Omega_0$ is asymptotically constant, more precisely assume $\sigma^n\omega$ is constant, then define :

$$\bar{G}_\omega = \langle \bar{a}, \bar{b}_\omega, \bar{c}_\omega, \bar{d}_\omega\rangle < \bar{G}_{\sigma^n\omega} \wr Aut(T^n), \text{ where :}$$

$$\begin{aligned}
\bar{a} &= (1,\dots,1)_n p_n(a),\\
\bar{b}_\omega &= (1,\dots,1,u^b_{\sigma^{n-1}\omega},\bar{b}_{\sigma^n\omega})_n p_n(b_{\sigma^n\omega}),\\
\bar{c}_\omega &= (1,\dots,1,u^c_{\sigma^{n-1}\omega},\bar{c}_{\sigma^n\omega})_n p_n(c_{\sigma^n\omega}),\\
\bar{d}_\omega &= (1,\dots,1,u^d_{\sigma^{n-1}\omega},\bar{d}_{\sigma^n\omega})_n p_n(d_{\sigma^n\omega}).
\end{aligned}$$

Note that this definition is independent of the n chosen and the construction implies :

Proposition 2.3.5. *The following application is continuous :*

$$\begin{aligned}
\bar{\Psi} : (\Omega, d) &\to (\bar{\mathcal{Y}}, d)\\
\omega &\mapsto \bar{\mathcal{G}}_\omega
\end{aligned}$$

The following Corollary of Lemma 2.3.1 now applies in this larger setting :

Corollary 2.3.6. *If $d(\omega,\omega') \leq \frac{1}{\log_2(R)+3}$ or equivalently if $\omega_0 = \omega'_0, \dots, \omega_i = \omega'_i$ for $i = \log_2(R)+2$, then $d(\mathcal{G},\mathcal{G}') \leq \frac{1}{R}$ and in particular $b_\omega(R) = b_{\omega'}(R)$.*

2.3.2 Classical estimates on growth functions

The family of groups G_ω is famous for the following Theorem. Torsion is due to Aleshin [Ale72], intermediate growth to Grigorchuk [Gri85] and the computations of exponents β_k appeared in [Bar98], [MP01], that of α_k in Chapter 1.

Theorem 2.3.7. *Let ω be a k-homogeneous sequence (which means every subsequence $(\omega_i, \omega_{i+1}, \dots, \omega_{i+k-1})$ contains the three different values $0, 1$ and 2), then the group G_ω is a torsion group and there exists constants $C_1, C_2 > 1$ such that its growth function satisfies :*

$$C_1 R^{\alpha_k} \leq \log(b_\omega(R)) \leq C_2 R^{\beta_k},$$

with $\alpha_k > \frac{1}{2}$ and $\beta_k < 1$ is the positive root of the polynomial $X^k + X^{k-1} + X - 2$. The numerical values for $k = 3$ are approximately $\alpha_3 \approx 0.52$ and $\beta_3 \approx 0.76$.

It is a natural open question whether if there are such groups for which there is an exponent γ and a constant C such that :

$$\frac{1}{C}R^{\gamma} \leq \log(b_{\omega}(R)) \leq CR^{\gamma},$$

and in particular whether if this is the case for the group (generated by an automaton) $G_{012012...}$. The main result stated below shows that there are groups (with ω far from periodic) for which it is not the case.

Remark 2.3.8. Erschler has given the following estimates for the growth of the group $G_{010101...}$ which is not torsion (see [Ers04a]). Let $\varepsilon > 0$, then there exists a constant C_{ε} such that :

$$\frac{1}{C_{\varepsilon}}\frac{R}{\log(R)^{2+\varepsilon}} \leq \log(b_{010101...}(R)) \leq C_{\varepsilon}\frac{R}{\log(R)^{1-\varepsilon}}.$$

The next Proposition relates the growth function of the group G_{ω} with that of the groups $G_{\sigma^n\omega}$ and will be crucial in our purpose.

Proposition 2.3.9. *Let $b_{\omega}(R)$ be the growth function of the group G_{ω} relatively to the generating set S_{ω} and n a positive integer, the following estimates hold :*

$$b_{\sigma^n\omega}\left(\frac{1}{2^{n+1}}R\right) \leq b_{\omega}(R) \leq 2^{2^{n+1}}\left(b_{\sigma^n\omega}\left(\frac{1}{2^{n-1}}R\right)\right)^{2^n}.$$

The Proposition follows straightforwardly from the :

Lemma 2.3.10. *Let $b_{\omega}(R)$ be the growth function of the group G_{ω} relatively to the generating set S_{ω}, the following estimates hold :*

$$b_{\sigma\omega}\left(\frac{R-1}{2}\right) \leq b_{\omega}(R) \leq 2\left(b_{\sigma\omega}\left(\frac{R+1}{2}\right)\right)^{2}.$$

Proof. If the image of $g \in G_{\omega}$ of word length $|g| \leq R$ in the wreath product has the form $g = (g_0, g_1)\sigma$, then g is determined by σ in S_2 and two elements g_i of length $|g_i| \leq \frac{R+1}{2}$ in $G_{\sigma\omega}$ (just notice that a minimal representative word of g has the from $w_g = a^{\tau}x_1ax_2...x_na^{\tau'}$ and use relations (2.3.1)). This proves right inequality.

Moreover, given an element g_0 of $G_{\sigma\omega}$ of word length $|g_0| \leq R$, there exists (in accordance with relations (2.3.1)) an element g in G_{ω} of length $|g| \leq 2R + 1$ such that $g = (g_0, g_1)\sigma$ for some g_1 and σ. This proves left inequality. □

2.4 Main result

2.4.1 Groups with oscillating growth function

Definition 2.4.1. Let $\rho, \tau : \mathbb{N} \to \mathbb{R}_{\geq 0}$ two functions, a function $f(R)$ is said to oscillate between $\rho(R)$ and $\tau(R)$ if there exists infinitely many integers R such that $f(R) \geq \rho(R)$ and infinitely many such that $f(R) \leq \tau(R)$.

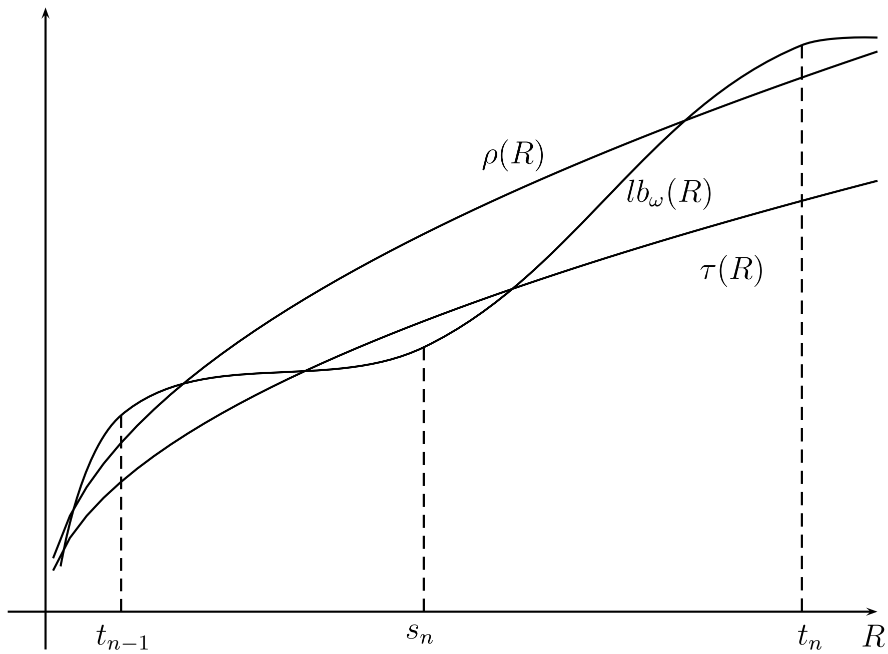

FIG. 2.1 – Function $lb_\omega(R) = \log(b_\omega(R))$ oscillating between $\rho(R)$ and $\tau(R)$.

Theorem 2.4.2. *There exists groups with oscillating growth functions.*

More precisely, given a sublinear function $\rho(R)$ (sublinear means $\rho(R) = o(R)$) and a function $\tau(R)$ such that $R^{\beta_3} = o(\tau(R))$, there exists a sequence ω in Ω such that the log growth function $\log(b_\omega(R))$ of the group G_ω relatively to the generating set S_ω oscillates between $\rho(R)$ and $\tau(R)$.

Moreover, there exists such groups (given $\rho(R)$ and $\tau(R)$) which are torsion.

Examples 2.4.3.
1. Taking $\rho(R) = R^{0.9}$ and $\tau(R) = R^{0.8}$ shows that there are groups satisfying $b(R) \leq R^\beta$ for infinitely many R but with no exponent γ such that $\log(b(R)) \approx R^\gamma$.
2. Taking $\rho(R) = \frac{R}{\log(\ldots(\log(R))\ldots)}$ and $\tau(R) = R^{\beta_3+\varepsilon}$ shows that the amplitude $\frac{\rho}{\tau}$ of the oscillations can be made almost as large as $R^{1-\beta_3}$.

Proof of Theorem 2.4.2. The main idea is that (torsion) groups G_ω with growth function satisfying $\log(b_\omega(R)) \precsim R^{\beta_3}$ are dense in the metric space $\bar{\mathcal{Y}}$ of Grigorchuk groups, as well as groups $\bar{G}_\omega$ with exponential growth function.

Proceed by induction on n to construct a sequence $\omega = \omega_0 \ldots \omega_{i_1} \ldots \omega_{j_1} \ldots \omega_{i_2} \ldots$ and integers s_n such that $b_\omega(s_n) \leq \tau(s_n)$ and $b_\omega(R)$ depends only on $\omega_0, \ldots, \omega_{i_n}$ (with $i_n = \log_2(s_n) + 2$) for $R \leq s_n$ (use Corollary 2.3.6), respectively integers t_n such that $b_\omega(t_n) \geq \rho(t_n)$ and $b_\omega(R)$ depends only on $\omega_0, \ldots, \omega_{j_n}$ for $R \leq t_n$. The induction is started taking $i_1 = 0$, $s_0 = 0$.

Assume already constructed a sequence $\omega_0 \ldots \omega_{i_n}$ and an integer s_n such that $\log(b_\omega(s_n)) \leq \tau(s_n)$ for any sequence ω starting with $\omega_0 \ldots \omega_{i_n}$. Consider the group

$\bar{G}_{\omega'}$ associated to the sequence $\omega' = \omega_0 \dots \omega_{i_n}\omega_u$ (which means $\sigma^{i_n+1}\omega' = \omega_u$ is a constant sequence) which has exponential growth (use Propositions 2.3.9 and 2.3.3) :

$$b_{\omega'}(R) \geq b_{\omega_u}\left(\frac{R}{2^{i_n+2}}\right) = b_{\omega_u}\left(\frac{R}{2^4 s_n}\right) \geq \theta^{\frac{R}{2^4 s_n}}, \tag{2.4.1}$$

this estimates and sublinearity of $\rho(R)$ imply :

$$\log(b_{\omega'}(R)) \geq R\frac{\log(\theta)}{2^4 s_n} \geq \rho(R),$$

where the last inequality holds only for R large enough. The minimal such R is called t_n, set $j_n = \log_2(t_n) + 2$ and note $\omega_{i_n+1} = \dots = \omega_{j_n} = 0$

Now consider the sequence $\omega' = \omega_0 \dots \omega_{j_n}\omega_l$ (which means $\sigma^{j_n+1}\omega' = \omega_l = 012012\dots$). Its growth function satisfies (use Proposition 2.3.9 and Theorem 2.3.7) :

$$b_{\omega'}(R) \leq 2^{2^{j_n+2}} b_{\omega_l}\left(\frac{R}{2^{j_n}}\right)^{2^{j_n+1}} \leq 2^{2^4 t_n}\left(e^{C_2\left(\frac{R}{2^2 t_n}\right)^{\beta_3}}\right)^{2^3 t_n} \leq C_n e^{C_n R^{\beta_3}}, \tag{2.4.2}$$

where C_n depends only on t_n. This estimate together with the assumption on $\tau(R)$ ensure :

$$\log(b_{\omega'}(R)) \leq C_n R^{\beta_3} + \log(C_n) \leq \tau(R),$$

where the last inequality holds only for R large enough. The minimal such R is called s_{n+1}, set $i_{n+1} = \log_2(s_{n+1}) + 2$ and note $(\omega_{j_n+1} \dots \omega_{i_{n+1}}) = (012012\dots012)$.

To ensure torsion, the number s_{n+1} constructed in the procedure should be replaced by an a priori larger number s'_{n+1}. As every element g of the group $g_{\omega'}$ is torsion (see Theorem 2.3.7) there exists an integer e_g such that $g^{e_g} = 1$. Let us set $e_n = \max\{e_g | g \in B_{\omega'}(t_n)\}$, and $s'_{n+1} = \max(s_{n+1}, t_n^{e_n})$. This guarantees that $\log(b_\omega(s'_{n+1})) \leq \tau(s'_{n+1})$ and that all elements of S_ω-word length less than t_n are torsion in G_ω.

The required sequence ω has the form :

$$\omega = \underbrace{0\dots0}_{i_1}\underbrace{012\dots012}_{j_1-i_1}\underbrace{0\dots0}_{i_2-j_1}\underbrace{012\dots012}_{j_2-i_2}\dots$$

□

2.4.2 Quantitative estimates

Given functions $\rho(R) = o(R)$ and $R^{\beta_3} = o(\tau(R))$, the previous Theorem 2.4.2 provides two sequences (s_n) and (t_n) of integers such that :

$$s_n \leq t_n \leq s_{n+1} \leq t_{n+1} \leq \dots,$$
$$\rho(t_n) \leq lb_\omega(t_n),$$
$$lb_\omega(s_n) \leq \tau(s_n),$$

where $lb_\omega(R) = \log(b_\omega(R))$ is the log growth function. Our aim is to estimate the frequency of the oscillations, more specifically, give estimates on the sequences :

$$\lambda_n = \frac{t_n}{s_n}, \mu_n = \frac{s_{n+1}}{t_n},$$

called respectively upper and lower pseudo period of the log growth function $lb_\omega(R)$ oscillating between $\rho(R)$ and $\tau(R)$. A key example of these estimates will be for the case $\rho(R) = R^\beta$ and $\tau(R) = R^\alpha$ with $\beta_3 < \alpha < \beta < 1$.

Proposition 2.4.4 (Estimates on the upper pseudo period)**.** *Assume $lb_\omega(s) \le \tau(s)$ and $\rho(t) \le lb_\omega(t)$ for some $t = \lambda s$, then :*

1. *the subadditivity condition implies $\frac{\rho(\lambda s)}{\lambda} \le \tau(s)$ (thus $\lambda \ge s^{\frac{\beta-\alpha}{1-\beta}}$ for the key exemple),*
2. *but in the scope of Theorem 2.4.2 it is sufficient to take λ satisfying*

$$\frac{\rho(\lambda s)}{\lambda} \le c$$

for some constant c independent of s (thus $\lambda = cs^{\frac{\beta}{1-\beta}}$ in the key exemple).

Remark 2.4.5. For the exemple $\rho(R) = R^{0.9}$ and $\tau(R) = R^{0.8}$, subadditivity implies only $\lambda_n \ge s_n$ whereas the group is constructed with an upper pseudo period $\lambda_n \approx s_n^9$.

Proof. By subadditivity compute :

$$\rho(t) \le lb_\omega(t) = lb_\omega(\lambda s) \le \lambda lb_\omega(s) \le \lambda\tau(s).$$

In practice, estimate (2.4.1) gives $b_\omega(t) = b_{\omega_0 \ldots b_i 000\ldots}(t) \ge b_{000\ldots}(\frac{t}{2^{i+2}}) \ge \theta^{\frac{t}{2^4 s}}$ (remind $i = \log_2(s) + 2$), so that $\frac{t}{2^4 s}\log(\theta) \ge \rho(t)$ which proves the second part with $c = \frac{\log(\theta)}{2^4}$. □

Proposition 2.4.6 (Estimate on the lower pseudo period)**.** *Assume $lb_\omega(s') \le \tau(s')$ and $\rho(t) \le lb_\omega(t)$ for some $s' = \mu t$, then it is sufficient to take μ satisfying*

$$\frac{\mu^{\beta_3}}{\tau(\mu t)} \le \frac{c'}{t}$$

for some constant c' independent of t (thus $\mu = c' t^{\frac{1-\alpha}{\alpha-\beta_3}}$ in the key exemple).

Remark 2.4.7. The factor $\frac{1}{t}$ in the above inequality shows that it is slightly harder to lower the growth rate than to increase it.

Proof. Estimate (2.4.2) provides $b_\omega(s') = b_{\omega_0 \ldots \omega_j 012012\ldots}(s') \le 2^{2^4 t} b_{012012\ldots}(\frac{s'}{2^2 t})^{2^3 t}$ where $j = \log_2(t) + 2$, so that :

$$lb_\omega(s') \le 2^3 t lb_{012012\ldots}(\frac{s'}{2^2 t}) + 2^4 t\log(2) \le 2^3 tC\left(\frac{s'}{2^2 t}\right)^{\beta_3} + 2^4 t\log(2),$$

so that it is sufficient to take μ such that $c' t\mu^{\beta_3} \le \tau(s')$, which proves the Proposition. □

2.5 Torsion free exemples

2.5.1 A space of torsion free groups

In his paper [Gri86], Grigorchuk constructs a torsion free group of intermediate growth, very similar to the group $G_{012012\dots}$. On this model, a whole family indexed by Ω of torsion free groups can be constructed. They are defined as groups acting on an infinite valency rooted tree $T_\infty = T_{\mathbb{Z}}$ (ie graph with vertices finite sequences $(i_1, \dots, i_k)$ of $i_j \in \mathbb{Z}$ and links between $(i_1, \dots, i_k)$ and $(i_1, \dots, i_k, i_{k+1})$. The automorphism group of such a graph satisfies :

$$Aut(T_{\mathbb{Z}}) \simeq Aut(T_{\mathbb{Z}}) \wr_{\mathbb{Z}} S_{\mathbb{Z}}.$$

To each sequence $\omega \in \Omega$ is associated as above a group $\tilde{G}_\omega$ of automorphisms of $T_{\mathbb{Z}}$ generated by a finite family $\tilde{S}_\omega = (\tilde{a}, \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega)$, where $\tilde{a}$ is a rooted permutation defined as $\tilde{a} = (\dots, 1, 1, 1, \dots)\tilde{\varepsilon}$ with $\tilde{\varepsilon} : \mathbb{Z} \to \mathbb{Z}$ the shift application $\tilde{\varepsilon}(n) = n + 1$, and the three other generators defined as :

$$\begin{aligned} \tilde{b}_\omega &= (\dots, \tilde{b}_{\sigma\omega}, \tilde{u}^b_\omega, \tilde{b}_{\sigma\omega}, \tilde{u}^b_\omega, \dots), \\ \tilde{c}_\omega &= (\dots, \tilde{c}_{\sigma\omega}, \tilde{u}^c_\omega, \tilde{c}_{\sigma\omega}, \tilde{u}^c_\omega, \dots), \\ \tilde{d}_\omega &= (\dots, \tilde{d}_{\sigma\omega}, \tilde{u}^d_\omega, \tilde{d}_{\sigma\omega}, \tilde{u}^d_\omega, \dots), \end{aligned}$$

with $\tilde{u}^x_\omega$ in odd positions, taking values :

$$(\tilde{u}^b_\omega, \tilde{u}^c_\omega, \tilde{u}^d_\omega) = \begin{cases} (\tilde{a}, \tilde{a}, 1) & \text{if } \omega_0 = 0, \\ (\tilde{a}, 1, \tilde{a}) & \text{if } \omega_0 = 1, \\ (1, \tilde{a}, \tilde{a}) & \text{if } \omega_0 = 2. \end{cases}$$

Note that this definition implies that :

$$\varphi : \tilde{G}_\omega \hookrightarrow (\tilde{G}_{\sigma\omega} \times \tilde{G}_{\sigma\omega}) \rtimes \langle \tilde{\varepsilon} \rangle,$$

where $\tilde{\varepsilon}$ acts on the two copies by permutation $S_2 \simeq \langle \tilde{\varepsilon} \rangle / \langle \tilde{\varepsilon}^2 \rangle$. Indeed, a priori $\tilde{G}_\omega$ injects only into a wreath product on infinite base $(\prod_{\mathbb{Z}} \tilde{G}_{\sigma\omega}) \rtimes \langle \tilde{\varepsilon} \rangle$, but the images on even (respectively odd) coordinates are canonically isomorphic by definition of the generators. As in the torsion case an element $g \in \tilde{G}_\omega$ is identified with its image via φ, written $g = (g_o, g_e)\tilde{\sigma}$ (note that $\tilde{\sigma}$ belongs to $\langle \tilde{\varepsilon} \rangle = \mathbb{Z}$).

The portrait of automorphisms of $T_{\mathbb{Z}}$ defined as in section 2.2.1 factorizes similarly when the automorphisms belong to $\tilde{G}_\omega$. Indeed, it is clear that $p_v(g) = p_{v'}(g)$ whenever $v = i_1 \dots i_k$ and $v' = i'_1 \dots i'_k$ and $i_j = i'_j \mod 2$ for each j. Thus an element of $\tilde{G}_\omega$ is described by a portrait :

$$\begin{aligned} \tilde{p}(g) : T_2 &\to \mathbb{Z} \\ v &\mapsto \tilde{p}_v(g). \end{aligned}$$

Property 2.5.1. *The three automorphisms $\tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega \in Aut(T_\mathbb{Z})$ satisfy :*

$$[\tilde{b}_\omega, \tilde{c}_\omega] = [\tilde{c}_\omega, \tilde{d}_\omega] = [\tilde{d}_\omega, \tilde{b}_\omega] = 1.$$

In particular they generate a free abelien group $\mathbb{Z}^3$ of rank 3 provided the three values 0, 1 and 2 appear in the sequence ω.

Proof. Compute $[\tilde{b}_\omega, \tilde{c}_\omega] = ([\tilde{b}_{\sigma\omega}, \tilde{c}_{\sigma\omega}], [\tilde{u}^b_{\sigma\omega}, \tilde{u}^c_{\sigma\omega}])$ and use a joint (on ω) induction on n to show the commutator acts trivially on the n first levels of the rooted infinite valency tree $T_\mathbb{Z}$. □

This property implies that the group $\tilde{G}_\omega$ is a quotient of the free product $\mathbb{Z} * \mathbb{Z}^3$, where the copy of $\mathbb{Z}^3$ generated by $(\tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega)$ degenerates to $\mathbb{Z}^2$ (respectively $\mathbb{Z}$) if only two of the three values 0, 1 and 2 appear in ω (respectively ω constant sequence).

Note that every element $g \in \tilde{G}_\omega$ admits a minimal representative word with respect to the generating set $\tilde{S}_\omega = (\tilde{a}, \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega)$ (and inverses) of the form :

$$g = a^{\varepsilon_1} b^{p_1} c^{q_1} d^{r_1} a^{\varepsilon_2} b^{p_2} c^{q_2} d^{r_2} \dots a^{\varepsilon_k} b^{p_k} c^{q_k} d^{r_k} a^m, \tag{2.5.1}$$

where $m, p_i, q_i, r_i \in \mathbb{Z}$ and $\varepsilon_i \in \{\pm 1\}$ (use the relation $a^2 x = x a^2$ for $x \in \{b, c, d\}$), which permits to prove :

Proposition 2.5.2. *The group $\tilde{G}_\omega$ is torsion free.*

Proof. Proceed by induction on $|g|_{\tilde{S}_\omega} = |m| + \sum_i |\varepsilon_i| + |p_i| + |q_i| + |r_i|$ to show that $g^n = 1$ implies $g = 1$, which is true for $|g|_{\tilde{S}_\omega} \leq 1$. Let $g = (g_o, g_e)\tilde{\sigma}$ with $\tilde{\sigma} \in \langle \tilde{\varepsilon} \rangle$. If $\tilde{\sigma} \neq 1$, then $g^n \neq 1$ for any n. Otherwise $g^n = (g_o^n, g_e^n)$ where $|g_o|_{\tilde{S}_\omega}, |g_e|_{\tilde{S}_\omega} \leq \sum |p_i| + |q_i| + |r_i| < |g|_{\tilde{S}_\omega}$ and induction applies unless $g \in \langle \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega \rangle$ which is torsion free by Property 2.5.1. □

Let us denote Ω_1 the space of sequences of Ω for which there is a value in $\{0, 1, 2\}$ that appears only finitely many times. A sequence in $\Omega \setminus \Omega_1$ guarantees that the free abelian group of Property 2.5.1 has indeed rank 3, which allows to initiate induction in the proof of the next lemma similar to Lemma 2.3.1 :

Lemma 2.5.3. *Let ω and ω' belong to $\Omega \setminus \Omega_1$ such that $d(\omega, \omega') \leq \frac{1}{n+1}$ (equivalently the n first entries of the two sequence coincide), then :*

$$d(Cay(\tilde{G}_\omega, \tilde{S}_\omega), Cay(\tilde{G}_{\omega'}, \tilde{S}_{\omega'})) \leq \frac{1}{2^{n-2}}.$$

Proof. Independently of the sequence ω every element g admits a representative w_g of the form (2.5.1) with $b_\omega, c_\omega, d_\omega$ mutually distincts as soon as $\omega \notin \Omega_1$. This shows that the oracle $\mathcal{O}(w_g)$ is independent of ω if $k \leq 1$. Indeed, $w_g = a^{\varepsilon_1} b^{b_1} c^{q_1} d^{r_1} a^m$ is trivial in $\tilde{G}_\omega$ if and only if $\varepsilon_1 + m = p_1 = q_1 = r_1 = 0$.

Note also that if $g = (g_o, g_e)\tilde{a}^{m'}$ has normal form (2.5.1), where $m' = m + \sum \varepsilon_i$, then :

$$g_o = a^{n_1} b^{p_2} c^{q_2} d^{r_2} a^{n_3} \dots b^{p_{k_o-1}} c^{q_{k_o-1}} d^{r_{k_o-1}} a^{n_{k_o}},$$
$$g_e = a^{n_0} b^{p_1} c^{q_1} d^{r_1} a^{n_2} \dots b^{p_{k_e-1}} c^{q_{k_e-1}} d^{r_{k_e-1}} a^{n_{k_e}},$$

where $k_o, k_e \leq \frac{k+1}{2}$. This permits to show (as for Lemma 2.3.1) by induction on k that the oracle $\mathcal{O}_{|\{w_g | k \leq 2^{n-1}+1\}}$ depends only on the n first values of ω. This proves the result using $k \leq |w_g| = |g|_{\tilde{S}_\omega}$ and Lemma 2.2.9. □

This Lemma provides us with a continuous application as in Proposition 2.3.5 :

$$\begin{array}{rcl} \tilde{\Psi} : (\Omega, d) & \to & (\tilde{\mathcal{Y}}, d) \\ \omega & \mapsto & \tilde{\mathcal{G}}_\omega, \end{array}$$

where $\tilde{\mathcal{G}}_\omega = Cay(\tilde{G}_\omega, \tilde{S}_\omega)$ when $\omega \in \Omega \setminus \Omega_1$, and $\tilde{\mathcal{G}}_\omega = \lim_n Cay(\tilde{G}_{\omega^{(n)}}, \tilde{S}_{\omega^{(n)}})$ for $\omega \in \Omega_1$ independent of the approximating sequence $\Omega \setminus \Omega_1 \ni \omega^{(n)} \to \omega$ and $\tilde{\mathcal{Y}} \subset Y_8$ is a compact subspace.

By abuse of notations, the group with Cayley graph $\tilde{\Psi}(\omega) = \tilde{\mathcal{G}}_\omega$ is denoted $\tilde{G}_\omega$ in the next sections even if $\omega \in \Omega_1$ (despite homogeneous notations would imply to denote it $\bar{\tilde{G}}_\omega$).

2.5.2 Growth properties of the groups $\tilde{G}_\omega$

Let us denote $\tilde{b}_\omega(R)$ the growth function of the group $\tilde{G}_\omega$ with respect to the generating set $\tilde{S}_\omega \cup \tilde{S}_\omega^{-1}$, then :

Proposition 2.5.4. *Given a fixed sequence $\omega \in \Omega$, for each integer R, the following holds :*

$$\tilde{b}_\omega(R) \geq b_\omega(R).$$

Proof. There are $b_\omega(R)$ different words of the form $w = a^\tau x_1 a x_2 \dots x_n a^{\tau'}$ with $x_i \in (b_\omega, c_\omega, d_\omega)$ of length less than R for which the portraits $p(w) : T_2 \to S_2$ are pairwise distincts.

The construction of $\tilde{G}_\omega$ implies that the words $\tilde{w} = \tilde{a}^\tau \tilde{x}_1 \tilde{a} \tilde{x}_2 \dots \tilde{x}_n \tilde{a}^{\tau'}$ with $\tilde{x}_i \in (\tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega)$ have portraits $\tilde{p}(\tilde{w}) : T_2 \to \mathbb{Z}$ which are pairwise distincts (more precisely the reductions mod 2 are pairwise distincts $(T_2 \overset{\tilde{p}(\tilde{w})}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}) = (T_2 \overset{p(w)}{\to} S_2)$). This proves $\tilde{b}_\omega(R) \geq b_\omega(R)$. □

As in the torsion case, the growth function of $\tilde{G}_\omega$ is equivalent to that of $\tilde{G}_{\sigma^n\omega}$.

Proposition 2.5.5. *For each integer R, the following holds :*

$$\tilde{b}_{\sigma^n\omega}\left(\frac{R}{2^{n+1}}\right) \leq \tilde{b}_\omega(R) \leq (2R+1)^{2^{n+1}} \tilde{b}_{\sigma^n\omega}(R)^{2^n}.$$

This follows straightforwardly from the :

Lemma 2.5.6. *For each integer R, the following holds :*

$$\tilde{b}_{\sigma\omega}\left(\frac{R-1}{2}\right) \leq \tilde{b}_{\omega}(R) \leq (2R+1)\tilde{b}_{\sigma\omega}(R)^2.$$

Proof. Let $g_o = a^{\varepsilon_1}b^{b_1}c^{q_1}d^{r_1}a^{\varepsilon_2}\dots a^m \in \tilde{B}_{\sigma\omega}(R)$ and assume for instance $\omega_0 = 0$. Set $g = b^{\varepsilon_1}(b^a)^{b_1}(c^a)^{q_1}(d^a)^{r_1}b^{\varepsilon_2}\dots b^m \in \tilde{B}_{\omega}(2R+1)$ and check that $g = (g_o, g_e)$ so that left inequality holds.

On the other hand, if $g = (g_o, g_e)\tilde{\sigma}$ belongs to $\tilde{B}_{\omega}(R)$ then $|g_o|_{\tilde{S}_{\sigma\omega}}, |g_e|_{\tilde{S}_{\sigma\omega}} \leq R$ and $\tilde{\sigma}$ belongs to $B_{\langle\tilde{\varepsilon}\rangle,\{\tilde{\varepsilon}^{\pm 1}\}}(R)$ hence can take at most $(2R+1)$ different values, which provides the right inequality. $\square$

Some torsion free groups $\tilde{G}_{\omega}$ also have intermediate growth :

Theorem 2.5.7 (Grigorchuk [Gri86])**.** *The group $\tilde{G}_{012012\dots}$ has intermediate growth.*

2.5.3 Torsion free groups with oscillating growth functions

Theorem 2.5.8. *There exists torsion free groups with oscillating growth function.*

More precisely, given a sublinear function $\rho(R)$ and a function $\tau(R)$ such that $\log(\tilde{b}_{012012\dots}(R)) = o(\tau(R))$, there exists a sequence ω in Ω such that the log growth function of the group $\tilde{G}_{\omega}$ with respect to the generating set $\tilde{S}_{\omega}$ oscillates between $\rho(R)$ and $\tau(R)$.

Proof. The proof of Theorem 2.4.2 applies here. If there is s_n such that $\log(b_{\omega}(s_n)) \leq \tau(s_n)$ for all sequence ω starting with $\omega_0, \dots, \omega_{i_n}$ for $i_n = \log_2(s_n)$, look at the group $\tilde{G}_{\omega_0,\dots\omega_{i_n}000\dots}$ which has exponential growth function (Proposition 2.5.4 and 2.5.5) to find t_n such that $\log(b_{\omega}(t_n)) \geq \rho(t_n)$ for all sequences ω starting with $\omega_0, \dots, \omega_{j_n}$ for $j_n = \log_2(t_n)$.

Then look at the group $\tilde{G}_{\omega_0,\dots\omega_{i_n}012012\dots}$ the growth function of which is $o(\tau(R))$ in accordance with Theorem 2.5.7 and Proposition 2.5.5 to find s_{n+1}. $\square$

Chapitre 3

Amenability and non uniform growth

3.1 Introduction

Given a finitely generated group Γ endowed with a generating set S the growth function, $b_{\Gamma,S}(r)$ is defined as the number of group elements which are products of less than a given number r of generators and their inverses. The growth of Γ is qualified exponential when the exponential growth rate $h_S(\Gamma) = \lim \sqrt[r]{b_{\Gamma,S}(r)}$ strictly exceeds 1 for some, hence for all, generating set S. The growth is said intermediate if $h_S(\Gamma) = 1$ and the growth function is not polynomial, that is when the group is not virtually nilpotent ([Gro81a]). The growth is qualified uniform when the infimum of the exponential growth rates over all generating sets strictly exceeds 1, non uniform when exponential but : $\inf_S h_S(\Gamma) = 1$.

The question of existence of groups of non uniform exponential growth was asked by Gromov in 1981 in the little green book [Gro81b]. It has been shown that such groups do not occur in several classes such as hyperbolic groups (see [Kou98]), linear groups (see [EMO02]), elementary amenable groups (see [Osi04]). A pleasant exposition is given in [dlH02]. The first examples of such groups have been provided by Wilson in [Wil04b] and [Wil04a]. They contain free subgroups. Another example is due to Bartholdi in [Bar03]. The main object of this paper is the following :

Theorem 3.1.1. *There exists uncountably many pairwise non isomorphic amenable groups of non uniform exponential growth.*

Those groups will appear as subgroups of the group $Aut(T_{\bar{d}})$ of automorphisms of a spherically homogeneous rooted tree, which is described in section 3.2. In section 3.3 a subgroup of $Aut(T_{\bar{d}})$ is proved to be amenable when the tree has bounded valency. This Main Theorem 3.3.1 implies in particular that the group considered in [Bar03] is amenable. Sections 3.4 and 3.5 are devoted to the proof of this Main Theorem. In section 3.6, using specific generating sets of the alternate group of permutation, some groups of intermediate growth are introduced. Those groups are proved to be dense in the profinite group of alternate automorphism of the rooted tree. The groups of Theorem 3.1.1 are constructed in section 3.7, using results of Wilson ([Wil04a]). Some part of Wilson Theorem 3.7.1, namely the convergence to 1 of the exponential growth rates associated to different generating sets, is reinterpreted as a convergence of the Cayley graphs to Cayley graphs of the groups of intermediate growth introduced in the previous section. The last section 3.8 deals with the question of subexponential amenability. The groups of non uniform exponential growth constructed are proved not to be in the class SG.

3.2 Automorphisms of rooted trees

3.2.1 Spherically homogeneous rooted tree

Given a sequence $\bar{d} = \{d_j\}_{j\geq 0}$ of integers $d_j \geq 2$, the associated spherically homogeneous rooted tree denoted $T_{\bar{d}}$ is defined as follows : the vertices are indexed by all finite sequences $v = (i_1 i_2 \dots i_k)$ with i_j in $\{1, 2, \dots, d_{j-1}\}$, including the empty sequence $\emptyset$ called the root, and the edges link the pairs $\{(i_1 i_2 \dots i_k), (i_1 i_2 \dots i_k i_{k+1})\}$. Note that the sequence $\bar{d}$ need not be infinite in which case the tree is finite.

The distance (each edge has length 1) from a vertex to the root is called the level of the vertex. The vertices of level $l(v) = n$ form the nth layer (or level) of cardinality $d_0 d_1 \dots d_{n-1}$.

Each vertex v of level n gives rise to a spherically homogeneous rooted subtree T_v when restricting to vertices of the form $(v i_n i_{n+1} \dots i_{n+k})$. The tree T_v is isomorphic to the tree $T_{\sigma^n \bar{d}}$ associated to the sequence $\sigma^n \bar{d} = \{d_j\}_{j\geq n}$ (with σ denoting the usual shift $\sigma : (d_0 d_1 d_2 \dots) \mapsto (d_1 d_2 d_3 \dots)$).

3.2.2 Automorphism group

An automorphism of $T_{\bar{d}}$ is a graph automorphism, that is a bijection of the set of vertices mapping edges to edges, which fixes the root. These properties imply that the layers are preserved, and an automorphism acts on a layer by permutation. The group of all such automorphisms will be denoted $Aut(T_{\bar{d}})$. Spherical homogeneity ensures that $Aut(T_{\bar{d}})$ and $Aut(T_{\sigma\bar{d}})$ are related by an isomorphism :

$$Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma\bar{d}}) \wr S_{d_0}. \tag{3.2.1}$$

Recall that $G \wr S_d \simeq (G \times \dots \times G) \rtimes S_d$ where S_d (the group of permutation of the set $\{1, 2, \dots, d\}$) acts on the d copies of G by permutation. This identification will allow to write extensively $f = (f_1, f_2, \dots, f_{d_0})\sigma$ with f in $Aut(T_{\bar{d}})$, the f_i in $Aut(T_{\sigma\bar{d}})$ and σ in S_{d_0}. The product rule is $fg = (f_1, f_2, \dots, f_{d_0})\sigma(g_1, g_2, \dots, g_{d_0})\tau = (f_1 g_{\sigma(1)}, \dots, f_{d_0} g_{\sigma(d_0)})\sigma\tau$. In particular, there is a projection $p : Aut(T_{\bar{d}}) \rightarrow S_{d_0}$ called restriction to the first level. The kernel of this projection is called the stabilizer of the first level, denoted $St_1(Aut(T_{\bar{d}}))$, easily checked to be isomorphic to the direct product $Aut(T_{\sigma\bar{d}}) \times \dots \times Aut(T_{\sigma\bar{d}})$ with d_0 factors.

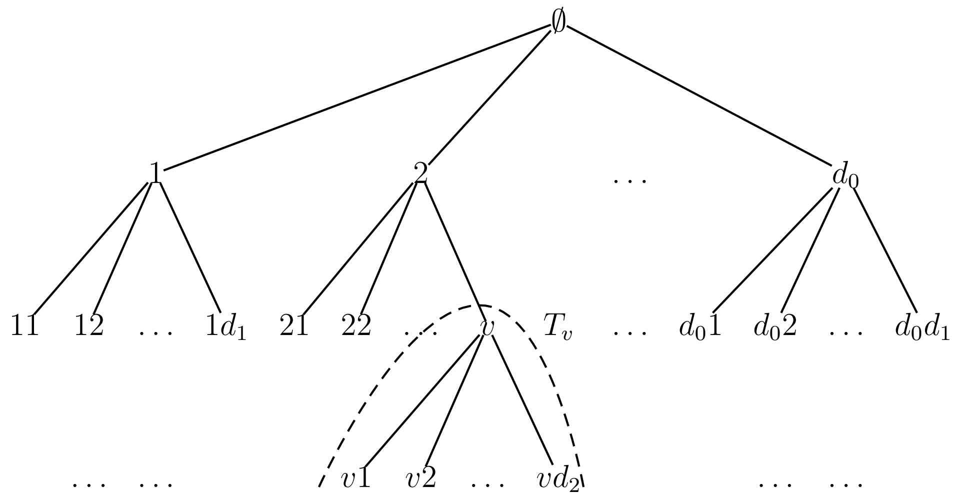

FIG. 3.1 – Spherically homogeneous rooted tree, subtree.

More generally for each integer n, there is an isomorphism :

$$Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma^n \bar{d}}) \wr Aut(T_{d_0 \dots d_{n-1}}), \tag{3.2.2}$$

where $Aut(T_{d_0 \dots d_{n-1}})$ acts by permutation on $d_0 \dots d_{n-1}$ copies of $Aut(T_{\sigma^n \bar{d}})$ the way it acts on the set of leaves (the boundary) of the finite tree $\partial T_{d_0 \dots d_{n-1}}$. There is also a projection $p_n : Aut(T_{\bar{d}}) \to Aut(T_{d_0 \dots d_{n-1}})$, the kernel of which constitutes the stabilizer $St_n(Aut(T_{\bar{d}}))$ of the nth level. This is a normal subgroup of $Aut(T_{\bar{d}})$ isomorphic to the direct product $St_n(Aut(T_{\bar{d}})) \simeq Aut(T_{\sigma^n \bar{d}}) \times \dots \times Aut(T_{\sigma^n \bar{d}})$, the elements of which will occasionally be written $g = (g_{1 \dots 1}, \dots, g_{d_0 \dots d_{n-1}})_n$.

The full group of automorphism can be viewed as a profinite group via :

$$Aut(T_{\bar{d}}) = \varprojlim_{n \to \infty} Aut(T_{d_0 \dots d_{n-1}}) = \varprojlim_{n \to \infty} (S_{d_{n-1}} \wr S_{d_{n-2}} \wr \dots \wr S_{d_0}). \tag{3.2.3}$$

A basis of open sets for the profinite topology associated is $\{St_n(Aut(T_{\bar{d}}))\}_{n \geq 0}$. This topology can also be defined as associated to any of the following metrics $\delta_{\bar{\lambda}}$ on $Aut(T_{\bar{d}})$. Given a decreasing sequence $\bar{\lambda} = \{\lambda_n\}_{n \geq 0}$ of positive numbers tending to zero, set :

$$\delta_{\bar{\lambda}}(g, h) = \inf\{\lambda_n | g(v) = h(v) \quad \text{for all vertices } v \text{ of level} \leq n\}.$$

A nice description of automorphisms of a rooted tree is to draw portraits. A portrait is a function g from the set of all vertices v of the tree $T_{\bar{d}}$ taking permutation values $g(v) \in S_{d_{l(v)}}$. A portrait gives rise to a unique automorphism via the formula :

$$g(i_1 i_2 i_3 \dots i_k) = (g(\emptyset) i_1)(g(i_1) i_2)(g(i_1 i_2) i_3) \dots (g(i_1 \dots i_{k-1}) i_k).$$

Conversely, every automorphism has a unique portrait. The metrics $\delta_{\bar{\lambda}}$ are such that two automorphisms are n-close if their portraits coincide on the n firsts layers.

An automorphism is said to be even (or alternate) if all the permutations $g(v) \in S_{d_{l(v)}}$ involved in the portrait are alternate permutations $g(v) \in \mathcal{A}_{d_{l(v)}}$. The group of alternate automorphisms will be denoted $Aut^e(T_{\bar{d}})$. It satisfies :

$$Aut^e(T_{\bar{d}}) = \varprojlim_{n\to\infty} Aut^e(T_{d_0\dots d_{n-1}}) = \varprojlim_{n\to\infty} \mathcal{A}_{d_{n-1}} \wr \mathcal{A}_{d_{n-2}} \wr \dots \wr \mathcal{A}_{d_0}, \qquad (3.2.4)$$

the profinite topology, the distances associated and the stabilizers of levels are defined in the same way as for the full automorphism group. Note that if T_2 is a 2-regular rooted tree, then $Aut^e(T_2)$ is the trivial group.

3.2.3 Directed automorphism subgroups

This paper focuses on specific subgroups of $Aut(T_{\bar{d}})$, those directed by a given infinite geodesic of the tree $T_{\bar{d}}$ starting from the root. Such a geodesic can always be chosen to be that passing at all vertices indexed by $11\dots1$ (the leftmost geodesic in the illustrations). First introduce actions of some permutation groups on $T_{\bar{d}}$. The group S_{d_0} acts on the rooted tree by permuting the subtrees of the first layer :

$$\iota_0 : S_{d_0} \hookrightarrow Aut(T_{\bar{d}}).$$

More precisely, ι_0 is defined by $\iota_0(\sigma)(i_1 i_2 \dots i_k) = \sigma(i_1) i_2 \dots i_k$. For simplicity of notations, we will identify $\sigma = \iota_0(\sigma) = (id_{T_{\sigma\bar{d}}}, \dots, id_{T_{\sigma\bar{d}}})\sigma$ and call those rooted automorphisms (their portrait is trivial outside of the root).

The infinite direct product $\bar{H} = S_{d_1} \times \dots \times S_{d_1} \times S_{d_2} \times \dots \times S_{d_2} \times \dots$ of permutation groups where S_{d_k} appears $d_{k-1} - 1$ times also acts on a canonical (once the geodesic is chosen) way on the rooted tree $T_{\bar{d}}$:

$$\iota : \bar{H} \hookrightarrow Aut(T_{\bar{d}}).$$

Indeed, consider the vertices $1_k i = 1\dots1i$ with k ones and i in $\{2, \dots, d_k\}$. They form the set P of vertices at distance exactly 1 of the leftmost geodesic $111\dots$. Each permutation group S_{d_k} acts on a subtree $T_{1_k i}$ via the above homomorphism ι_0 (corresponding to the rooted tree $T_{\sigma^k \bar{d}}$). More precisely, the action is recursively defined through the wreath product by :

$$\iota(\sigma_2, \dots, \sigma_{d_0}, \sigma_{12}, \dots, \sigma_{1d_1}, \dots) = (\iota'(\sigma_{12}, \dots, \sigma_{1d_1}, \dots), \sigma_2, \dots, \sigma_{d_0}),$$

where $\iota'(\sigma_{12}, \dots, \sigma_{1d_1}, \dots)$ represents the action of the restriction $\bar{H} \twoheadrightarrow \bar{H}_1$ via :

$$\iota' : \bar{H}_1 = S_{d_2}^{d_1 - 1} \times S_{d_3}^{d_2 - 1} \times \dots \hookrightarrow Aut(T_1) \simeq Aut(T_{\sigma\bar{d}})$$

The geometry of the set P ensures that the action of different factors commute, thus ι is a well defined injection. This is best understood by Figure 3.2, showing the portrait is non trivial only on P. The automorphisms obtained in $\iota(\bar{H})$ are said to be directed by the geodesic $111\dots$.

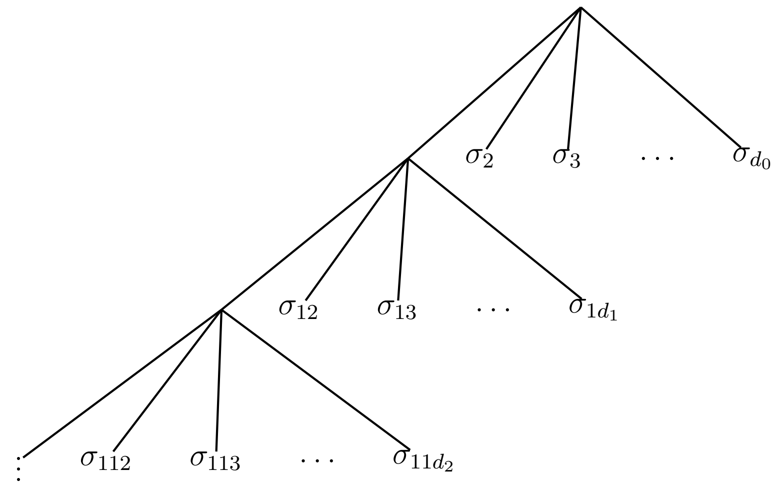

FIG. 3.2 – The ι-action of $\bar{H}$.

Given a subgroup A of S_{d_0} and a subgroup H of $\bar{H}$, denote by $G(A,H)$ the subgroup of $Aut(T_{\bar{d}})$ generated by $\iota_0(A)$ and $\iota(H)$. Such a group will be called a directed group of automorphisms. Note that the group H might not be countable as $\bar{H}$ is not. The group $G(S_{d_0},\bar{H})$ will be called full group of directed automorphisms. Note that the full tree automorphism group isomorphism (3.2.1) descends to the full directed automorphism groups :

$$G(S_{d_0},\bar{H}) \simeq G(S_{d_1},\bar{H}_1) \wr S_{d_0}. \tag{3.2.5}$$

The class of groups of the form $G(A,H)$ has been considered in [Gri00]. It gathers many famous examples such as the family of Aleshin-Grigorchuk groups known to be torsion (see [Ale72]) and of intermediate growth (see [Gri85]). Other interesting examples are some groups of non uniform growth constructed by Wilson ([Wil04b], [Wil04a]) and Bartholdi ([Bar03]), to which section 3.7 is devoted.

3.3 The Main Theorem

In this section the Main theorem on full directed automorphism groups is stated and its proof is reduced to the proof of the a priori weaker Theorem 3.3.2.

Theorem 3.3.1 (Main Theorem). *Let $\bar{d}=(d_i)_{i\geq 0}$ be a sequence of integers $d_i\geq 2$, let S_{d_0}, $\bar{H}$ and $G(S_{d_0},\bar{H})$ be the full directed subgroup of $Aut(T_{\bar{d}})$, then :*

1) if the sequence $\bar{d}$ is bounded, the group $G(S_{d_0},\bar{H})$ is amenable.

2) if the sequence $\bar{d}$ is unbounded, the group $G(S_{d_0},\bar{H})$ contains a free group $\mathbb{F}_2$ on two generators.

The proof of part 1) of the Main Theorem 3.3.1 reduces to proving the following, which will be the object of sections 3.4 and 3.5.

Theorem 3.3.2. *Let $\bar{d} = (d_i)_{i\geq 0}$ be a bounded sequence of integers $2 \leq d_i \leq D$, let $H < \bar{H}$ be a finite saturated subgroup, then the directed subgroup $G(S_{d_0}, H)$ of $Aut(T_{\bar{d}})$ is amenable.*

Proof of part 2). The second part of the Main Theorem is an immediate consequence of the following lemma stated in [Wil04a] (see also [TW84]).

Lemma 3.3.3 ([Wil04a]). *Let F be the free product of two non-trivial finite groups which are not both of order 2, and $\mathcal{S}$ be any infinite subset of $\mathbb{N}$. Then the alternate permutation group $\mathcal{A}_d$ is a homomorphic image of F for all sufficiently large d and the intersection of the kernels of all epimorphisms from F to groups $\mathcal{A}_d$ with $d \in \mathcal{S}$ is the trivial subgroup.*

This implies that if $\bar{d}$ is unbounded then the group $\bar{H}$ already contains a free group $\mathbb{F}_2$ on two generators. Indeed, let $F = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ the free group generated by elements x of order 2 and y of order 3. Let D be such that there is an onto homomorphism $\varphi_d : F \to \mathcal{A}_d$ when $d \geq D$. Define :

$$h_1 = (\varphi_{d_1}(x), \ldots, \varphi_{d_1}(x), \varphi_{d_2}(x), \ldots, \varphi_{d_2}(x), \ldots) \in \bar{H}$$
$$h_2 = (\varphi_{d_1}(y), \ldots, \varphi_{d_1}(y), \varphi_{d_2}(y), \ldots, \varphi_{d_2}(y), \ldots) \in \bar{H}$$

where $\varphi_d(x) = \varphi_d(y) = 1 \in \mathcal{A}_d \subset S_d$ if $d < D$. Then Lemma 3.3.3 ensures that the subgroup $\langle h_1, h_2\rangle < \bar{H}$ is isomorphic to F which contains $\mathbb{F}_2$ as a subgroup of finite index. □

If the sequence $\bar{d}$ is bounded then the properties of the group $\bar{H}$ are much different.

Fact 3.3.4. *Let $\bar{H} = T_1 \times T_2 \times \ldots$ where the groups T_i belong to a finite family $\mathcal{F} = \{F_1, \ldots F_D\}$ of finite groups, then every finitely generated subgroup H' of $\bar{H}$ is finite.*

Proof. Let $h_1, \ldots, h_k$ be generators of H', they are of the form $h_j = (h_j^1, h_j^2, \ldots)$ with $h_j^i \in T_i$. There are at most $M = D.(max\{\#F_i\})^k$ different $k+1$-tuples $(h_1^i, h_2^i, \ldots, h_k^i, F_i)$. Let I be a subset of $\mathbb{N}$ of size less than M such that all different $(k+1)$-tuples appear when i describes I. Then the projection $\pi_I : \bar{H} \to \times_{i\in I} T_i$ is injective, so that H' is finite. □

Definition 3.3.5. A finite subgroup H of the group $\bar{H} = T_1 \times T_2 \times \ldots$ where the T_i belong to a finite family $\mathcal{F}$ of finite group is said to be *saturated* if the equidistributed probability measure q_H on H projects on each coordinate i to the equidistributed probability measure q_{T_i} on T_i, that is if $h = (h_1, h_2, \ldots) \in H$ then $q_H(h_i = t) = q_{T_i}(t) = \frac{1}{\#T_i}$.

Fact 3.3.6. *Every finite subgroup H' of $\bar{H}$ is included in a finite saturated group H.*

Proof. With the above notations set for each i in I :

$$J_i = \{j \in \mathbb{N} | (h_1^j, h_2^j, \ldots, h_k^j, F_j) = (h_1^i, h_2^i, \ldots, h_k^i, F_i)\}.$$

There is a diagonal embedding $T_i \to \times_{j\in J_i} T_j$ and as $\cup_{i\in I} J_i = \mathbb{N}$ we get a diagonal injection :

$$\times_{i\in I} T_i \hookrightarrow \bar{H}$$

the image H of which contains H' and is saturated by construction, knowing a finite direct product is always saturated. □

Proof that Theorem 3.3.2 implies the Main Theorem. To prove the group $G(S_{d_0}, \bar{H})$ is amenable, it is sufficient to prove amenability for every finitely generated subgroup G_f (Theorem 1.2.7. in [Gre69]), which reduces, assuming Theorem 3.3.2, to show that G_f is included in some $G(S_{d_0}, H)$ for H finite saturated. Indeed, let $s_1, \ldots, s_k$ be generators of G_f, each s_j is of the form $s_j = a_j^1 h_j^2 a_j^3 \ldots h_j^{n_j}$, with $a_j^i \in S_{d_0}$ and $\langle (h_j^i)_{i,j} \rangle < \bar{H}$ finitely generate a subgroup H' which is included in some finite saturated subgroup H by Facts 3.3.4 and 3.3.6. □

3.4 Scheme of the proof of Theorem 3.3.2

This section is devoted to the scheme of the proof of Theorem 3.3.2 which implies the Main Theorem 3.3.1. The details are given in section 3.5. Groups of the form $G(S_{d_0}, H)$ share similarities with the Basilica group defined by a three state automaton introduced by Grigorchuk and Zuk in [GZ02a]. The Basilica group was shown to be amenable by Bartholdi and Virag (see [BV05]) using selfsimilarity of some random walks. This method, called the "Münchhausen trick", has been used to show amenability of a few other groups (see [Kai05] and [Muc05]). We proceed with the same methods, using Kesten's criterion on symmetric random walks.

As H is a finite saturated subgroup of $\bar{H} = S_{d_1}^{d_0-1} \times S_{d_2}^{d_1-1} \times \ldots$, let us denote H_k its restriction to $\bar{H}_k = S_{d_{k+1}}^{d_k-1} \times S_{d_{k+2}}^{d_{k+1}-1} \times \ldots$ which is also a finite saturated subgroup and it follows from (3.2.5) that $G(S_{d_0}, H) \hookrightarrow G(S_{d_1}, H_1) \wr S_{d_0}$, and more generally the group $G(S_{d_k}, H_k)$ is a directed subgroup of $Aut(T_{\sigma^k \bar{d}})$ satisfying the crucial :

$$G(S_{d_k}, H_k) \hookrightarrow G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}.$$

The word metric does not behave appropriately enough through this wreath product embedding, rather use :

Proposition 3.4.1 (A fractal family of pseudo norms of exponential growth)**.** *There exists a family of pseudo norms ν^k on $G(S_{d_k}, H_k)$ (which means symmetric positive functions $\nu^k : G(S_{d_k}, H_k) \to \mathbb{R}^+$ satisfying the triangle inequality) such that :*

a) if g belongs to $G(S_{d_k}, H_k)$ and has image $g = (g_1, \ldots, g_{d_k})\sigma$ in $G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}$, then $\nu^k(g) \le \nu^{k+1}(g_1) + \cdots + \nu^{k+1}(g_{d_k})$, and

b) if $B_{\nu^k}(r) = \{g \in G(S_{d_k}, H_k) | \nu^k(g) \le r\}$, then $\# B_{\nu^k}(r) \le C^r$ where C is a constant depending only on the bound D on the valencies of the tree and the size of the finite group H (which contains H_k for every k).

Let p denote the symmetric probability measure on the finite generating set $S_{d_0} \cup H$ of $G(S_{d_0}, H)$ defined by $p(a) = \frac{1}{2\#S_{d_0}}$ for $a \in S_{d_0}$ and $p(h) = \frac{1}{2\#H}$ for $h \in H$. The random walk associated is $Z_N = s_1 \dots s_N$ where the s_i are independent random variables identically p-distributed. The set of random sequences $(Z_N)_{N\in\mathbb{N}}$ is endowed with the product measure (defined on the sigma algebra generated by cylinders) $P = p^{\otimes\infty}$. The drift of this random walk with respect to the pseudo norm $\nu = \nu^0$ vanishes :

Proposition 3.4.2. *The random walk (Z_N) satisfies :*

$$\frac{\nu(Z_N)}{N} \underset{N\to+\infty}{\longrightarrow} 0, \quad P \text{ a.s.}.$$

To prove this proposition, another (non symmetric) random walk is usefull. Let us define $Y_n = t_0 t_1 t_2 \dots t_n$ where t_{2i} are random variables equidistributed on S_{d_0} and t_{2i+1} are equidistributed on H and all the t_i are independent. Denote $Q = (q_{S_{d_0}} \otimes q_H)^{\otimes\infty}$ the associated measure on the set of sequences $(Y_n)_{n\in\mathbb{N}}$ (with respect to the cylindrical sigma algebra), then :

Proposition 3.4.3. *The random walk (Y_n) satisfies :*

$$\frac{\nu(Y_n)}{n} \underset{n\to+\infty}{\longrightarrow} 0, \quad Q \text{ a.s.}.$$

The key argument to prove Proposition 3.4.3 is the next Lemma 3.4.6 together with Proposition 3.4.1 a).

Remark 3.4.4 (On the dependence on t_0). The pseudo norm $\nu = \nu^0$ satisfies $\nu(ah) = \nu(h)$ for every a in S_{d_0} and h in H (Proposition 3.5.2 (2)), which ensures $\nu(Y_n) = \nu(t_0^{-1} Y_n) = \nu(t_1 t_2 \dots t_n)$, showing that $\nu(Y_n)$ is independent of t_0. This will be of importance and justifies the :

Definition 3.4.5. Two random variables U and V on the group $G(S_{d_0}, H)$ are said ν-equivalent if $\nu(U)$ and $\nu(V)$ have the same distribution law on $\mathbb{N}$, which will be denoted :

$$U \sim_{\nu\text{-law}} V.$$

Consider the random walk $(Y_n)_{n\in\mathbb{N}}$ and its image in the wreath product of the form $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ where σ_n is a random variable in S_{d_0} and the coordinates Y_n^t for $t \in \{1, \dots, d_0\}$ are random variables in $G(S_{d_1}, H_1)$. The point is that $(Y_n^t)_n$ follows the law of the similarly defined random walk $(Y'_m)_{m\in\mathbb{N}}$ on $G(S_{d_1}, H_1)$ (which is taking independent equidistributed increments alternatively in S_{d_1} and H_1), but at a slower speed. More precisely :

Lemma 3.4.6 (Similarity of the random walks (Y_n) and (Y'_m))**.** *Let $(Y_n)_{n\in\mathbb{N}}$ the random walk defined above and $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ its image in the wreath product. For each coordinate $(Y_n^t)_n$ the sequence $(Y_n)_n$ defines a sequence of random integers $(m_t(n))_n$ and a random sequence $(\varepsilon_t(n))_n$ taking values in $\{0, 1\}$ such that :*

1. *For every integer n the values of $m_t(n)$ and $\varepsilon_t(n)$ depend only on $(Y_{n'})_{n' \leq n}$.*
2. *For every integer n the coordinate Y_n^t belonging to $G(S_{d_1}, H_1)$ has the same ν^1-distribution law as the random variable $Y'_{m_t(n)+\varepsilon_t(n)}$. More precisely the conditional law :*
$$(Y_n^t | m_t(n), \varepsilon_t(n)) \sim_{\nu^1\text{-law}} Y'_{m_t(n)+\varepsilon_t(n)}.$$
3. *The random sequence $(m_t(n))_n$ satisfies :*
$$m_t(n) \sim_{n \to +\infty} \left(\frac{d_0 - 1}{d_0}\right) \frac{n}{d_0}, \quad Q \text{ a.s.}.$$

Propositions 3.4.1 and 3.4.2 are sufficient to apply the :

Theorem 3.4.7 (Kesten criterion of amenability [Kes59a])**.** *Let Γ be a finitely generated group and (Z_N) a symmetric random walk on Γ. The group Γ is amenable if and only if the sequence $(P(Z_{2N} = id_\Gamma))_N$ does not decay exponentially fast with N.*

The following fact is also usefull :

Fact 3.4.8. *Let (Z_N) a symmetric random walk on a finitely generated group Γ, then for any fixed integer N the function $\Gamma \to [0,1] : g \mapsto P(Z_{2N} = g)$ is maximal for $g = id_\Gamma$.*

Proof of the Fact 3.4.8. Let $p_k(x,y)$ denote the probability to go from x to y in k steps, let δ_x denote the function on Γ taking values 1 on x and 0 elsewhere and M the symmetric random walk operator on the space $l^2(\Gamma)$. Then Cauchy inequality implies :

$$\begin{aligned} p_{2N}(id,x)^2 &= \langle M^{2N}\delta_{id}, \delta_x\rangle^2 = \langle M^N\delta_{id}, M^N\delta_x\rangle^2 \\ &\leq ||M^N\delta_{id}||.||M^N\delta_x|| = p_{2N}(id,id).p_{2N}(x,x) = p_{2N}(id,id)^2. \end{aligned}$$

□

Note that Theorem 3.4.7 and Fact 3.4.8 only apply to symmetric random walks.

Proof of Theorem 3.3.2. Given an arbitrary positive ε the previous Fact 3.4.8 applied to the symmetric random walk (Z_N) constructed above raises :

$$P(\nu(Z_{2N}) \leq \varepsilon 2N) = \sum_{\nu(g) \leq \varepsilon 2N} P(Z_{2N} = g) \leq P(Z_{2N} = id_{G(S_{d_0},H)}) \# B_\nu(\varepsilon 2N),$$

and the Propositions 3.4.1 b) and 3.4.2 ensure :

$$P(Z_{2N} = id) \geq P\left(\frac{\nu(Z_{2N})}{2N} \leq \varepsilon\right) C^{-\varepsilon 2N} \sim_{N\to\infty} C^{-\varepsilon 2N}.$$

Thus $P(Z_{2N} = id)$ does not decrease exponentially fast and Kesten's criterion proves Theorem 3.3.2 and thus the Main Theorem. □

3.5 Details of the proof of Theorem 3.3.2

3.5.1 Fractal pseudo norms of exponential growth (proof of Proposition 3.4.1).

To the symmetric generating set $S=(S_{d_0}\cup H)\backslash\{1\}$ of $G(S_{d_0},H)$ is associated the word norm on $G(S_{d_0},H)$ by :

$$|g|=\min\{r|g=z_1\dots z_r, z_i\in S\}.$$

Denote $B_S(r)$ the ball of radius r associated to this norm (that is the set of all g such that $|g|\leq r$), then $\#B_S(r)\leq(\#S)^r$.

Note that since $G(S_{d_0},H)$ is a quotient of the free product $S_{d_0}*H$ a word $z_1\dots z_r$ is a minimal representative of g (that is $r=|g|$) only in the following cases : either $z_{2j}\in S_{d_0}\backslash\{1\}$ and $z_{2j+1}\in H\backslash\{1\}$, or conversely. This brings another definition :

$$||g||_0=\min\{r|g=a_1h_1a_2h_2\dots h_ra_{r+1}, a_i\in S_{d_0}, h_j\in H\}. \tag{3.5.1}$$

The following is straightforward :

Properties 3.5.1. *The function $||.||_0$ is a norm when restricted to the stabilizer of the first level $St_1(G(S_{d_0},H))$, namely it satisfies :*

1. $||gh||_0\leq||g||_0+||h||_0$ *for all* g,h *in* $G(S_{d_0},H)$,
2. $||g^{-1}||_0=||g||_0$ *for all* g *in* $G(S_{d_0},H)$,
3. $||g||_0=0$ *if and only if* $g\in S_{d_0}$,

This function $||.||_0$ is related to the usual word norm since for g in $G(S_{d_0},H)$:

$$2||g||_0-1\leq|g|\leq2||g||_0+1,$$

which implies that if $B_{||.||_0}(r)$ is the ball of radius r associated to $||.||_0$ in $G(S_{d_0},H)$, then :

$$\#B_{||.||_0}(r)\leq(\#S)^{2r+1}.$$

Following [BV05], let us introduce a new function ν on $G(S_{d_0},H)$ which is to be thought of as a fractal distance. For $g\in G(S_{d_0},H)$ and a vertex v on layer $k=l(v)$ of $T_{\bar{d}}$, denote by g_v the action of g on the descendant subtree $T_v\simeq T_{\sigma^k\bar{d}}$ of $T_{\bar{d}}$ and $g(v)\in S_{d_k}$ the action on the d_k children of v. The automorphism g_v of the rooted tree T_v belongs to the group $G(S_{d_k},H_k)$. The function defined by (3.5.1) for $G(S_{d_k},H_k)$ will be denoted by $||.||_k$.

A subtree T of $T_{\bar{d}}$ is said to be rooted if it contains the root $\emptyset$ of $T_{\bar{d}}$. It is said regular if for every vertex $v\in T$, either T contains the $d_{l(n)}$ descendant of v, either it contains none of them.

Given a finite regular rooted subtree T of $T_{\bar{d}}$ with set of leaves ∂T, define a function ν_T on $G(S_{d_0}, H)$ by :

$$\nu_T(g) = \sum_{v\in\partial T}(1 + ||g_v||_{l(v)}).$$

and a function $\nu : G(S_{d_0}, H) \to \mathbb{N}$ as :

$$\nu(g) = \min\{\nu_T(g)|T \text{ is a finite regular rooted subtree of } T_{\bar{d}}\}. \quad (3.5.2)$$

The construction (3.5.2) defines similarly a function $\nu^k : G(S_{d_k}, H_k) \to \mathbb{N}$ for the subgroup $G(S_{d_k}, H_k) < Aut(T_{\sigma^k\bar{d}}) \simeq Aut(T_v)$ for any vertex v on the kth layer. Note that $\nu = \nu^0$ and that the following proposition is still true replacing ν by ν^k and ν^1 by ν^{k+1}.

Proposition 3.5.2. *The function ν satisfies :*

1. *Let g in $G(S_{d_0}, H)$ and $g = (g_1, \dots, g_{d_0})\sigma$ be its embedded image in the wreath product $G(S_{d_0}, H) \hookrightarrow G(S_{d_1}, H_1) \wr S_{d_0}$, then :*

$$\nu(g) = \min\{\nu^1(g_1) + \dots + \nu^1(g_{d_0}), 1 + ||g||_0\}.$$

2. *Let g in $G(S_{d_0}, H)$, then $\nu(g) = \nu(g^{-1})$.*
3. *Let g, g' be in $G(S_{d_0}, H)$, then $\nu(gg') \leq ||g||_0 + \nu(g')$.*
4. *Let g, g' be in $G(S_{d_0}, H)$, then $\nu(gg') \leq \nu(g) + \nu(g')$.*

In particular, this function ν is a pseudo-norm on $G(S_{d_0}, H)$.

The use of induction in the proof of Proposition 3.5.2 requires the :

Property 3.5.3. *Let g in $G(S_{d_k}, H_k)$ have image $g = (g_1, \dots, g_{d_k})\sigma$ in the wreath product $G(S_{d_{k+1}}, H_{k+1}) \wr S_{d_k}$ and assume $||g||_k \geq 2$ then $||g_t||_{k+1} < ||g||_k$ for any coordinate t.*

Proof of Property 3.5.3. An element g admits a minimal representative of the form $g = h_1^{\sigma_1} \dots h_r^{\sigma_r}\sigma_{r+1}$ with σ_i in S_{d_k} and h_i in H_k (remind $x^y = yxy^{-1}$). Moreover by construction $h = (h', a_2, \dots, a_{d_k})$ with h' in H_{k+1} and a_i in $S_{d_{k+1}}$ and the conjugate h^σ is the same d_k-tuple where the coordinates are σ permuted. This ensures $||g_1||_{k+1} + \dots + ||g_{d_0}||_{k+1} \leq ||g||_k$. It is sufficient to prove the property for $||g||_k = 2$, that is $g = h_1^{\sigma_1}h_2^{\sigma_2}$. If $\sigma_1(1) \neq \sigma_2(1)$ the property is obvious. If $\sigma_1(1) = \sigma_2(1)$ then $||g_i||_{k+1} = 0$ if $i \neq \sigma_1(1)$ and $||g_{\sigma_1(1)}||_{k+1} = ||h_1'h_2'||_{k+1} = 1$ because $h_1'h_2'$ is an element of H_{k+1}. □

Proof of Proposition 3.5.2. Note that $1 + ||g||_0 = \nu_{\{\emptyset\}}(g)$ and assume the minimum in definition (3.5.2) is obtained for a finite regular rooted tree $T \neq \{\emptyset\}$. Clearly $\partial T = \partial T(1) \cup \dots \cup \partial T(d_0)$ where $T(v)$ denotes the intersection of T with the subtree T_v of $T_{\bar{d}}$ hung on vertex v, thus :

$$\nu_T(g) = \sum_{v\in\partial T}(1 + ||g_v||_{l(v)}) = \sum_{t=1}^{d_0}\sum_{v\in\partial T(t)}(1 + ||g_v||_{l(v)}) = \sum_{t=1}^{d_0}\nu_{T(t)}(g_t),$$

which is minimal if and only if $\nu_{T(t)}(g_t) = \nu^1(g_t)$ is minimal for all t. This implies part (1).

It follows that if $||g||_0 = 1$ then $\nu(g) = 2 = \nu(g^{-1})$. Similarly if $||g||_k = 1$ for $g \in G(S_{d_k}, H_k)$ then $\nu^k(g) = 2 = \nu^k(g^{-1})$. Assume by induction on r that $\nu^k(g) = \nu^k(g^{-1})$ if $||g||_k \leq r$ and this jointly for every level k, then the inverse formula $g^{-1} = \sigma^{-1}(g_1^{-1}, \dots, g_{d_0}^{-1})$ and the induction hypothesis ensuring $\nu^1((g^{-1})_1) + \dots + \nu^1((g^{-1})_{d_0}) = \nu^1(g_1) + \dots + \nu^1(g_{d_0})$ (as $||g_t||_{k+1} < ||g||_k$ by Property 3.5.3) together with part (1) show part (2).

To prove part (3), note first that $\nu(ag) = \nu(g)$ for all $a \in S_{d_0}$. Indeed, a only permutes the subtrees of the first level and does not increase any of the $||g_v||_{l(v)}$. To conclude, it is sufficient to show that when h is in H, we have $\nu_T(hg) \leq 1 + \nu_T(g)$ for any finite regular subtree T. Proceed by induction on the size of T. Indeed, this is true for $T = \{\emptyset\}$ by Property 3.5.1 (1). More generally, denoting $g = (g_1, \dots, g_{d_0})\sigma_0$ and $h = (h_1, a_2, \dots, a_{d_0})$ with g_t in $G(S_{d_1}, H_1)$, h_1 in H_1 and a_t in S_{d_1}, we get $hg = (h_1 g_1, a_2 g_2, \dots, a_{d_0} g_{d_0})\sigma_0$ and :

$$\nu_T(hg) = \nu_{T(1)}(h_1 g_1) + \sum_{t=2}^{d_0} \nu_{T(t)}(a_t g_t) \leq 1 + \nu_{T(1)}(g_1) + \sum_{t=2}^{d_0} \nu_{T(t)}(g_t) = 1 + \nu_T(g)$$

using the induction hypothesis on $T(1)$.

Part (4) is implied by part (3) in case $\nu(g) = 1 + ||g||_0$ or $\nu(g') = 1 + ||g'||_0$. Otherwise :

$$\nu(gg') \leq \sum_{t=1}^{d_0} \nu^1((gg')_t) = \sum_{t=1}^{d_0} \nu^1(g_t g'_{\sigma(t)}) \leq \sum_{t=1}^{d_0} \nu^1(g_t) + \nu^1(g_{\sigma(t)}) = \nu(g) + \nu(g'),$$

where the second inequality comes by joint induction on $||g||_k$ using Property 3.5.3. □

Let $B_\nu(r) = \{g \in G(S_{d_0}, H) | \nu(g) \leq r\}$ denote the ball of radius r associated to the function ν. The next proposition is crucial for our purpose.

Proposition 3.5.4. *Consider a spherically homogeneous rooted tree $T_{\bar{d}}$ of bounded valency $2 \leq d_i \leq D$, a finite subgroup H of $\bar{H}$ and the function ν constructed above, then the balls $B_\nu(r) \subset G(S_{d_0}, H)$ grow at most exponentially fast. Namely, there exists a constant C depending only on D and the size of H such that :*

$$\# B_\nu(r) \leq (C)^r, \quad \text{for all } r \text{ sufficiently large.}$$

In order to prove this proposition, recall classical estimates on the number of rooted subtrees of a rooted tree. The formula below can be found in [PR87], the equivalent is derived from Stirling's formula.

Proposition 3.5.5. *The number of (not necessarily regular) rooted subtrees of a D-regular tree T_D containing r vertices is :*

$$s_r^{(D)} = \frac{1}{r} C_{Dr}^{r-1} \sim_{r \to +\infty} \frac{1}{D-1} \sqrt{\frac{D}{2(D-1)\pi}} r^{-\frac{3}{2}} \left(\frac{D^D}{(D-1)^{(D-1)}} \right)^r .$$

More precisely the following is sufficient :

Corollary 3.5.6. *The number $t_r^{(D)}$ of regular rooted subtrees of $T_{\bar{d}}$ (with $\bar{d}$ bounded by D) containing at most r leaves satisfies :*

$$t_r^{(D)} \leq (K_D)^r, \qquad \textit{for } K_D = \frac{D^{2D}}{(D-1)^{2(D-1)}},$$

provided r is sufficiently large.

Proof. It is well known that a subtree with at most r leaves contains at most $2r-1$ vertices and the asymptotic equivalent of $s_r^{(D)}$ gives the corollary. □

Proof of Proposition 3.5.4. If $\nu(g) \leq r$ then there exists a regular rooted subtree T such that $\nu_T(g) \leq r$. In particular, such a subtree has less than r leaves so that there are at most $(K_D)^r$ choices for T (corollary 3.5.6). Given T, the element g is described by all $g(v) \in S_{d_{l(v)}}$ where $v \in \mathring{T}$, which allow at most $(D!)^{\#\mathring{T}} \leq (D!)^r$ choices, and all $g_v \in G(S_{d_{l(v)}}, H_{l(v)})$ with $v \in \partial T$, which satisfy :

$$\sum_{v \in \partial T} ||g_v||_{l(v)} \leq r.$$

The number of possibilities for this last choice is less than $(M+1)^{2r}$ where $M = max\{\#B_{||.||_k}(1)\}$ (finite because the size of the generating set $S_{d_k} \cup H_k$ on layer k depends only on $d_k \leq D$ and $\#H_k \leq \#H$) bounds the number of symbols which represent an automorphism of norm 1 on a given leaf. An extra symbol (a coma) is added to denote passing to the next leaf. All in all, taking $C = K_D D!(M+1)^2$ gives the desired result. □

3.5.2 Similarity of random walks (proof of Lemma 3.4.6).

First recall elementary probabilistic facts which will be usefull.

Fact 3.5.7. *Let $(z_i)_{i \geq 1}$ be independent random variables equidistributed on a finite group F. Then the sequence $(X_k)_{k \geq 1}$ of products $X_k = z_1 \dots z_k$ is a family of independent random variables equidistributed on F.*

Proof of Fact 3.5.7. Denote by q_F the equidistribution measure on the finite group F. It is sufficient to prove by induction that :

$$q_F^{\otimes \infty}(X_i = f_i, i \leq k) = \prod_{i=1}^{k} q_F^{\otimes \infty}(X_i = f_i) = \prod_{i=1}^{k} q_F(X_i = f_i),$$

for arbitrary $f_1, \dots, f_k$ in F, which comes from :

$$\begin{aligned} q_F^{\otimes\infty}(X_i = f_i, i \leq k) &= q_F^{\otimes\infty}(X_k = f_k | X_i = f_i, i \leq k-1) q_F^{\otimes\infty}(X_i = f_i, i \leq k-1) \\ &= q_F^{\otimes\infty}(z_k = f_{k-1}^{-1} f_k) \prod_{j=1}^{k-1} q_F^{\otimes\infty}(X_j = f_j) \\ &= q_F(z_k = f_{k-1}^{-1} f_k) \prod_{j=1}^{k-1} q_F(X_j = f_j). \end{aligned}$$

□

Fact 3.5.8. *Let z be a random variable equidistributed on a finite group F acting transitively on a finite set A, then $q_F(z(t) = t') = \frac{1}{\#A}$ for all t, t' in A.*

Proof of Fact 3.5.8. The quotient $F/Stab_F(t)$ is of size $\#A$. If $z_0(t) = t'$ (transitivity) then $z_0 Stab_F(t) = \{z | z(t) = t'\}$ has the same size as $Stab_F(t)$ by injectivity of left translation in F. □

Fact 3.5.9. *Let $(u_i)_{i \in \mathbb{N}}$ be independent random Bernoulli variables on $\{0,1\}$ (say $p(u_i = 0) = p$ and $p(u_i = 1) = 1 - p$ for some p in $]0,1[$). Let $f(w_N)$ be the number of alternations in the subsequence $w_N = u_1 \dots u_N$, that is the number of indexes i such that $u_i \neq u_{i+1}$. Equivalently, $1 + f(w_N)$ is the number of maximal packs of constant successive terms. Then :*

$$f(w_N) \sim_{N \to +\infty} 2p(1-p)N, \qquad P = p^{\otimes\infty} \ a.s..$$

Proof of Fact 3.5.9. Apply the law of large numbers to $f(w_N) = \sum_{i=1}^{N-1} 1_{\{u_i \neq u_{i+1}\}}$ knowing that $E(1_{\{u_i \neq u_{i+1}\}}) = 2p(1-p)$ and that the terms are independent. □

Proof of Lemma 3.4.6. Consider the random walk Y_n at step n as :

$$Y_n = t_0 t_1 \dots t_n = a_1 h_1 a_2 h_2 \dots a_s h_s a_{s+1}$$

with $s = [\frac{n}{2}]$ (a_{s+1} empty if n even), where the terms a_i (resp. h_i) are random variables equidistributed in S_{d_0} (resp. in H), all being independent. This can be rewritten $Y_n = h_1^{\sigma_1} \dots h_s^{\sigma_s} \sigma_{s+1}$ (remind the conjugate notation $h^\sigma = \sigma h \sigma^{-1}$) where the $\sigma_i = a_1 a_2 \dots a_i$ are independent random variables equidistributed in S_{d_0} by Fact 3.5.7.

Using coordinates in the wreath product an element h of H has the form $h = (h^1, a^2, \dots, a^{d_0})$ with h^1 in H_1 and a^i in S_{d_1} and each of them is equidistributed for h equidistributed in H by saturation (note that the coordinates are not independent). Conjugating by a rooted automorphism σ raises $h^\sigma = (a^{\sigma(1)}, \dots, a^{\sigma(d_0)})$ with h^1 in position $\sigma(1)$.

Consider now the random walk $Y_n = (Y_n^1, \dots, Y_n^{d_0})\sigma_n$ at time n and focus on coordinate t, which is a product $Y_n^t = u_1 \dots u_s$ of s independent terms such that

u_i belongs to and is equidistributed in S_{d_1} (resp. H_1) if $\sigma_i(t)$ belongs to $\{2,\ldots,d_0\}$ (resp. $\sigma_i(t)=1$). Since the σ_i are equidistributed in S_{d_0} the probability that u_i is in S_{d_1} (resp. H_1) for a given i is $\frac{d_0-1}{d_0}$ (resp. $\frac{1}{d_0}$) by Fact 3.5.8. This is summarized in :

$$Q(u_i=g)=\begin{cases}\frac{d_0-1}{d_0}\frac{1}{\#S_{d_1}} & \text{if } g\in S_{d_1},\\ \frac{1}{d_0}\frac{1}{\#H_1} & \text{if } g\in H_1,\end{cases}$$

and the terms u_i are independent. Define $m_t(n)$ to be the number of maximal packs of successive u_i belonging either to S_{d_1}, or to H_1 in the sequence $Y_n^t=u_1\ldots u_s$. Fact 3.5.9 ensures that :

$$m_t(n)\sim_{n\to+\infty} 2\frac{1}{d_0}\left(1-\frac{1}{d_0}\right)s\sim_{n\to+\infty}\left(\frac{d_0-1}{d_0}\right)\frac{n}{d_0}.$$

Given an integer n, assume we know the distribution $\mathcal{D}$ of which terms u_i are in S_{d_1} and H_1, then the kth pack of terms $v_k=u_{i_k}u_{i_k+1}\ldots u_{j_k}$ of constant belonging is a product of equidistributed independent elements in the finite group S_{d_1} or H_1 hence is equidistributed. In this situation $Y_n^t=v_0v_1\ldots v_{m_t(n)}$ where two cases are possible : either u_1 belongs to S_{d_1} (set $\varepsilon_t(n)=0$), the terms v_{2k+1} are equidistributed in H_1 and v_{2k} are equidistributed in S_{d_1}, which is of the form $Y'_{m_t(n)}$; or u_1 belongs to H_1 (set $\varepsilon_t(n)=1$), then re index the v_i as $Y_n^t=id_{S_{d_1}}v_1\ldots v_{m_t(n)+1}$ which is of the form $Y'_{m_t(n)+1}$ except for v_0 which follows the Dirac law on $id_{S_{d_1}}$; this has no influence on the ν-distribution of the sequences (Remark 3.4.4). In both cases :

$$(Y_n^t|\mathcal{D})\sim_{\nu\text{-law}} Y'_{m_t(n)+\varepsilon_t(n)},$$

where the condition depends only on the number of alternations $m_t(n)$ and the starting condition $\varepsilon_t(n)$ of the distribution $\mathcal{D}$. □

3.5.3 Zero drift of (Y_n) (proof of Proposition 3.4.3)

Note that the Kolmogorov 01-law implies almost sure constance of $\limsup\frac{\nu(Y_n)}{n}$:

Lemma 3.5.10. *For every integer k denote $(Y_n^{(k)})_n$ the random walk on $G(S_{d_k},H_k)$ which is taking independent equidistributed increments alternatively in S_{d_k} and H_k, in particular $(Y_n)=(Y_n^{(0)})$ and $(Y'_n)=(Y_n^{(1)})$. Then there exists l_k in $[0,\frac{1}{2}]$ such that :*

$$\limsup_{n\to+\infty}\frac{\nu^k(Y_n^{(k)})}{n}=l_k,\qquad Q_k=(q_{S_{d_k}}\otimes q_{H_k})^{\otimes\infty}\ a.s..$$

Proof. Proposition 3.5.2 (1) implies $\nu^k(Y_n^{(k)})\leq 1+||Y_n^{(k)}||_k\leq\frac{n+1}{2}$ so that the $\limsup$ is $\leq\frac{1}{2}$. Given l in $[0,\frac{1}{2}]$ the event $E_l=\{\limsup\frac{\nu^k(Y_n^{(k)})}{n}\leq l\}$ is a tail event, that is an event which is independent of any finite subsequence $(Y_n^{(k)})_{n\leq N}$, hence has probability 0 or 1 by the 01-Kolmogorov law. The function $l\mapsto Q_k(E_l)$ is increasing, right continuous and takes values in $\{0,1\}$, so that there exists l_k such that $Q_k(E_l) = 0$ for $l < l_k$ and $Q_k(E_l) = 1$ for $l \geq l_k$. Then :

$$Q_k\left(\{\limsup \frac{\nu^k(Y_n^{(k)})}{n} = l_k\}\right) = Q_k(E_{l_k} \setminus \cup_{n\geq 1} E_{l_k - \frac{1}{n}}) = 1.$$

□

Proof of Proposition 3.4.3. To show $l_0 = 0$, prove $l_k \leq \frac{(D-1)}{D} l_{k+1}$ where D is the bound on the valencies of the spherically homogeneous rooted tree $T_{\bar{d}}$. This is sufficient as $l_k \leq \frac{1}{2}$ for every k. To ease notations, compute for $k = 0$. Proposition 3.4.1 (a) ensures :

$$\limsup_{n\to+\infty} \frac{\nu(Y_n)}{n} \leq \limsup_{n\to+\infty} \sum_{t=1}^{d_0} \frac{\nu^1(Y_n^t)}{n} \leq \sum_{t=1}^{d_0} \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{n}. \tag{3.5.3}$$

To compute the right side introduce the condition $(m_t(n))$:

$$\limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{n} = \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{m_t(n)} \frac{m_t(n)}{n} \leq \limsup_{n\to+\infty} \frac{\nu^1(Y_n^t)}{m_t(n)} \limsup_{n\to+\infty} \frac{m_t(n)}{n},$$

where Lemma 3.4.6 gives $\limsup \frac{m_t(n)}{n} = (\frac{d_0-1}{d_0})\frac{1}{d_0}$, Q a.s. and

$$\limsup \frac{\nu^1(Y_n^t)}{m_t(n)} = \limsup \frac{\nu^1(Y'_{m_t(n)+\varepsilon_n^t})}{m_t(n)} = l_1, \quad Q \text{ a.s.}$$

because $m_t(n) \to +\infty$ Q a.s.. The last estimates gathered together on a Q probability one event show that :

$$l_0 \leq \sum_{t=1}^{d_0} l_1 \left(\frac{d_0-1}{d_0}\right)\frac{1}{d_0} = \left(\frac{d_0-1}{d_0}\right) l_1 \leq \left(\frac{D-1}{D}\right) l_1.$$

□

3.5.4 Zero drift of (Z_N) (proof of Proposition 3.4.2)

Recall the :

Fact 3.5.11. *Let $(a_i)_{i\in\mathbb{N}}$ be a random sequence in $\{0,1\}^{\mathbb{N}}$ endowed with a probability measure μ. Assume that there exists an infinite subset I of $\mathbb{N}$ such that $\mu(a_i = 1) \geq \delta > 0$ for all $i \in I$, then $\mu(a_i = 1$ for infinitely many $i) \geq \delta$.*

Proof of Fact 3.5.11. Let $E = \{(a_i) | a_i = 1 \text{ infinitely often}\}$ and assume by contradiction $\mu(E) = \delta' < \delta$, this implies $\mu(E^c \cap \{a_i = 1\}) \geq \delta - \delta'$ for all i in I. However the complement of E is the infinite increasing union :

$$E^c = \cup_{n\in\mathbb{N}}\{(a_i) | a_i = 0 \text{ for } i \geq n\} = \cup_{n\in\mathbb{N}} F_n,$$

so that $\mu(F_N) \geq 1 - \frac{\delta+\delta'}{2}$ for some N. But the case $i \geq N$ raises the contradiction :

$$\mu(E^c \cap \{a_i = 1\}) = \mu((E^c \setminus F_N) \cap \{a_i = 1\}) \leq \mu(E^c \setminus F_N) \leq \frac{\delta + \delta'}{2} - \delta' = \frac{\delta - \delta'}{2}.$$

□

Proposition 3.4.3 will be used in the (a priori) weaker form :

Corollary 3.5.12. *For every positive ε and α, there exists N_0 such that for $n \geq N_0$:*

$$Q\left(\frac{\nu(Y_n)}{n} \leq \varepsilon\right) \geq 1 - \alpha.$$

Proof. Assume the statement does not hold, then there exists ε_0, α_0 and infinitely many integers n_k with $Q\left(\frac{\nu(Y_{n_k})}{n_k} \geq \varepsilon_0\right) \geq \alpha_0$ and then

$$Q\left(\limsup \frac{\nu(Y_n)}{n} \geq \varepsilon_0\right) \geq \alpha_0$$

by Fact 3.5.11, contradicting Proposition 3.4.3. □

The random walks (Z_N) and (Y_n) are closely related by :

Fact 3.5.13. *Let N be a fixed integer. To each walk $Z_N = s_1 \ldots s_N$ is associated the number of alternations $a(N)$ from s_i in S_{d_0} to s_{i+1} in H or vice versa. Then the conditional law of Z_N satisfies :*

$$(Z_N | a(N)) \sim_{\nu-law} Y_{a(N)}.$$

Proof. Conditioning by the distribution $\mathcal{D}$ of which terms s_i are in S_{d_0} and in H, the walk is rewritten : $Z_N = s_1 \ldots s_{i_0} s_{i_0+1} \ldots s_{i_1} \ldots s_{i_{a(N)}} = t_0 t_1 \ldots t_{a(N)}$ where $t_{2j} = s_{i_{2j-1}} \ldots s_{i_{2j}}$ are equidistributed in S_{d_0} (except maybe t_0 which could be empty) and $t_{2j+1} = s_{i_{2j}} \ldots s_{i_{2j+1}}$ are equidistributed in H, all factors being independent, which is the definition of the random walk $Y_{a(N)}$. The condition matters only on $a(N)$ and not $\mathcal{D}$. □

This Fact 3.5.13 allows us to show a weak form :

Lemma 3.5.14. *For every positive ε and α, there exists M such that for $N \geq M$:*

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) \geq 1 - \alpha.$$

Proof. Fact 3.5.9 ensures that the conditioning term $a(N)$ satisfies $\lim_{N\to\infty} \frac{a(N)}{N} = \frac{1}{2}$, P almost surely. In particular for every positive α there exists an integer N_1 such that $P(a(N) \geq \frac{N}{3}) \geq 1 - \alpha$ for all $N \geq N_1$.

Now compute under the condition $a(N)$:

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) = \sum_{a(N)} P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon | a(N)\right) P(a(N)),$$

but if $N \geq N_1$ then $P(a(N) \leq \frac{N}{3}) \leq \alpha$. Moreover for $N \geq 3N_0$ (defined by Corollary 3.5.12) the condition $a(N) \geq \frac{N}{3} \geq N_0$ ensures via Fact 3.5.13 :

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon | a(N)\right) = Q\left(\frac{\nu(Y_{a(N)})}{a(N)} \frac{a(N)}{N} \leq \varepsilon\right) \geq 1 - \alpha,$$

because $\frac{a(N)}{N} \leq 1$. All in all, when $N \geq \max\{N_1, 3N_0\}$:

$$P\left(\frac{\nu(Z_N)}{N} \leq \varepsilon\right) \geq \sum_{a(N) \geq \frac{N}{3}} (1-\alpha) P(a(N)) \geq (1-\alpha)^2,$$

which proves Lemma 3.5.14. □

The previous Lemma ensures that P almost surely : $\liminf \frac{\nu(Z_N)}{N} = 0$ (Fact 3.5.11). To get Proposition 3.4.2 use :

Theorem 3.5.15 (Kingman subadditive Theorem ([Kal02] 9.14))**.** *Let $(X_{m,n})$ be random variables such that :*

1. $X_{0,n} \leq X_{0,m} + X_{m,n}$ *for all* $0 < m < n$,
2. $(X_{m+1,n+1})$ *has the same law as* $(X_{m,n})$,
3. $E(X_{0,1}^+) < +\infty$,

then the random sequence $(\frac{X_{0,n}}{n})$ *converges almost surely.*

Applying this to $X_{n,m} = \nu(Z_m^{-1} Z_n)$ shows that the inferior limit is in fact a limit, proving Proposition 3.4.2. The interested reader will remark that Lemma 3.5.14 is sufficient for our purpose and thus the Main Theorem does not rely on Kingman's Theorem.

3.6 Groups of intermediate growth

3.6.1 Generating pairs for alternate groups

In his paper [Wil04a] (Proposition 2.1), Wilson constructs interesting generating pairs of alternate groups $\mathcal{A}_d$:

Proposition 3.6.1 (Wilson [Wil04a])**.** *Let $d \geq 29$, then the alternate group of permutation $\mathcal{A}_d$ of the finite set $\{1, \ldots, d\}$ contains an eligible (see [Wil04a] for the full definition) pair of elements x_d, y_d. In particular :*

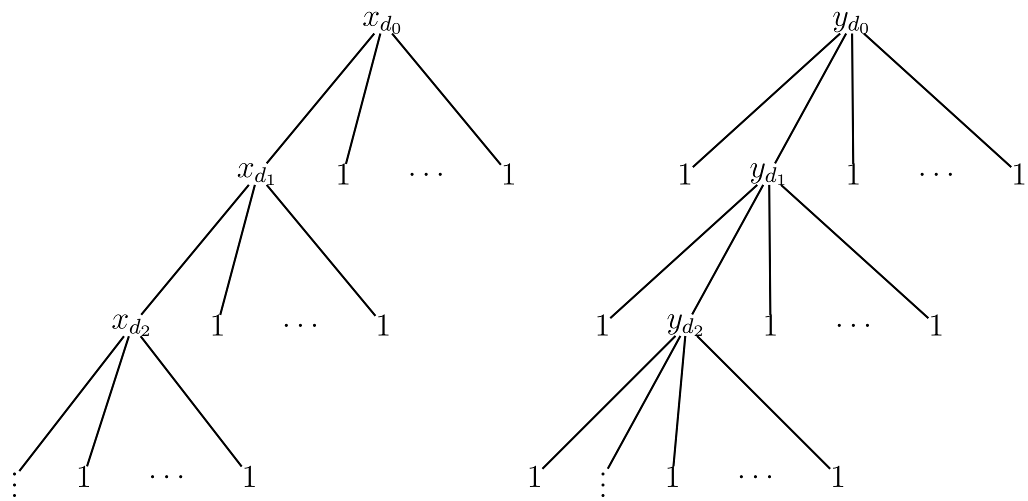

FIG. 3.3 – Portraits of the elements $x_{\bar{d}}$ and $y_{\bar{d}}$.

1) the pair is generating : $\langle x_d, y_d\rangle = \mathcal{A}_d$*, the elements have order 2 and 3 :* $x_d^2 = y_d^3 = 1$*, and a fixed point property that there exists* α *and* β *in* $\{1,\dots,d\}$ *such that :* $x_d(\alpha) = y_d x_d y_d^{-1}(\alpha) = \alpha$ *and* $y_d(\beta) = \beta$ *(up to re index we assume* $\alpha = 1$ *and* $\beta = 2$*).*

2) let $\hat{x} = (u, 1, \dots, 1)x_{d_0}$ *and* $\hat{y} = (1, v, 1, \dots, 1)y_{d_0}$ *belong to* $Aut(T_{\bar{d}})$ *with* $d_0 \geq 29$ *and* u, v *in* $Aut(T_{\sigma\bar{d}})$ *with* $u^2 = v^3 = 1$*, then the group generated by* $\hat{x}$ *and* $\hat{y}$ *contains the whole group of alternate rooted automorphisms* $\mathcal{A}_{d_0}$*. More precisely :*

$$\langle \hat{x}, \hat{y}\rangle \simeq \langle u, v\rangle \wr \mathcal{A}_{d_0}.$$

Given a (not necessarily bounded) sequence $\bar{d}$ of integers ≥ 29, the above Proposition 3.6.1 allows to define recursively the following pair of automorphisms of the spherically homogeneous rooted tree $T_{\bar{d}}$ (remind the assumption on fixed points $\alpha = 1$ and $\beta = 2$) :

$$\begin{aligned} x_{\bar{d}} &= (x_{\sigma\bar{d}}, 1, \dots, 1)x_{d_0}, \\ y_{\bar{d}} &= (1, y_{\sigma\bar{d}}, 1, \dots, 1)y_{d_0}. \end{aligned} \tag{3.6.1}$$

This definition is best understood by looking at the portraits on Figure 3.3. The automorphism subgroup generated is denoted $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}}\rangle$. Note that in the case $\bar{d} = \sigma\bar{d}$ is a constant sequence the group $H_{\bar{d}}$ is generated by a three state automaton.

Property 3.6.2. *The alternate automorphism* $x_{\bar{d}}$ *has order 2, and* $y_{\bar{d}}$ *has order 3.*

Proof. Show by joint (on $x_{\sigma^i\bar{d}}$ for i in $\mathbb{N}$) induction on k that $x^2_{\sigma^i\bar{d}}$ acts trivially on the k first levels of $T_{\sigma^i\bar{d}}$. This implies it acts trivially on the whole tree hence is trivial automorphism. Proposition 3.6.1 1) ensures :

$$x^2_{\sigma^i\bar{d}} = (x^2_{\sigma^i\bar{d}}, 1, \dots, 1)x^2_{d_i} = (x^2_{\sigma^i\bar{d}}, 1, \dots, 1),$$

which initiates the induction. Moreover $x^2_{\sigma^i\bar{d}}$ acts trivially on the subtrees $T_2, \dots, T_{d_i}$ of $T_{\sigma^i\bar{d}}$ and as $x^2_{\sigma^{i+1}\bar{d}}$ on T_1 which acts trivially on the k first level of T_1 by induction hypothesis. This proves $x^2_{\sigma^i\bar{d}}$ acts trivially on the $k+1$ first levels of $T_{\sigma^i\bar{d}}$. □

3.6.2 Density properties

Proposition 3.6.3. *The subgroup $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}}\rangle < Aut^e(T_{\bar{d}})$ is dense in $Aut^e(T_{\bar{d}})$ endowed with the profinite topology from (3.2.4).*

Proof. It is sufficient to show that the subgroup $\mathcal{A}_{d_k} \wr \cdots \wr \mathcal{A}_{d_0} < Aut^e(T_{\bar{d}})$ of alternate automorphisms of portrait supported on the k firsts levels is included in $H_{\bar{d}}$ for arbitrary k. Proceed by joint (on $H_{\sigma^i\bar{d}}$ for $i \in \mathbb{N}$) induction on k to show :

$$H_{\sigma^i\bar{d}} \simeq H_{\sigma^{i+k}\bar{d}} \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_i}, \tag{3.6.2}$$

which will be sufficient taking $i = 0$ and the trivial subgroup of $H_{\sigma^k\bar{d}}$. The case $k = 0$ follows from Proposition 3.6.1 2) :

$$H_{\sigma^i\bar{d}} = \langle x_{\sigma^i\bar{d}}, y_{\sigma^i\bar{d}}\rangle \simeq \langle x_{\sigma^{i+1}\bar{d}}, y_{\sigma^{i+1}\bar{d}}\rangle \wr \mathcal{A}_{d_i} = H_{\sigma^{i+1}\bar{d}} \wr \mathcal{A}_{d_i}. \tag{3.6.3}$$

Assuming isomorphism (3.6.2) then isomorphism (3.6.3) for $i+k$ proves step $k+1$:

$$H_{\sigma^i\bar{d}} \simeq H_{\sigma^{i+k}\bar{d}} \wr \mathcal{A}_{d_{i+k-1}} \wr \cdots \wr \mathcal{A}_{d_i} \simeq H_{\sigma^{i+k+1}\bar{d}} \wr \mathcal{A}_{d_{i+k}} \wr \mathcal{A}_{d_{i+k-1}} \wr \cdots \wr \mathcal{A}_{d_i}.$$

□

This density property is in contrast with the case of the full (non alternate) automorphism group of a rooted tree :

Proposition 3.6.4. *The group $Aut(T_{\bar{d}})$ endowed with the profinite topology from (3.2.3) admits no finitely generated dense subgroup.*

Proof. Denote $sgn : S_d \to \mathbb{Z}/2\mathbb{Z}$ the signature morphism of permutations. Given an element g in $Aut(T_{\bar{d}})$, recall that $g(v)$ is the permutation in $S_{d_{l(v)}}$ associated to vertex v in the portrait of g.

(Recall $g = (g_{1\dots1}, \dots, g_v, \dots, g_{d_0\dots d_{l(v)-1}})\tau_{l(v)-1}$ with g_v in $Aut(T_v) \simeq Aut(T_{\sigma^{l(v)}\bar{d}})$ and $\tau_{l(v)-1} \in Aut(T_{d_0\dots d_{l(v)-1}})$, then g_v has image $g_v = (g_{v1}, \dots, g_{vd_{l(v)}})g(v)$ via the isomorphism $Aut(T_{\sigma^{l(v)}\bar{d}}) \simeq Aut(T_{\sigma^{l(v)+1}\bar{d}}) \wr S_{d_{l(v)}}$.)

Similarly to Lemma 1. in [Ale83], define for each integer k the following morphism (of products of signatures of permutations on level k in the portraits) :

$$\begin{array}{rccl} R_k : & Aut(T_{\bar{d}}) & \to & \mathbb{Z}/2\mathbb{Z} \\ & g & \mapsto & R_k(g) = \prod_{v \in Level(k)} sgn(g(v)). \end{array}$$

The computations via the isomorphism (3.2.2) show this is a group morphism. The product morphism $\varphi : Aut(T_{\bar{d}}) \to (\mathbb{Z}/2\mathbb{Z})^\infty$ defined as $\varphi(g) = (R_0(g), R_1(g), \dots)$ is then a surjective group morphism continuous for the profinite topologies. Assume now there exists a finitely generated dense subgroup G of $Aut(T_{\bar{d}})$, then $\varphi(G)$ is a finitely generated dense subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\infty}$. This is impossible since any finitely generated subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\infty}$ is finite thanks to Fact 3.3.4. □

Density in $Aut^e(T_{\bar{d}})$ of a finitely generated subgroup implies superpolynomial growth :

Proposition 3.6.5. *Let $\bar{d} = (d_i)_{i\in\mathbb{N}}$ a sequence of integers $d_i \geq 3$, then any dense finitely generated subgroup of $Aut^e(T_{\bar{d}})$ has superpolynomial growth.*

Proof. Let G be such a group and k an arbitrary integer, then the level k stabilizer $St_k(G) \simeq G_{1\ldots 1} \times \cdots \times G_{d_0\ldots d_{k-1}}$ is a direct product of $d_0 \ldots d_{k-1}$ subgroups of $Aut^e(T_{\sigma^k\bar{d}})$ each of which inherits the property to be dense and finitely generated. In particular each of the groups G_v is infinite ($d_i \geq 3$) and thus has at least linear growth, so that the subgroup $St_k(G)$ of finite index and thus G have growth function at least $b(r) \gtrsim r^{d_0\ldots d_{k-1}}$, hence superpolynomial. □

3.6.3 Intermediate growth

Proposition 3.6.6. *The group $H_{\bar{d}} < Aut(T_{\bar{d}})$ has intermediate growth.*

Proof of Proposition 3.6.6. Superpolynomial growth follows from Propositions 3.6.3 and 3.6.5, so there remains to prove subexponential growth. Proceed as in [Gri85]. Denote $B_k(r)$ the ball of radius r in $H_{\sigma^k\bar{d}}$ for the word metric $|.|_k$ associated with the generating set $\langle x_{\sigma^k\bar{d}}, y_{\sigma^k\bar{d}}\rangle$, denote $b_k(r)$ its cardinal and $c_k = \lim \sqrt[r]{b_k(r)} = h_{\{x_{\sigma^k\bar{d}}, y_{\sigma^k\bar{d}}\}}(H_{\sigma^k\bar{d}})$ its exponential growth rate. The fixed point condition on eligible pairs ensures :

$$x_{\bar{d}}y_{\bar{d}}x_{\bar{d}}y_{\bar{d}}^{-1}x_{\bar{d}} = (x_{\sigma\bar{d}}, 1, \ldots, y_{\sigma\bar{d}}, y_{\sigma\bar{d}}^{-1}, \ldots, 1)x_{d_0}y_{d_0}x_{d_0}y_{d_0}^{-1}x_{d_0}, \tag{3.6.4}$$

with $y_{\sigma\bar{d}}$ and $y_{\sigma\bar{d}}^{-1}$ in positions $x_{d_0}(2)$ and $x_{d_0}y_{d_0}x_{d_0}(2)$ and the second and third $x_{\sigma\bar{d}}$ cancel out. As the generators are of order 2 and 3 every element $g = (g_1, \ldots, g_{d_0})\sigma$ in $B_0(r)$ admits a minimal representative word of the form $g = x_{\bar{d}}y_{\bar{d}}^{\varepsilon_1}x_{\bar{d}}y_{\bar{d}}^{\varepsilon_2}\ldots x_{\bar{d}}y_{\bar{d}}^{\varepsilon_n}x_{\bar{d}}$, with ε_i in $\{-1, 1\}$. Given g (more precisely given a fixed minimal representative word), denote $a(g)$ the number of alternations in the sequence (ε_i), equality (3.6.4) implies :

$$|g_1|_1 + \cdots + |g_{d_0}|_1 \leq |g|_0 - a(g). \tag{3.6.5}$$

Given any parameter $t \geq 2$, split the ball $B_0(r)$ into :

$$B_0^+(r) = \{g \in B_0(r) | a(g) \geq \tfrac{r}{t}\},$$
$$B_0^-(r) = \{g \in B_0(r) | a(g) \leq \tfrac{r}{t}\}.$$

The size of the first part of the ball is bounded by :

$$b_0^+(r) \leq \#\mathcal{A}_{d_0} \sum_{r_1+\cdots+r_{d_0}\leq(1-\frac{1}{t})r} b_1(r_1)\ldots b_1(r_{d_0}). \tag{3.6.6}$$

Indeed, each element $g = (g_1, \dots, g_{d_0})\sigma$ of $B_0(r)$ is injectively described by the permutation σ in $\mathcal{A}_{d_0}$ and the coordinates $g_1, \dots, g_{d_0}$ the sum of the $|.|_1$ length is bounded by $r - a(g) \leq (1 - \frac{1}{t})r$ thanks to computation (3.6.5). The size of the second part of the ball is bounded by (recall notation C_n^k for the number of subsets of size k in $\{1, \dots, n\}$) :

$$b_0^-(r) \leq 4 \sum_{s \leq \frac{r}{t}} C_r^s \leq 4\frac{r}{t} C_r^{\frac{r}{t}}. \tag{3.6.7}$$

Indeed the term 4 corresponds to choosing the start of the representative word (y, y^{-1}, xy or xy^{-1}), s represents the number of alternation $a(g)$ and C_r^s the number of choice for the positions of such alternations.

The size is estimated by $b_0(r) \leq b_0^+(r) + b_0^-(r) \leq \max\{2b_0^+(r), 2b_0^-(r)\}$, and taking limits of r-roots raises $c_0 \leq \max\{c_1^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{(\frac{1}{t}-1)}\}$, since Stirling formula ensures :

$$(4\frac{r}{t} C_r^{\frac{r}{t}})^{\frac{1}{r}} \sim_{r\to\infty} \left(4\frac{r}{t}\frac{\sqrt{(2\pi r)}}{\sqrt{2\pi\frac{r}{t}}\sqrt{2\pi(1-\frac{1}{t})r}}\right)^{\frac{1}{r}} \frac{\frac{r}{e}}{(\frac{r}{et})^{\frac{1}{t}}((1-\frac{1}{t})\frac{r}{e})^{1-\frac{1}{t}}} \sim_{r\to\infty} t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}.$$

The estimate is valid for any level k so that for all parameter $t \geq 2$:

$$c_k \leq \max\{c_{k+1}^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\}.$$

In particular, this shows the sequence $(c_k)_k$ increases (note $t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1} \to 1$ for $t \to \infty$). Moreover the sequence is bounded by 2 (the groups are quotients of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$), hence admits a limit c_∞, which satisfies by continuity :

$$c_\infty \leq \max\{c_\infty^{1-\frac{1}{t}}, t^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\}$$

for any parameter $t \geq 2$, which is impossible unless $c_\infty = 1$ (otherwise take t large enough). This shows subexponential growth of the groups $H_{\sigma^k \bar{d}}$. □

Remark 3.6.7. When the tree has bounded valency, set $f(r) = \max\{b_k(r) | k \in \mathbb{N}\}$ the estimate (3.6.6) can be made homogeneous on $d_k \leq D$. This together with estimate (3.6.7) applied for a parameter t of the form $t = \frac{K}{\log(r)}$ raises inequality :

$$f(r) \leq K\left(\sum_{r_1+\dots+r_D \leq (1-\frac{K}{\log(r)})r} \prod_{i=1}^{D} f(r_i)\right) + KC_r^{\frac{K}{\log(r)}}.$$

A computation due to Erschler (Lemma 6.4 in [Ers04a]) gives the explicit upper bound on the growth :

$$b_0(r) \leq f(r) \leq \exp\left(\frac{K\log(\log(r))r}{\log(r)}\right).$$

3.7 Groups of non uniform growth

3.7.1 A Theorem of Wilson

The first examples of groups with non uniform exponential growth have been constructed by Wilson in [Wil04b]. The following Theorem from [Wil04a] is a generalization.

Theorem 3.7.1 (Wilson [Wil04a]). *Let k be a positive integer and χ_k a class of groups with the two properties :*

1. *each group G in χ_k is perfect (that is $G = [G, G]$) and can be generated by k involutions ;*
2. *each group G in χ_k is isomorphic to a permutational wreath product $G_1 \wr \mathcal{A}_d$ with $G_1 \in \chi_k$ and $d \geq 29$.*

Then each group G in χ_k contains two sequences of elements $(a^{(n)})$, $(b^{(n)})$ such that :
(a) $(a^{(n)})^2 = (b^{(n)})^3 = 1$ and $\langle a^{(n)}, b^{(n)} \rangle = G$ for each n and,
(b) $h_{\{a^{(n)},b^{(n)}\}}(G) \to 1$ as $n \to \infty$.

In section 4. of [Wil04a], Wilson constructs subgroups of $Aut^e(T_{\bar{d}})$ for unbounded sequences $\bar{d} = (d_i)_i$ in the classes χ_k. Unboundedness of the sequence permits to construct such groups with a subgroup isomorphic to the free group $\mathbb{F}_2$ on two generators. This ensures exponential growth, but prevents amenability.

In the next section groups in the class χ_k are constructed similarly but acting on bounded valency rooted tree. The Main Theorem 3.3.1 will apply to show amenability. Exponential growth is due to the presence of free semigroups. Note however that in [Wil04b] Wilson constructs groups of automorphism of a regular (in particular bounded valency) rooted tree which have non uniform growth and contain a free group.

3.7.2 Amenable groups of non uniform growth

Let $\bar{d} = (d_i)_i$ be a bounded sequence of integers $5 \leq d_i \leq D$, define a subgroup of the group $\bar{H}$ (constructed in section 3.2.3) as $\bar{A} < \bar{H} = S_{d_1}^{d_0 - 1} \times S_{d_2}^{d_1 - 1} \times \dots$ where $\bar{A} = \mathcal{A}_{d_1} \times \mathcal{A}_{d_2} \times \dots$ as an abstract group and each group $\mathcal{A}_{d_k}$ is acting as a rooted automorphism on $T_{1_{k-1}2}$; this is best understood by Figure 3.4.

Now for each integer d in $\{5, \dots, D\}$, denote $E_d = \{i \geq 1 | d_i = d\}$. There is a diagonal injection :

$$j_d : \mathcal{A}_d \hookrightarrow \prod_{i \in E_d} \mathcal{A}_{d_i} < \bar{A},$$

and the diagonal product of those injections :

$$j : A_{\bar{d}} = \prod_{d=5}^{D} \mathcal{A}_d \hookrightarrow \bar{A}.$$

To ease notations the image subgroup of $A_{\bar{d}}$ is still denoted $A_{\bar{d}}$. It is a finite saturated subgroup of $\bar{A}$. The subgroup of $Aut^e(T_{\bar{d}})$ generated by alternate rooted automorphisms $\mathcal{A}_{d_0}$ and $A_{\bar{d}}$ is denoted $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}}) < Aut^e(T_{\bar{d}})$. Note that when $\bar{d} = \sigma\bar{d}$ is a constant sequence, then $A_{\bar{d}}$ is the diagonal injection of $\mathcal{A}_d$ in $\bar{A}$ and the group $G(\mathcal{A}_d, A_{\bar{d}})$ is generated by a finite automaton.

Proposition 3.7.2. *Let $\bar{d} = (d_i)_i$ a bounded sequence of integers $29 \leq d_i \leq D$, the group $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}})$ belongs to the class χ_k where k depends only on D.*

Proof. Show this Proposition simultaneously for all groups $G_i = G(\mathcal{A}_{d_i}, A_{\sigma^i\bar{d}}) < Aut^e(T_{\sigma^i\bar{d}})$. The group G_i is perfect because generated by copies of the groups $\mathcal{A}_{d_i}$, $\mathcal{A}_d, d \in \{5, \dots, D\}$ which are perfect (even simple). Moreover, those groups (hence G_i) are generated by double transpositions, in particular by involutions the number of which depends only on D, so that the condition (1) of definition of groups in the class χ_k is satisfied for some k depending only on D.

To check condition (2), note first that the injection in the wreath product (3.2.1) has image in :

$$G_i = G(\mathcal{A}_{d_i}, A_{\sigma^i\bar{d}}) \hookrightarrow G(\mathcal{A}_{d_{i+1}}, A_{\sigma^{i+1}\bar{d}}) \wr \mathcal{A}_{d_i} = G_{i+1} \wr \mathcal{A}_{d_i}. \tag{3.7.1}$$

This is clear for the generators in $\mathcal{A}_{d_i}$ and the generators b in $A_{\sigma^i\bar{d}}$ have image $b = (b', a, 1, \dots, 1)$ where a belongs to $\mathcal{A}_{d_{i+1}}$ and b' to $A_{\sigma^{i+1}\bar{d}}$ by construction. Now remains to prove this injection is onto hence an isomorphism.

Given any two elements a_1, a_2 in $\mathcal{A}_{d_{i+1}}$ there exists $b_1 = (b'_1, a_1, 1, \dots, 1)$ and $b_2 = (b'_2, a_2, 1, \dots, 1)$ in $A_{\sigma^i\bar{d}}$. Moreover the double transposition $\sigma = (13)(45)$ belongs to $\mathcal{A}_{d_i}$, so that G_i contains $b_2^\sigma = \sigma b_2 \sigma^{-1} = (1, a_2, b'_2, 1, \dots, 1)$, hence $[b_1, b_2^\sigma] = (1, [a_1, a_2], 1, \dots, 1)$ and then $1 \times \mathcal{A}_{d_{i+1}} \times \dots \times 1$ by perfection. Similarly given any two b'_1, b'_2 in $A_{\sigma^{i+1}\bar{d}}$, the group G_i contains $[b_1, b_2^\tau] = ([b'_1, b'_2], 1, \dots, 1)$ where $\tau = (23)(45)$, hence $A_{\sigma^{i+1}\bar{d}} \times 1 \times \dots \times 1$. Since $\mathcal{A}_{d_i}$ acts transitively by conjugation on the coordinates, this proves injection (3.7.1) is onto. □

Proposition 3.7.3. *Let $\bar{d} = (d_i)_i$ a bounded sequence of integers $5 \leq d_i \leq D$, the group $G_0 = G(\mathcal{A}_{d_0}, A_{\bar{d}})$ has exponential growth.*

Proof. Each group $\mathcal{A}_d$ contains the double transpositions $u = (12)(34)$ and $v = (12)(35)$. Moreover each of the groups $A_{\sigma^i\bar{d}} \simeq \mathcal{A}_{d_i} \times \mathcal{A}_{d_{i+1}} \times \dots$ contains the diagonal elements $\bar{u} = (u, u, u, \dots)$ and $\bar{v} = (v, v, v, \dots)$. The following Lemma due to Bartholdi (Proposition 2.3 in [Bar03]) ensures that $\langle \bar{u}u, \bar{v}v \rangle \simeq \mathbb{S}_2$ is a free semigroup. More precisely :

Lemma 3.7.4. *[Bartholdi [Bar03]] The quotient semigroup*

$$\langle \bar{u}u, \bar{u}v, \bar{v}u, \bar{v}v \rangle / (\bar{u}u = \bar{u}v, \bar{v}u = \bar{v}v) \simeq \mathbb{S}_2$$

is freely generated by $\{\bar{u}u, \bar{v}v\}$.

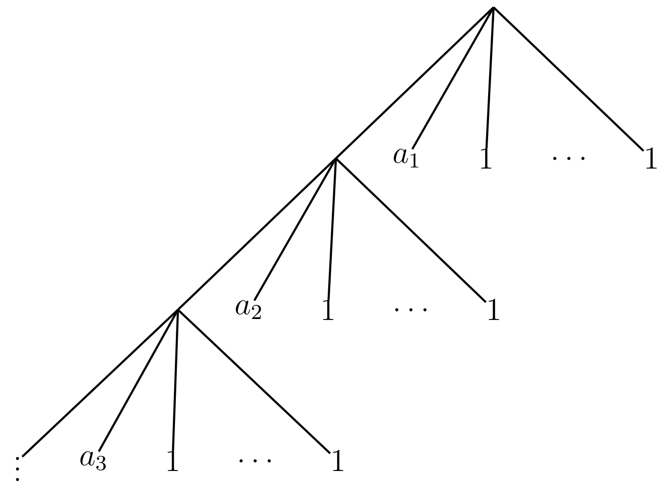

FIG. 3.4 – The group $\bar{A}$.

This ensures exponential growth of the group G_0. □

Corollary 3.7.5 (Theorem 3.1.1). *The groups $G(\mathcal{A}_{d_0}, A_{\bar{d}})$ associated to sequences $\bar{d} = (d_i)_i$ of integers $29 \leq d_i \leq D$ are (uncountably many pairwise non isomorphic) amenable groups of non uniform exponential growth.*

Proof. This follows from the Main Theorem 3.3.1, Wilson's Theorem 3.7.1, Proposition 3.7.2 and Proposition 3.7.3. The bracketted part follows from Corollary 3.8.2. □

3.7.3 Convergence of the Cayley graphs

This section is devoted to give another proof of some part of Wilson Theorem 3.7.1. Namely the convergence to 1 of the exponential growth rate of the generating sets $\{a^{(n)}, b^{(n)}\}$ can be understood as the convergence of the associated Cayley graphs of the group to the Cayley graph of a group $H_{\bar{d}}$ of intermediate growth introduced in section 3.6.

More precisely, let $G = G_0$ belong to some class χ_k, then by definition of the class there exists a sequence of groups G_i in χ_k and integers $d_i \geq 29$ such that $G_i \simeq G_{i+1} \wr \mathcal{A}_{d_i}$. The Theorem 3.7.1 of Wilson ensures in particular that for each integer i there exists a generating pair of elements $\langle a_i^{(0)}, b_i^{(0)} \rangle = G_i$ such that $(a_i^{(0)})^2 = (b_i^{(0)})^3 = 1$. Out of this first generating pair, Wilson constructs a sequence of generating pairs for G_i, defined inductively as (also see Figure 3.5 and compare with Figure 3.3) :

$$\begin{aligned} a_i^{(n+1)} &= (a_{i+1}^{(n)}, 1, \ldots, 1) x_{d_i}, \\ b_i^{(n+1)} &= (1, b_{i+1}^{(n+1)}, 1, \ldots, 1) y_{d_i}. \end{aligned} \tag{3.7.2}$$

The fact that $a_i^{(n)}$ and $b_i^{(n)}$ have order 2 and 3 and that they generate G_i is a

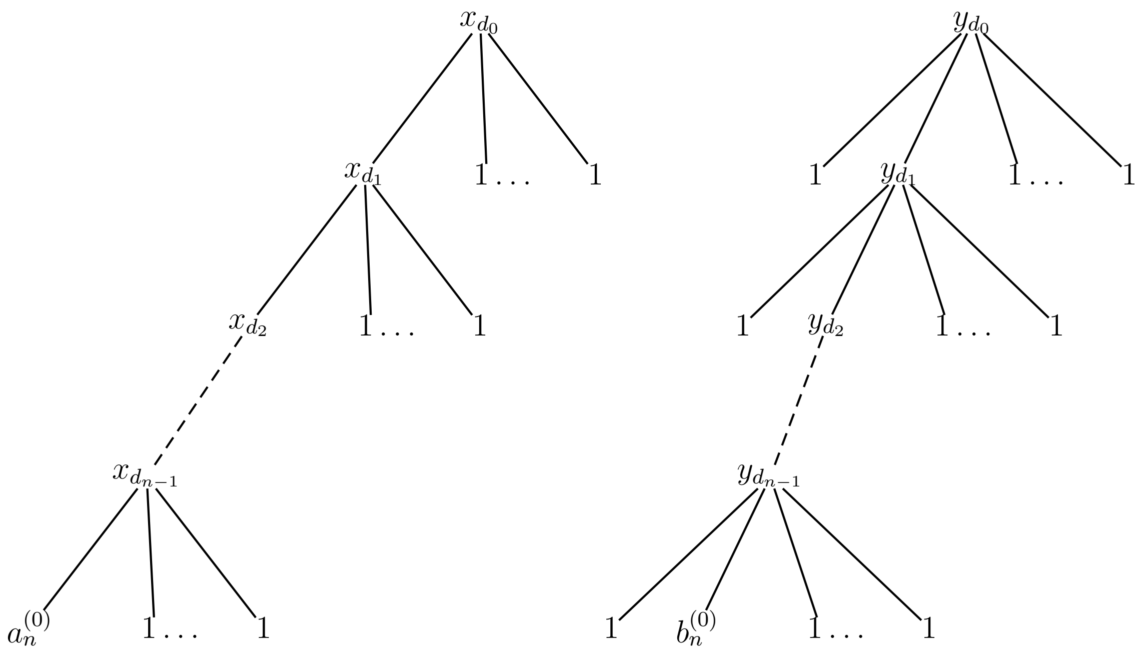

FIG. 3.5 – Portraits of the elements $a_0^{(n)}$ and $b_0^{(n)}$.

direct consequence of the properties of the generating pairs x_{d_i}, y_{d_i} of the alternate group $\mathcal{A}_{d_i}$ (see Proposition 3.6.1).

Definition 3.7.6 (Distance between Cayley graphs)**.** Let (Γ, S) and (Δ, T) be two groups with generating sets, denote $B_{\Gamma,S}(R)$ the restriction of the Cayley graph of Γ relatively to the generating set S to vertices at distance less than R of the neutral element (for the word distance in S). The distance between (Γ, S) and (Δ, T) is defined as :

$$d((\Gamma, S), (\Delta, T)) = \inf\{\frac{1}{R} | B_{\Gamma,S}(R) \sim_G B_{\Delta,T}(R)\},$$

where $Gr_1 \sim_G Gr_2$ if Gr_1 and Gr_2 are isometric as colored graphs.

Non uniform growth of the group G_0 comes from the two next propositions, since intermediate growth of $H_{\bar{d}}$ implies $h_{\{x_{\bar{d}}, y_{\bar{d}}\}}(H_{\bar{d}}) = 1$.

Proposition 3.7.7. *With the notations above :*

$$d((G_0, \{a_0^{(n)}, b_0^{(n)}\}), (H_{\bar{d}}, \{x_{\bar{d}}, y_{\bar{d}}\})) \underset{n \to +\infty}{\longrightarrow} 0.$$

Note that this Proposition is true independently of the amenability or not of the group G_0 in a class χ_k. In particular, such a convergence is also true for the non amenable groups constructed by Wilson in [Wil04b], [Wil04a].

Proposition 3.7.8. *If* $d((\Gamma, S_n), (\Delta, T)) \to 0$*, then :*

$$\limsup_{n \to \infty} h_{S_n}(\Gamma) \leq h_T(\Delta).$$

Proof of Proposition 3.7.8. Given a positive ε the definition of $h_T(\Delta)$ ensures that for $R \geq R_0$ large enough the ball $B_{\Delta,T}(R)$ has size $\#B_{\Delta,T}(R) \leq (h_T(\Delta)+\varepsilon)^R$. Now the convergence of Cayley graphs shows that for $n \geq N$ large enough $\#B_{\Gamma,S_n}(R) \leq (h_T(\Delta)+\varepsilon)^R$, and by subadditivity $\#B_{\Gamma,S_n}(kR) \leq \#B_{\Gamma,S_n}(R)^k \leq (h_T(\Delta)+\varepsilon)^{kR}$, so that :

$$h_{S_n}(\Gamma) = \lim \sqrt[kR]{\#B_{\Gamma,S_n}(kR)} \leq h_T(\Delta)+\varepsilon,$$

which was required. ☐

The proof of Proposition 3.7.7 uses the following :

Lemma 3.7.9 (of contraction)**.** *If $\hat{x} = (u,1,\dots,1)x_d$ and $\hat{y} = (1,v,1,\dots,1)y_d$ are as in Proposition 3.6.1, then for elements $g = (g_1,\dots,g_d)\sigma$ in the wreath product isomorphism $\langle \hat{x},\hat{y}\rangle \simeq \langle u,v\rangle \wr \mathcal{A}_d$, one has for each coordinate t :*

$$|g_t|_{\{u,v\}} \leq \frac{1}{2}(|g|_{\{\hat{x},\hat{y}\}}+1),$$

where $|.|_S$ denotes the word norm associated to the generating set S (inverses of elements of S have length 1).

Proof. It is sufficient to check that $\hat{x}\hat{y}^\varepsilon = (u,1,\dots,v^\varepsilon,1,\dots,1)x_d y_d$ with v^ε on coordinate $x_d(2) \neq 1$. ☐

Proof of Proposition 3.7.7. Introduce other relations depending on integer $l \geq 1$ on groups with generating sets : $(\Gamma,S) \sim_l (\Delta,T)$ if for every free word w of length less than l in S (elements and inverses) one has $w(S) = id_\Gamma$ if and only if $w(T) = id_\Delta$ (for $l = 1$ the relation $\sim_1$ just means $S \cup S^{-1}$ and $T \cup T^{-1}$ have the same size). If the relation $(\Gamma,S) \sim_{2l+1} (\Delta,T)$ is satisfied then $d((\Gamma,S),(\Delta,T)) \leq \frac{1}{l}$ because to describe $B_{\Gamma,S}(R)$ it is sufficient to know when $g'g^{-1} = s$ for every g,g' in $B_{\Gamma,S}(R)$ and s in $S \cup S^{-1}$.

To ease notations set $S_i^{(n)} = \{a_i^{(n)}, b_i^{(n)}\}$ and $T_i = \{x_{\sigma^i\bar{d}}, y_{\sigma^i\bar{d}}\}$. It is sufficient to show for all integers i : $(G_i, S_i^{(n)}) \sim_{l_n} (H_{\sigma^i\bar{d}}, T_i)$ with a sequence $l_n \to \infty$. Proceed by induction on n, using :

$$\begin{aligned} w(S_i^{(n+1)}) &= (w_1(S_{i+1}^{(n)}),\dots,w_{d_i}(S_{i+1}^{(n)}))w(x_{d_i},y_{d_i}),\\ w(T_i) &= (w_1(T_{i+1}),\dots,w_{d_i}(T_{i+1}))w(x_{d_i},y_{d_i}),\end{aligned}$$

where for each coordinate t the elements $w_t(S_{i+1}^{(n)})$ and $w_t(T_{i+1})$ involve the same word w_t because the permutations on the first level are the same for generators in $S_i^{(n+1)}$ or in T_i (namely x_{d_i} and y_{d_i}). Lemma 3.7.9 ensures that $|w_t(T_{i+1})| \leq \frac{1}{2}(|w(T_i)|+1)$ (and $|w_t(S_{i+1}^{(n)})| \leq \frac{1}{2}(|w(S_i^{(n+1)})|+1)$) so that if w has length less than $l_{n+1} = 2l_n - 1$ one has $w(S_i^{(n+1)}) = id_{G_i}$ if and only if $w(T_i) = id_{H_{\sigma^i\bar{d}}}$. The result follows since the sequence (l_n) starts with $l_0 = 1$ and $l_1 = 2$. ☐

Corollary 3.7.10 (of Proposition 3.7.7)**.** *The group $H_{\bar{d}}$ of intermediate growth is not finitely presented.*

Proof. Assume the contrary $H_{\bar{d}} = \langle x_{\bar{d}}, y_{\bar{d}} | r_1, \ldots, r_k \rangle$. Let R be bigger than the maximal length of the relations $r_1, \ldots, r_k$, and n large enough so that :

$$d((G_0, \{a_0^{(n)}, b_0^{(n)}\}), (H_{\bar{d}}, \{x_{\bar{d}}, y_{\bar{d}}\})) \leq \frac{1}{R}.$$

Then the automorphisms $a_0^{(n)}$ and $b_0^{(n)}$ satisfy all relation $r_1, \ldots, r_k$. In particular, G_0 is a quotient of $H_{\bar{d}}$ hence has subexponential growth. This contradicts Proposition 3.7.3. □

3.8 Non subexponential amenability

3.8.1 Description of normal subgroups

The normal subgroups of finite index of groups in a class χ_k are completely described by the :

Proposition 3.8.1 (Neumann [Neu86]). *Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of finitely generated perfect groups such that for each i there exists an integer $d_i \geq 5$ such that $G_i \simeq G_{i+1} \wr \mathcal{A}_{d_i}$. Consider the isomorphisms :*

$$G_0 \simeq G_i \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0} \simeq \underbrace{(G_i \times \cdots \times G_i)}_{d_0 \ldots d_{i-1} times} \rtimes Aut^e(T_{d_0 \ldots d_{i-1}}),$$

then the subgroups $K_i = (G_i \times \cdots \times G_i)$ for $i \in \mathbb{N}$ are the only normal subgroups of G_0 of finite index. Moreover if one (hence all) of the groups G_i is residually finite, then $(K_i)_{i \in \mathbb{N}}$ are the only non trivial normal subgroups of G_0 ; in particular G_0 is just infinite.

Proposition 3.8.1 (as well as Lemma 3.8.3) is a slight generalization of Theorem 5.1. in [Neu86]. The proof is given here for the sake of completeness and to avoid the reader multiple references and notations. The second part is also similar to Theorem 4. in [Gri00]. Note that all examples in this paper are groups of automorphism of a rooted tree. In particular they satisfy the assumption of residual finiteness.

Corollary 3.8.2. *Two groups G_0 and H_0 satisfying the hypothesis of Proposition 3.8.1 (in particular groups in a class χ_k) for two different sequences of integers $(d_i)_i$ and $(e_i)_i$ are non isomorphic.*

Proof. The index of K_i in G_0 has value :

$$[G_0 : K_i] = \#Aut^e(T_{d_0 \ldots d_{i-1}}) = \#(\mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0}) = a(d_{i-1})^{d_{i-2} \ldots d_0} \ldots a(d_1)^{d_0} a(d_0),$$

where $a(d) = \frac{d!}{2} = \#\mathcal{A}_d$. In particular the sequence of index of subgroups $([G_0 : K_i])_i$ is an isomorphism invariant from which the sequence $(d_i)_i$ can be recovered. □

Lemma 3.8.3. *Under the hypothesis of Proposition 3.8.1, the only normal subgroups of G_0 containing K_m are $K_0, K_1, \dots, K_m$.*

The proof of this lemma will use the :

Fact 3.8.4. *Given a finite group Γ, assume $\Delta \lhd \Gamma$ is a minimal normal subgroup (minimal means the only normal subgroup of Γ strictly contained in Δ is trivial) and that the centralizer $Cent_\Gamma(\Delta)$ of Δ is trivial. Then Δ is the unique minimal normal subgroup of Γ.*

Proof. Assume Δ' is another such subgroup, then $\Delta \cap \Delta'$ is trivial by minimality. In particular, for every $\delta \in \Delta$ and $\delta' \in \Delta'$ the commutator $\Delta \cap \Delta' \ni [\delta, \delta'] = 1$, which ensures $\Delta' \subset Cent_\Gamma(\Delta) = \{1\}$, contradiction. □

Proof of Lemma 3.8.3. By induction on m and using Fact 3.8.4, it is sufficient to prove that :

$$\mathcal{A}_{d_{m-1}}^{(1\dots1)} \times \dots \times \mathcal{A}_{d_{m-1}}^{(d_0\dots d_{m-2})} \simeq K_{m-1}/K_m \lhd G_0/K_m \simeq Aut^e(T_{d_0\dots d_{m-1}})$$

is minimal and has trivial centralizer, which shows K_{m-1} is the only minimal subgroup of G_0 containing K_m.

Let U a non trivial subgroup normal in G_0/K_m and included in K_{m-1}/K_m. Then $1 \neq y \in U$ can be written $y = (y_{1\dots1}, \dots, y_{d_0\dots d_{m-1}})$ in the wreath product $G_0/K_m \simeq \mathcal{A}_{d_{m-1}} \wr Aut^e(T_{d_0\dots d_{m-2}})$, with some coordinate $1 \neq y_v \in \mathcal{A}_{d_{m-1}}^{(v)}$. By simplicity, the normal closure of y_v is the full alternate group $\langle y_v \rangle_{\mathcal{A}_{d_{m-1}}} = \mathcal{A}_{d_{m-1}}$. Moreover the group $Aut^e(T_{d_0\dots d_{m-2}})$ acts by conjugation transitively on the coordinates, so that $U > \langle y \rangle_{G_0/K_m} = \mathcal{A}_{d_{m-1}}^{(1\dots1)} \times \dots \times \mathcal{A}_{d_{m-1}}^{(d_0\dots d_{m-2})} = K_{m-1}/K_m$, proving minimality. Transitivity also shows that the centralizer $Cent_{G_0/K_m}(K_{m-1}/K_m)$ is included in $St_{m-1}(G_0/K_m) = K_{m-1}/K_m$, which has trivial center, hence the centralizer is trivial. □

Proof of Proposition 3.8.1. Let X be a finite group and $f : G_0 \to X$ a homomorphism. Restricting to factors of the subgroups $K_m = G_m^{(1\dots1)} \times \dots \times G_m^{(v)} \times \dots \times G_m^{(d_0\dots d_{m-1})}$, it appears that for m large enough there exists $v \neq v'$ such that the associated factors have the same image $f(G_m^{(v)}) = f(G_m^{(v')}) = Y$, which must be abelian because $[G_m^{(v)}, G_m^{(v')}] = 1$, hence $Y = \{1\}$ because $G_m^{(v)} \simeq G_m$ is perfect. This shows $G_m^{(v)} \subset \operatorname{Ker}(f)$.

Moreover for each coordinate v' there exists $\varphi \in Aut^e(T_{d_0\dots d_{m-1}})$ such that $\varphi(v) = v'$, so that $\varphi G_m^{(v)} \varphi^{-1} = G_m^{(v')} \subset \operatorname{Ker}(f)$ and consequently K_m lies in the kernel of f. Applying Lemma 3.8.3 shows $\operatorname{Ker}(f) = K_i$ for some $i \leq m$, which proves the first part.

Now assume G_0 is residually finite, and $N \lhd G_0$ is an arbitrary normal subgroup. The description of the first part ensures that $\cap_{m \geq 0} K_m = \{1\}$, and as the sequence of subgroups $(K_m)_m$ is strictly decreasing there exists an integer n such that $N \leq K_n$

and $N \nleq K_{n+1}$. To get the second part, it is sufficient to prove $N \geq K_{n+1}$ since the first part will force $K_n = N$.

Consider $x \in N \setminus K_{n+1}$ and its image $x = (x_{1\ldots 1}, \ldots x_{d_0\ldots d_{n-1}})_n$ in the factor decomposition of K_n. There exists v such that :

$$x_v = (x_{v1}, \ldots, x_{vd_n})\sigma_v \in G_n^{(v)} \simeq G_{n+1} \wr \mathcal{A}_{d_n},$$

with a non trivial permutation σ_v, and in particular there are $s \neq t$ in $\{1, \ldots d_n\}$ with $\sigma_v(s) = t$. Now given any two elements ξ, η in G_{n+1}, define f, g in $K_n = (G_n \times \cdots \times G_n)$ as :

$$\begin{aligned} f &= (1, \ldots, 1, f_v, 1 \ldots, 1)_n, \quad f_v = (1, \ldots, 1, \xi, 1, \ldots, 1) \in G_{n+1} \wr \mathcal{A}_{d_n}, \\ g &= (1, \ldots, 1, g_v, 1 \ldots, 1)_n, \quad g_v = (1, \ldots, 1, \eta, 1, \ldots, 1) \in G_{n+1} \wr \mathcal{A}_{d_n}, \end{aligned}$$

with ξ, η on coordinate s. The normal subgroup N contains the commutator $[f, x] = fxf^{-1}x^{-1} = (1, \ldots, 1, [f_v, x_v], 1, \ldots, 1)_n$, where :

$$\begin{aligned} [f_v, x_v] &= (1, \ldots, \xi, \ldots, 1)(x_{v1}, \ldots, x_{vd_n})\sigma_v(1, \ldots, \xi^{-1}, \ldots, 1)\sigma_v^{-1}(x_{v1}^{-1}, \ldots, x_{vd_n}^{-1}) \\ &= (1, \ldots, 1, \xi, 1, \ldots, 1, x_{vt}\xi^{-1}x_{vt}^{-1}, 1, \ldots, 1), \end{aligned}$$

with ξ in coordinate s and $x_{vt}\xi^{-1}x_{vt}^{-1}$ in coordinate t. Taking another commutator, the subgroup N contains $[g, [f, x]] = (1, \ldots, 1, [g_v, [f_v, x_v]], 1, \ldots, 1)_n$ with :

$$[g_v, [f_v, x_v]] = (1, \ldots, 1, [\eta, \xi], 1, \ldots, 1),$$

and this for ξ, η in G_{n+1} arbitrary, which can be rewritten :

$$N \ni (1, \ldots, 1, [\eta, \xi], 1, \ldots, 1)_{n+1} \in G_{n+1} \wr Aut^e(T_{d_0 \ldots d_n}),$$

with $[\eta, \xi]$ in position vs. As this group is perfect, the subgroup N contains $1 \times \cdots \times G_{n+1}^{(vs)} \times \cdots \times 1$. The transitivity of the action of $Aut^e(T_{d_0 \ldots d_n})$ on level $n+1$ by conjugation ensures that N contains $(G_{n+1} \times \cdots \times G_{n+1}) = K_{n+1}$ as required. □

3.8.2 Non subexponential amenability

Denote SG_0 (respectively EG_0) the class of groups such that all finitely generated subgroups have subexponential growth (respectively are abelian). Assume that for an ordinal $\alpha > 0$ the classes SG_β and EG_β are defined for every ordinal $\beta < \alpha$. When α is a limit ordinal, set $SG_\alpha = \cup_{\beta<\alpha} SG_\beta$ (respectively $EG_\alpha = \cup_{\beta<\alpha} EG_\beta$). When α is a successor ordinal, define SG_α (respectively EG_α) to be the class of groups that can be obtained from groups in the class $SG_{\alpha-1}$ (respectively $EG_{\alpha-1}$) either by taking direct limits, or by taking extension by a group from the class SG_0 (respectively EG_0).

Each class SG_α (respectively EG_α) is closed under taking quotients and subgroups. Moreover, the class $SG = \cup_\alpha SG_\alpha$ (respectively $EG = \cup_\alpha EG_\alpha$) where the union runs over all ordinals α, is the smallest class of groups containing SG_0 (respectively EG_0) which is closed under the operations of taking subgroups, quotients, extensions and direct limits. As these operations preserve amenability, which is satisfied in SG_0 (respectively EG_0), the class SG (respectively EG) is called class of subexponentially (respectively elementary) amenable groups.

This construction of classes of groups is detailled in [Osi02]. It is obvious that EG_α is a subclass of SG_α for each ordinal α and that the class SG contains the class EG. This inclusion is strict (see [Cho80] and [Gri85]) and the Basilica group introduced in [GZ02a] was the first example of an amenable group out of SG. Osin has shown in [Osi04] that the class EG contains no group of non uniform growth. In particular, groups in the class χ such as the groups $G(\mathcal{A}_{d_0}, A_{\bar{d}})$ of non uniform exponential growth introduced in section 3.7.2 are not in EG. The following Proposition shows these groups are not even in SG, providing new examples of amenable groups outside of the class SG.

Proposition 3.8.5. *Consider a residually finite group G belonging to a class χ_k (see section 3.7), then one of the two following holds :*

1) either G belongs to the class SG_0 of groups of subexponential growth,

2) or G does not belong to the class SG of subexponentially amenable groups.

In particular, residually finite groups of exponential growth in a class χ_k are not in SG.

Recall an elementary property of ordinals :

Fact 3.8.6 (Theorem 7.3 (5) in [Kun80])**.** *Let C be a non empty set of ordinals, then there exists $x \in C$ such that for every $y \in C$, one has $x \leq y$. In other words, C has a minimum.*

Proof of Proposition 3.8.5. The proof is similar to that in [GZ02a]. Let G a group in a class χ_k having exponential growth, in particular not in the class SG_0. Denote G_i the group in the class χ_k such that $G = G_0 \simeq G_i \wr \mathcal{A}_{d_{i-1}} \wr \cdots \wr \mathcal{A}_{d_0}$. In particular all groups G_i have exponential growth. Assume G_0 lies in the class SG, then all the groups G_i (which are subgroups of G_0) lie in SG. For each integer i define α_i to be the minimal ordinal for which G_i belongs to SG_{α_i} (exists by Fact 3.8.6). The family $\{\alpha_i\}_{i\in\mathbb{N}}$ admits a minimum α_{i_0}. Now the ordinal α_{i_0} is not a limit ordinal otherwise G_{i_0} would belong to SG_β for some $\beta < \alpha_{i_0}$. Moreover, the group G_{i_0} is not a direct limit of a strictly increasing infinite sequence of groups because it is finitely generated. This forces the existence of N and H in $SG_{\alpha_{i_0}-1}$ such that the sequence $1 \to N \to G_{i_0} \to H \to 1$ is exact. But as G hence G_{i_0} is residually finite, Proposition 3.8.1 implies that $N = G_{i_0+m}$ for some integer m, so that $\alpha_{i_0+m} \leq \alpha_{i_0} - 1$ which contradicts minimality of α_{i_0}, proving G is not in SG. □

Chapitre 4

Discussion du Théorème 3.3.1 de moyennabilité

Dans un article soumis récemment sur arXiv, Bartholdi, Kaimanovich et Nekrashevych ont démontré la moyennabilité d'une large classe de groupes d'automates (voir [BKN08]). Pour cela, ils traitent d'abord le cas d'un groupe qu'ils nomment "groupe mère", similaire au groupes considérés dans le Théorème 3.3.1. On cherche dans la première partie de ce chapitre à donner le résultat de moyennabilité le plus large possible dans ce contexte.

Dans une deuxième partie, on montrera un renforcement du Théorème 3.3.1 dans le cas de l'arbre binaire, pour lequel le groupe $G(S_2, \bar{H})$ est plus que moyennable, il a croissance intermédiaire.

4.1 Généralisations du Théorème 3.3.1

4.1.1 Automorphismes δ-dirigés par une géodésique

On se donne une géodésique $\mathcal{G}$ de l'arbre enraciné $T_{\bar{d}}$. On peut supposer pour fixer les idées que cette géodésique est constituée des sommets de la forme $(1\dots1)$ (c'est-à-dire qu'il s'agit de la géodésique la plus à gauche de l'arbre $T_{\bar{d}}$.

Étant donné un entier $\delta \geq 1$, on construit un groupe $\bar{M}(\delta)$ contenant le groupe $\bar{H}$ du Théorème 3.3.1. Il s'agit du groupe des automorphismes de $T_{\bar{d}}$ qui fixent la géodésique $\mathcal{G}$ et dont le portrait est contenu dans un δ-voisinage de $\mathcal{G}$.

C'est-à-dire que si $d(v, \mathcal{G}) \geq \delta+1$ on a $g_v = id_{Aut(T_{\sigma^{l(v)}\bar{d}})}$ pour tous les éléments g de $\bar{M}(\delta)$, et par ailleurs si v et vi sont deux sommets successifs situés sur la géodésique $\mathcal{G}$ alors $g_v(vi) = vi$. En d'autres termes la permutation $p_v(g) \in S_{\{v1,v2,\dots,vd_{l(v)}\}}$ à la place v du portrait de g fixe vi. Dans le choix canonique $\mathcal{G} = (111\dots)$ cela s'écrit $p_{1\dots1}(g) \in S_{\{2,3,\dots,d_{l(v)}\}}$.

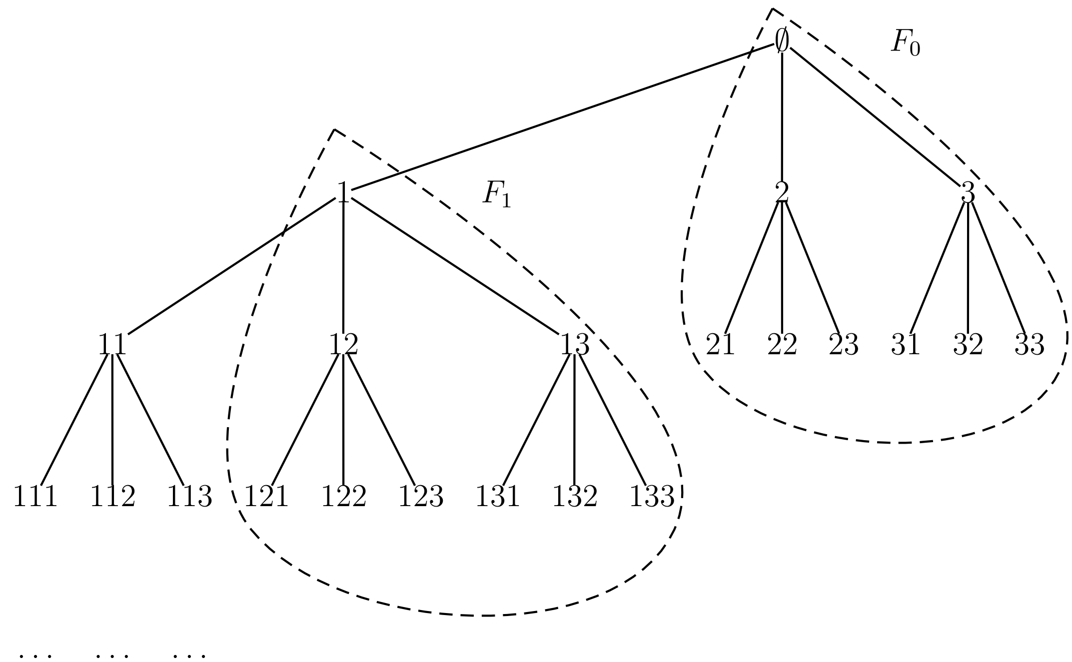

FIG. 4.1 – Le groupe $\bar{M}(2) < Aut(T_3)$ dirigé par $\mathcal{G} = (111\ldots)$.

Par construction, le groupe $\bar{M}(\delta)$ est un produit direct infini

$$\bar{M}(\delta) = F_0 \times F_1 \times F_2 \times \ldots \tag{4.1.1}$$

où chaque F_i est un groupe fini de la forme $F_i = Aut(T_{(d_i-1)d_{i+1}\ldots d_{i+\delta-1}})$. On rappelle que l'on note $Aut(T_{\bar{d}}^{\delta})$ le groupe (fini) des automorphismes de $T_{\bar{d}}$ dont le portrait est supporté par les δ premiers niveaux. Il s'identifie au groupe d'automorphismes $Aut(T_{d_0\ldots d_\delta})$ de l'arbre fini $T_{d_0\ldots d_\delta}$.

4.1.2 Une première généralisation

On note $G(Aut(T_{\bar{d}}^{\delta-1}), \bar{M}(\delta))$ le sous groupe de $Aut(T_{\bar{d}})$ engendré par les automorphismes appartenant à $Aut(T_{\bar{d}}^{\delta-1})$ et ceux appartenant à $\bar{M}(\delta))$. Le Théorème 3.3.1 admet la généralisation suivante :

Théorème 4.1.1. *Soit $\bar{d}$ une suite bornée d'entiers et $\delta \geq 1$ fixé, alors le groupe $G(Aut(T_{\bar{d}}^{\delta-1}), \bar{M}(\delta))$ est moyennable.*

Si l'on note $\bar{M}_i(\delta)$ le groupe $\bar{M}(\delta)$ obtenu pour l'arbre $T_{\sigma^i\bar{d}}$ et la géodésique $\sigma^i\mathcal{G}$, l'isomorphisme (3.2.5) admet la généralisation suivante :

$$G(Aut(T_{\bar{d}}^{\delta-1}), \bar{M}(\delta)) \simeq G(Aut(T_{\sigma\bar{d}}^{\delta-1}), \bar{M}_1(\delta)) \wr S_{d_0} \simeq G(Aut(T_{\sigma^n\bar{d}}^{\delta-1}), \bar{M}_n(\delta)) \wr Aut(T_{\bar{d}}^{n-1}).$$

Notons $D_0 = d_0 \ldots d_{\delta-1}$ et plus généralement $D_i = d_{i\delta} \ldots d_{(i+1)\delta-1}$ et considérons la suite $\bar{D} = (D_i)_{i\in\mathbb{N}}$. L'isomorphisme (cf 3.2.2)

$$Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma^\delta\bar{d}}) \wr Aut(T_{\bar{d}}^{\delta-1}),$$

montre en restreignant la structure d'arbre de $T_{\bar{d}}$ aux niveaux multiples de δ que l'on dispose d'une injection qui descend aux sous groupes de type $\bar{M}(\delta)$ comme suit :

$$\begin{array}{ccc} Aut(T_{\bar{d}}) & \hookrightarrow & Aut(T_{\bar{D}}) \\ \vee & & \vee \\ \bar{M}_{\bar{d}}(\delta) & \hookrightarrow & \bar{M}_{\bar{D}}(1), \end{array} \tag{4.1.2}$$

et toujours sous la même injection $Aut(T_{\bar{d}}^{\delta-1}) \hookrightarrow S_{D_0}$ ce qui assure :

$$G(Aut(T_{\bar{d}}^{\delta-1}), \bar{M}_{\bar{d}}(\delta)) \hookrightarrow G(S_{D_0}, \bar{M}_{\bar{D}}(1)),$$

ce qui montre qu'il suffit de prouver le Théorème 4.1.1 pour $\delta = 1$ pour le déduire avec δ arbitraire.

Preuve du Théorème 4.1.1. Il s'agit donc de prouver le Théorème pour $\delta = 1$, ou plus précisement de s'assurer que la preuve du Théorème 3.3.1 se généralise dans ce cas, où l'on a remplacé la relation recursive $h = (h_1, a_2, \ldots, a_{d_0})$ sur $\bar{H}$ par $m = (m_1, a_2, \ldots, a_{d_0})\tau$ sur $\bar{M}$ où $\tau \in S_{\{2,3,\ldots,d_0\}} \subset S_{d_0}$.

La réduction au cas $G(S_{d_0}, M)$ où M est un sous groupe fini saturé de $\bar{M}$ découle de l'égalité (4.1.1). La définition des normes $||.||_k$ est inchangée ainsi que la Propriété de réduction 3.5.3, ce qui permet de définir la fonction ν. La preuve de la partie 3. de la Proposition 3.5.2 demande le soin de remplacer l'égalité $hg = (h_1g_1, a_2, g_2, \ldots, h_{d_0}g_{d_0})\sigma_0$ par $mg = (m_1g_1, a_2g_{\tau(2)}, \ldots, a_{d_0}g_{\tau(d_0)})\tau\sigma_0$. La croissance des ν-boules n'est pas modifiée. On dispose donc bien d'une famille fractale de pseudo normes à croissance exponentielle.

Les preuves des Propositions 3.4.2 et 3.4.3 sont également préservées dès lors qu'on dispose du Lemme 3.4.6 d'auto-similarité de la marche aléatoire. Il s'agit de la partie de la preuve qui requiert le plus de soin pour être généralisée. En effet le produit aléatoire $Y_n = a_1m_1a_2m_2\ldots a_sm_sa_{s+1}$ se réécrit $Y_n = m_1^{\sigma_1}\ldots m_s^{\sigma_s}\sigma_{s+1}$ où $\sigma_i = a_1\tau_1a_2\tau_2\ldots a_{i-1}\tau_{i-1}a_i$ avec $a_i \in S_{d_0}$ et $\tau_i \in S_{\{2,\ldots,d_0\}}$ variables aléatoires indépendantes et équidistribuées.

Il suffit de vérifier que la famille $\{\sigma_i\}_{i\in\mathbb{N}}$ est une famille de variables aléatoires indépendantes et équidistribuée dans S_{d_0}, et indépendante de la famille $\{m_i\}_i$, ce qui se montre par récurrence via le :

Fait 4.1.2. *Soit b une variable aléatoire suivant une loi de probabilité μ sur un groupe fini F et a une variable aléatoire sur F indépendante de b suivant la mesure d'équidistribution q_F, alors $\sigma = ba$ est équidistribuée sur F et indépendante de b.*

Preuve. On note $P = \mu \otimes q_F$ la mesure obtenue pour $\sigma = ba$. Vérifions d'abord l'équidistribution :

$$P(\sigma = \sigma_0) = P(ba = \sigma_0) = \sum_b P(a = b^{-1}\sigma_0|b)P(b) = \sum_b \frac{1}{\#F}P(b) = \frac{1}{\#F}.$$

On peut alors vérifier l'indépendance :

$$\begin{aligned}
P(\sigma=\sigma_0 \text{ et } b=b_0) &= P(a=b_0^{-1}\sigma_0 \text{ et } b=b_0)\\
&= P(a=b_0^{-1}\sigma_0)P(b=b_0) \qquad \text{par indépendance de } a \text{ et } b,\\
&= \frac{1}{\#F}P(b=b_0)\\
&= P(\sigma=\sigma_0)P(b=b_0).
\end{aligned}$$

□

Le reste de la preuve du Lemme 3.4.6 d'auto-similarité n'est pas modifiée.

□

4.1.3 Une deuxième généralisation

Notons $D_\delta(T_{\bar{d}})$ le groupe des automorphismes de l'arbre enraciné avec un portrait dont le support est contenu dans le δ-voisinage d'une famille finie de géodésiques $(\mathcal{G}_1,\dots,\mathcal{G}_k)$. On appelle de tels automorphismes δ-dirigés, on note $D_\delta(T_{\bar{d}})$ le groupe qu'ils forment. Il est aisé de voir qu'un tel automorphisme est produit d'automorphismes δ-dirigés par une seule géodésique.

Une géodésique $\mathcal{G}=(i_1i_2i_3\dots)$ est dite rationnelle si la suite $(i_k)_{k\in\mathbb{N}}$ est périodique à partir d'un certain rang. Un automorphisme δ-dirigé g est dit rationnel si l'on peut trouver une famille finie $(\mathcal{G}_1,\dots,\mathcal{G}_k)$ de géodésiques rationnelles dont un δ-voisinage contient le support de g et si le portrait de g est asymptotiquement périodique le long de ces géodésiques. C'est-à-dire si, identifiant la géodésique à la suite de sommets de l'arbre qu'elle parcourt $\mathcal{G}_j=(\emptyset,v_1,v_2,v_3,\dots)$, l'application $i\mapsto p_{v_i}(g)\in S_{d_i}$ est périodique à partir d'un certain rang. On note $DR_\delta(T_{\bar{d}})$ le groupe des automorphismes δ-dirigés rationnels.

Fait 4.1.3. *Les ensembles $D_\delta(T_{\bar{d}})$ et $DR_\delta(T_{\bar{d}})$ sont des groupes.*

Preuve. Quitte à utiliser l'isomorphisme $Aut(T_{\bar{d}})\simeq Aut(T_{\sigma^n\bar{d}})\wr Aut(T^n_{\bar{d}})$ pour n assez grand, il suffit de montrer que si g (respectivement h) est δ dirigé par une géodésique $\mathcal{G}$ (respectivement $\mathcal{H}$), alors le produit gh est δ-dirigé par un nombre fini de géodésiques. Les automorphismes s'écrivent récursivement sous la forme :

$$\begin{aligned}
g_k &= (a_1(k),\dots,g_{k+1},\dots,a_{d_k}(k))\sigma(k),\\
h_k &= (b_1(k),\dots,h_{k+1},\dots,b_{d_k}(k))\tau(k),
\end{aligned}$$

pour $a_i(k),b_i(k)\in Aut(T^{\delta-1}_{\sigma^k\bar{d}})$, $\sigma(k),\tau(k)\in S_{d_k-1}$ et avec g_{k+1} et h_{k+1} en positions respectives x_k et y_k dans $\{1,\dots,d_k\}$. Posons $k_0=\min\{k|\sigma(y_k)\neq x_k\}$, on a alors :

$$\begin{aligned}
&g_kh_k=(a_1b_{\sigma(1)},\dots,g_{k+1}h_{k+1},\dots,a_{d_k}b_{\sigma(d_k)})\sigma\tau, \text{ pour tous } k<k_0 \text{ et :}\\
&g_{k_0}h_{k_0}=(a_1b_{\sigma(1)},\dots,g_{k+1}b_{\sigma(x_{k_0})},\dots,a_{\sigma^{-1}(y_k)}h_{k+1},\dots,a_{d_k}b_{\sigma(d_k)})\sigma\tau.
\end{aligned}$$

On en déduit que le produit gh est δ-dirigé par deux géodésiques si k_0 est fini, par la seule géodésique $\mathcal{G}$ sinon, et ceci rationnellement si c'était le cas pour g et h. □

On considèrera plus généralement les groupes :

$$D(T_{\bar d}) = \bigcup_{\delta\in\mathbb{N}} D_\delta(T_{\bar d}) \text{ et } DR(T_{\bar d}) = \bigcup_{\delta\in\mathbb{N}} DR_\delta(T_{\bar d})$$

d'automorphismes dirigés et rationnellement dirigés. Nekrashevych a démontré récemment le :

Théorème 4.1.4. *[Nekrashevych [Nek08a]] Si la suite $\bar d$ est bornée, alors le groupe $D(T_{\bar d})$ n'admet aucun sous-groupe libre non cyclique.*

La question de savoir si le groupe d'automorphismes dirigé $D(T_{\bar d})$ est moyennable est à ma connaissance ouverte. On obtient ici un résultat de moyennabilité dans cette direction englobant le théorème 3.3.1 et le théorème sur les automates bornés de Bartholdi, Kaimanovich et Nekrashevych ([BKN08]).

Théorème 4.1.5. *Si la suite $\bar d$ est bornée, alors le groupe $DR(T_{\bar d})$ est moyennable.*

L'hypothèse de rationnalité contient et affaiblit l'hypothèse d'automaticité de l'article [BKN08]. Elle est suffisante pour utiliser le lemme suivant, qui permet de conjuguer *conjointement* une famille d'automorphismes rationnellement dirigés à une famille dans $G(Aut(T_{\bar d}^{\delta-1}), \bar M(\delta))$.

Lemme 4.1.6 ([BKN08])**.** *Notons $c = (1, 2, \dots, d)$ un d-cycle, $c_i = c^{i-1}$ sa puissance envoyant 1 sur i, et δ l'automorphisme de l'arbre enraciné d-régulier T_d défini récursivement par la formule $\delta = (\delta c_1^{-1}, \delta c_2^{-1}, \dots, \delta c_d^{-1})$. Soit $g = g_0$ satisfaisant la propriété récursive*

$$g_k = (a_1(k), \dots, a_{x-1}(k), g_{k+1}, a_{x+1}(k), \dots, a_d(k))\sigma,$$

où $a_i(k) \in S_d$ sont enracinés et σ et x sont indépendants de k (en d'autres termes, g est 1-dirigé 1-rationnellement). Sous ces hypothèses, le conjugué $\delta^{-1}g\delta$ appartient à $G(Aut(T_d^2), \bar M_d(2))$.

Preuve du Lemme. Cela découle du calcul suivant. Sachant $\delta^{-1} = (c_1\delta^{-1}, \dots, c_d\delta^{-1})$ et $\sigma\delta = (\delta c_{\sigma(1)}^{-1}, \dots, \delta c_{\sigma(d)}^{-1})\sigma$, on calcule :

$$\delta^{-1}g_k\delta = (c_1\delta^{-1}a_1(k)\delta c_{\sigma(1)}^{-1}, \dots, c_x\delta^{-1}g_{k+1}\delta c_{\sigma(x)}^{-1}, \dots, c_d\delta^{-1}a_d(k)\delta c_{\sigma(d)}^{-1})\sigma.$$

Il s'en suit que pour tout entier k :

$$c_x\delta^{-1}g_k\delta c_{\sigma(x)}^{-1} = (c_x\delta^{-1}g_{k+1}\delta c_{\sigma(x)}^{-1}, c_{c_x(2)}\delta^{-1}a_{c_x(2)}(k)\delta c_{c_x(2)}^{-1}, \dots)c_x\sigma c_{\sigma(x)}^{-1},$$

et donc $c_x\delta^{-1}g\delta c_{\sigma(x)}^{-1}$ est dans $\bar M_d(2)$ dés que l'on remarque que

$$c_x\sigma c_{\sigma(x)}^{-1}(1) = c_{\sigma(x)}^{-1}(\sigma(c_x(1))) = 1$$

(action à droite) et que

$$\delta^{-1}a\delta = (c_1\delta^{-1}\delta c_{a(1)}^{-1}, \dots, c_d\delta^{-1}\delta c_{a(d)}^{-1})a = (c_1c_{a(1)}^{-1}, \dots, c_dc_{a(d)}^{-1})a \in Aut(T_d^2)$$

agit seulement sur les deux premiers niveaux. □

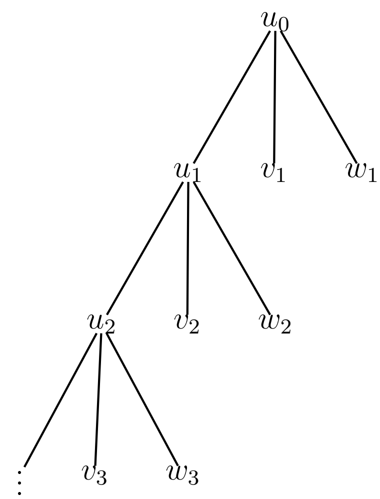

FIG. 4.2 – Exemple de portraits dirigés.

Preuve du Théorème 4.1.5. Il s'agit de prouver qu'un sous-groupe Γ de type fini de $DR_\delta(T_{\bar{d}})$ est moyennable pour un δ fixe commun aux générateurs de Γ, et où l'on peut supposer la suite $\bar{d}$ constante quitte à augmenter artificiellement la valence.

On peut aussi supposer qu'un tel sous-groupe est engendré par des éléments $g^{(j)}$ rationellement dirigés chacun par une géodésique périodique $\mathcal{G}_j$.

En utilisant l'isomorphisme $Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma^n \bar{d}}) \wr Aut(T_{\bar{d}}^n)$ pour n assez grand, on peut supposer les géodésiques et les portraits les longeant périodiques (on retrouvera alors la moyennabilité par produit direct et extension finie).

Quitte à utiliser l'injection (4.1.2), on peut enfin supposer que $\delta = 1$ et que les périodes des géodésiques et des applications de portrait en jeu valent 1.

Le lemme précédent permet alors de conjuguer le groupe Γ à un sous groupe de $G(Aut(T_{\bar{d}}^2), \bar{M}_d(2))$, ce qui conclut la preuve via le Théorème 4.1.1. □

Example 4.1.7. On liste sous quelles hypothèses sur le portrait de la figure 4.2 on peut appliquer quel théorème de moyennabilité :

1. Si $u_i = 1$ et $(v_i), (w_i)$ sont périodiques, le théorème sur le "groupe mère" de l'article [BKN08] s'applique.
2. Si $u_i = 1$ et $(v_i), (w_i)$ sont quelconques, notre théorème 3.3.1 du chapitre 3 s'applique.
3. Si (u_i) et $(v_i), (w_i)$ sont périodiques, le théorème sur les automates bornés de [BKN08] s'applique.
4. Si (u_i) est périodique et $(v_i), (w_i)$ sont quelconques, notre théorème 4.1.5 sur les automorphismes rationellement dirigés s'applique.
5. Si (u_i) et $(v_i), (w_i)$ sont quelconques, le théorème 4.1.4 de Nekrashevych sur l'abscence de sous-groupe libre s'applique, mais on ignore si le groupe est moyennable.

4.2 Le cas de l'arbre binaire

Dans le cadre d'un arbre binaire, le groupe $G(S_2, \bar{H})$ et le groupe $G(S_2, \bar{M}(1))$ coïncident. De plus, $\bar{H} = \bar{M}(1)$ coïncide avec l'ensembles des générateurs de la forme $b_\omega, c_\omega, d_\omega$ pour $\omega \in \Omega$ (cf section 2.2.2), et l'on sait que le groupe $G(S_2, \langle b_\omega, c_\omega, d_\omega \rangle)$ est à croissance sous exponentielle (et intermédiaire dés que la suite ω n'est pas asymptotiquement constante). Le théorème plus général suivant est énoncé dans [Ers05]. Il renforce le théorème 3.3.1 de moyennabilité dans ce cas particulier.

Théorème 4.2.1. *Le groupe $G(S_2, \bar{H}) < Aut(T_2)$ est à croissance intermédiaire au sens où tous ses sous-groupes de type fini le sont.*

Preuve. On sait que tout sous-groupe de type fini de $G(S_2, \bar{H})$ est contenu dans un groupe $G(S_2, H)$ pour un certain sous groupe H de $\bar{H}$ de type fini, et donc fini en vertu du fait 3.3.4. Comme le groupe $\bar{H}$ est isomorphe à un produit infini $\bar{H} \simeq \prod_{\mathbb{N}} S_2$, chaque élément h de H s'écrit :

$$h = (u_1^h, u_2^h, u_3^h, \dots),$$

où chaque u_i^h est à valeur dans $S_2 = \{1, \varepsilon\}$. Il y a un unique élément noté b dans $\bar{H}$ vérifiant $u_i^b = \varepsilon$ pour tout i.

On va utiliser les injections usuelles :

$$G(S_2, H) \hookrightarrow G(S_2, H_i) \wr Aut(T_2^{i-1}),$$

où $H_i = \prod_{j \geq i+1} S_2$ est l'image de H sur une coordonnée du produit en couronne itéré i fois.

Notons a le générateur du groupe d'automorphismes enracinés S_2, tout élément g de $G(S_2, H)$ admet une écriture minimale de la forme :

$$w_g = a^\tau h_1 a h_2 a \dots a h_k a^{\tau'},$$

où $\tau, \tau' \in \{0, 1\}$, $h_j \in H$ et $k \leq \frac{|g|+1}{2}$ avec $|g|$ la norme de g pour la longueur des mots pour le système générateur $S_2 \cup H$. On note que si $b \neq h \in H$ est fixé, il existe un entier i tel que $u_i^h = 1$. Comme chaque h_j contribue à au plus un terme $a = \varepsilon = u_i^{h_j}$, on en déduit que le nombre total de a apparraissant dans l'écriture du produit en couronne itéré i fois de $g = w_g = (g_1, \dots, g_{2^i})_i \sigma$, (où $g_t \in G(S_2, H_i)$ et $\sigma \in Aut(T_2^{i-1})$) est borné par $k - |w_g|_h$ (où $|w|_h$ est le nombre de h dans le mot w). On a donc :

$$|g_1| + \dots + |g_{2^i}| \leq \frac{|g_1|_a + \dots + |g_{2^i}|_a + 2^i}{2} \leq |g| - |w_g|_h + K_i. \qquad (4.2.1)$$

Cette inégalité de réduction va nous permettre de montrer le théorème par la discussion suivante. On note $B_{H_i}(R)$ la boule de rayon R du groupe $G(S_2, H_i)$ pour la partie génératrice $S_2 \cup H_i$, $b_{H_i}(R)$ son cardinal et $c_i = \lim \sqrt[R]{b_{H_i}(R)}$ son taux de croissance exponentielle. On discute suivant la fréquence d'apparition de l'élément b dans les mots réduits. Pour un paramètre $t \geq 2$, on note :

$$B_H^+(R) = \{g \in B_H(R) | |w_g|_b \leq (1 - \frac{1}{t})R\},$$

et $B_H^-(R) = B_H(R) \setminus B_H^+(R)$. Si b apparait moins de $(1 - \frac{1}{t})R$ fois dans un mot réduit w, alors il existe $b \neq h \in H$ qui apparait au moins $|w|_h \geq \frac{R}{t\#H}$ fois, et donc en utilisant (4.2.1), on obtient une partie de taille au moins $\frac{b_H^+(R)}{\#H}$ et un entier i tels que :

$$b_H^+(R) \leq \#H \sum_{R_1+\cdots+R_{2^i} \leq (1-\frac{1}{t\#H})R+K_i} b_{H_i}(R_1)\ldots b_{H_i}(R_{2^i}). \tag{4.2.2}$$

D'autre part, on peut estimer $b_H^-(R)$ en notant que ses éléments admettent des représentants où apparaissent moins de $\frac{R}{t}$ termes $h \neq b$. Il y a donc au plus $\sum_{s\leq \frac{R}{t}} C_R^s$ possibilités pour choisir leurs positions dans le mot réduit et $\#H^{\frac{R}{t}}$ choix des termes à position donnée. On a donc :

$$b_H^-(R) \leq \#H^{\frac{R}{t}} \sum_{s\leq\frac{R}{t}} C_R^s \leq \frac{R}{t} C_R^{\frac{R}{t}} \#H^{\frac{R}{t}}. \tag{4.2.3}$$

Les inégalités (4.2.2) et (4.2.3) entrainent que pour tout R, il existe i tel que :

$$b_H(R) \leq \max\{2\#H \sum_{R_1+\cdots+R_{2^i} \leq (1-\frac{1}{t\#H})R+K_i} b_{H_i}(R_1)\ldots b_{H_i}(R_{2^i}), 2\frac{R}{t}C_R^{\frac{R}{t}}\#H^{\frac{R}{t}}\}.$$

On en déduit qu'il existe un certain i tel que (on utilise la formule de Stirling pour évaluer le deuxième terme) :

$$c_0 \leq \max\{c_i^{(1-\frac{1}{t\#H})}, (\#Ht)^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\}$$

Notons que $(\#Ht)^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1} \underset{t\to\infty}{\longrightarrow} 1$, et donc en laissant tendre t vers l'infini dans l'inégalité on trouve i_0 tel que $c_0 \leq c_{i_0}$.

On itère maintenant le procédé pour trouver une suite i_j telle que $c_{i_j} \leq c_{i_{j+1}}$. Par ailleurs, comme $\#H_i \leq \#H$ pour tout i, la suite c_i est majorée. Au total, la suite $(c_{i_j})_j$ est croissante, majorée, donc convergente et sa limite λ satisfait par continuité :

$$\lambda \leq \max\{\lambda^{(1-\frac{1}{t\#H})}, (\#Ht)^{\frac{1}{t}}(1-\frac{1}{t})^{\frac{1}{t}-1}\},$$

pour une infinité de paramètres $t \to \infty$. Ceci implique $\lambda = 1$, et donc la croissance intermédiaire de $G(S_2, H)$. □

Remarque 4.2.2. Le lemme 3.7.4 de Bartholdi [Bar03] s'applique pour $u = (1,2)(3,4)$ et $v = (1,2)$. On en déduit que le groupe $G(S_4, \bar{H}) < Aut(T_4)$ contient des sous-groupes de type fini à croissance exponentielle. Je ne sais pas si $G(S_3, \bar{H}) < Aut(T_3)$ en contient.

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