Proof of De Smit's conjecture: a freeness criterion

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Abstract

Let $A \to B$ be a morphism of Artin local rings with the same embedding dimension. We prove that any A-flat B-module is B-flat. This freeness criterion was conjectured by de Smit in 1997 and improves Diamond's criterion [8, Theorem 2.1]. We also prove that if there is a nonzero A-flat B-module, then $A \to B$ is flat and is a relative complete intersection. Then we explain how this result allows one to simplify Wiles's proof of Fermat's Last Theorem: we do not need the so-called "Taylor-Wiles systems" any more.

Notations.

If A is a local ring, we denote by $\mathfrak{m}_A$ its maximal ideal and by $\kappa(A)$ its residue field. The embedding dimension of A, i.e. the minimal number of generators of $\mathfrak{m}_A$, is denoted by $\mathrm{edim}(A)$. A local morphism $A \to B$ of Noetherian local rings is a relative complete intersection if the ring $B/\mathfrak{m}_A B$ is a complete intersection.

1. Introduction

A crucial step in the proof of the modularity of semistable elliptic curves over $\mathbb{Q}$ by Taylor and Wiles is to show that certain Hecke algebras are complete intersections. Their method, now commonly referred-to as a "patching method", heuristically consists of considering the limit of algebras arising from forms of different levels. In their construction, they used in particular the following (fundamental) "multiplicity one" result: the homology of the modular curve is a free module over the Hecke algebra (after localization at some maximal ideal). Diamond reversed the argument in [8]: "multiplicity one" becomes a byproduct of the proof rather than an ingredient. The key input from commutative algebra that allowed Diamond's improvement was the freeness criterion [8, Thm 2.1]. Its proof still relies on a patching argument.

In the hypotheses of Diamond's freeness criterion, there is a condition that has to hold for all positive integers n. The condition for n implies the condition for all smaller integers. In [8, Remark 2.2], Diamond asks whether a sufficiently large n would be enough, and whether there could exist a lower bound for such an n depending only on the number r of variables and the rank of the module. Around 1997, de Smit conjectured that if $A \to B$ is a morphism of Artin local rings with the same embedding dimension, then any A-flat B-module is B-flat (see Corollary 1.2 below). In [4] we proved de Smit's conjecture when the embedding dimension r is $\leqslant 2$, and with the additional assumption that $A \to B$ was flat. Here we prove the following theorem, which in particular proves the full conjecture, and answers Diamond's question: $n = 2$ already suffices, for any r and any module. So, the freeness criterion will be much easier to apply.

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Theorem 1.1. *Let $\varphi : A \to B$ be a local morphism of Noetherian local rings. Assume the following:*

(i) $\operatorname{edim}(B) \leqslant \operatorname{edim}(A)$

(ii) There is a nonzero A-flat B-module M.

(iii) A and B are Artin or M is of finite type as a B-module.

Then:

(1) The morphism $A \to B$ is flat and $\operatorname{edim}(B) = \operatorname{edim}(A)$.

(2) The ring $B/\mathfrak{m}_A B$ is a complete intersection of dimension zero.

(3) M is flat over B.

Corollary 1.2 (de Smit's conjecture). *Let $A \to B$ be a morphism of Artin local rings with the same embedding dimension. Then any A-flat B-module is B-flat.*

Corollary 1.3 (see Theorem 2.5). *Let $f : X \to S$ be a morphism of locally Noetherian schemes such that for any $x \in X$, the tangent spaces satisfy* $\dim T_x \leqslant \dim T_{f(x)}$. *Then any S-flat coherent $\mathcal{O}_X$-module is also flat over X.*

Since its publication, Diamond's freeness criterion and the related patching method have been used quite a lot. For instance, here is a list of papers (nonexhaustive, in chronological order) that directly use Diamond's Theorem 2.1: [6], [9], [12], [1], [20], [5], [10], [18], [13], [3]. A variant of this freeness criterion, due to Kisin (see [17, (3.3.1)]), was also used in several major results, for instance the proof of Serre's modularity conjecture (see [17], [14] and [16]) or the Fontaine–Mazur conjecture for GL_2 (see [15]). A more functorial approach of the patching method is proposed in [11].

The paper is organized as follows: in section 2 we prove Theorem 1.1. We also give a global version for schemes that are not necessarily local (see Theorem 2.5). In section 3 we briefly explain how this new freeness criterion allows one to simplify Wiles's proof of Fermat's Last Theorem and avoid the use of the patching method.

2. Proof of Theorem 1.1

The following lemma is the heart of the proof.

Lemma 2.1. *Let B be a ring. Let $(x_1, \dots, x_n)$ and $(u_1, \dots, u_n)$ be sequences of elements of B and let us denote by J_x and J_u the ideals they generate. Let M be a B-module. Assume that:*

(i) We have the inclusion of ideals $J_x \subset J_u$, in other words, if x and u denote the column matrices $(x_1, \dots, x_n)^T$ and $(u_1, \dots, u_n)^T$, there is a square $n \times n$ matrix $W = (c_{ij})$ with entries in B such that $x = Wu$. Let $\Delta = \det W$.

(ii) For any $m_1, \dots, m_n \in M$, if we have the relation $\sum_{i=1}^n x_i m_i = 0$, then $m_i \in J_x M$ for all i.

Then, for any $m \in M$, if $\Delta m \in J_x M$, then $m \in J_u M$.

Proof. We will need some notations for the minors of W. For any integer ℓ, let E_ℓ be the set $\{1, \dots, \ell\}$ and let $\mathcal{P}_\ell$ be the set of subsets $I \subset E_n$ such that $|I| = \ell$. For $I, J \in \mathcal{P}_\ell$, let Δ^I_J denote the minor of W obtained by deleting the rows (respectively the columns) whose index is in I (respectively J). In other words, $\Delta^I_J = \det(c_{ij})_{\substack{i \in E_n \smallsetminus I \\ j \in E_n \smallsetminus J}}$. In particular, $\Delta^\varnothing_\varnothing = \Delta$ and $\Delta^{E_n}_{E_n} = 1$.

The comatrix is $\mathrm{Com}\,(W) = ((-1)^{i+j}\Delta^i_j)$. If $I \subset E_n$ and $i \in E_n$, we write $I - i$ for $I \smallsetminus \{i\}$. For $i \neq j \in E_n$, let $\overline{\varepsilon}(i,j) = 1$ if $i < j$ and $\overline{\varepsilon}(i,j) = -1$ if $i > j$. For $I \subset E_n$ and $i \in I$, let $\varepsilon(i,I) = (-1)^p$, where p is the position of i in the ordered set $E_n \smallsetminus (I - i)$. Note that $\varepsilon(i,I) = (-1)^i \prod_{j \in I-i} \overline{\varepsilon}(i,j)$. For any $i, j \in I$ with $i \neq j$, we have the relation

$$\varepsilon(i,I)\varepsilon(i,I-j) = \overline{\varepsilon}(i,j) \tag{1}$$

Now, for any $0 \leqslant \ell \leqslant n-1$, if $I \in \mathcal{P}_{\ell+1}$ and $i \in I$, the expansion of the minor $\Delta^{E_\ell}_{I-i}$ along the column i is

$$\Delta^{E_\ell}_{I-i} = (-1)^\ell \varepsilon(i,I) \sum_{k=\ell+1}^{n} (-1)^k c_{ki} \Delta_I^{E_\ell \cup \{k\}} \tag{2}$$

If $i \notin I$, pick an element $j \in I$ and consider the expansion along the column j of the minor $\Delta^{E_\ell}_{I-j}$ in which we have replaced the column j by the column i. Since the latter minor has two identical columns, it is zero and we see that for any $i \notin I$,

$$\sum_{k=\ell+1}^{n} (-1)^k c_{ki} \Delta_I^{E_\ell \cup \{k\}} = 0 \tag{3}$$

Now let us come back to our module M. Let $m \in M$ be such that $\Delta m \in J_x M$. For $0 \leqslant \ell \leqslant n$, let (A_ℓ) denote the following statement.

(A_ℓ): There is a family $(a^\ell_I)_{I \in \mathcal{P}_{\ell-1}}$ of elements of M such that for any $I \in \mathcal{P}_\ell$, the element

$$g_I = \Delta_I^{E_\ell} m + \sum_{i \in I} \varepsilon(i,I) u_i a^\ell_{I-i}$$

belongs to $J_x M$.

We will prove that (A_ℓ) holds for any $0 \leqslant \ell \leqslant n$ by induction on ℓ. The statement (A_0) means that $\Delta m \in J_x M$ and holds by assumption. Assume that (A_ℓ) holds for some $0 \leqslant \ell \leqslant n-1$ and let us prove the statement $(A_{\ell+1})$. For any $I \in \mathcal{P}_\ell$, since $g_I \in J_x M$, we can write $g_I = \sum_{k \in E_n} x_k a^k_I$ for some elements $a^k_I \in M$. Let $I \in \mathcal{P}_{\ell+1}$. We will compute $m' = \sum_{i \in I} \varepsilon(i,I) u_i g_{I-i}$. For this, we first compute

$$\begin{aligned} \sum_{i \in I} \varepsilon(i,I) u_i (g_{I-i} - \Delta^{E_\ell}_{I-i} m) &= \sum_{i \in I} \varepsilon(i,I) u_i \sum_{j \in I-i} \varepsilon(j, I-i) u_j a^\ell_{I \smallsetminus \{i,j\}} \\ &= \sum_{\substack{i,j \in I \\ i \neq j}} u_i u_j a^\ell_{I \smallsetminus \{i,j\}} S(i,j) \end{aligned}$$

where $S(i,j) = \varepsilon(i,I)\varepsilon(j,I-i)$. Now the sum vanishes because $S(i,j) = -S(j,i)$. Indeed, $S(i,j)S(j,i) = \overline{\varepsilon}(i,j)\overline{\varepsilon}(j,i)$ by equation (1) above. Hence,

$$m' = \sum_{i \in I} \varepsilon(i,I) u_i \Delta^{E_\ell}_{I-i} m$$

We expand $\Delta_{I-i}^{E_\ell}$ along the column i:

$$\begin{aligned}
m' &= \sum_{i\in I} \varepsilon(i,I) u_i (-1)^\ell \varepsilon(i,I) \sum_{k=\ell+1}^{n} (-1)^k c_{ki} \Delta_I^{E_\ell\cup\{k\}} m \\
&= (-1)^\ell \sum_{k=\ell+1}^{n} (-1)^k \Delta_I^{E_\ell\cup\{k\}} \left(\sum_{i\in I} c_{ki} u_i \right) m \\
&= (-1)^\ell \sum_{k=\ell+1}^{n} (-1)^k \Delta_I^{E_\ell\cup\{k\}} \left(x_k - \sum_{i\notin I} c_{ki} u_i \right) m \qquad \text{(because } x = Wu\text{)} \\
&= (-1)^\ell \sum_{k=\ell+1}^{n} (-1)^k \Delta_I^{E_\ell\cup\{k\}} x_k m \qquad \text{(using (3) for each } i \notin I\text{)}
\end{aligned}$$

On the other hand, by definition

$$m' = \sum_{i\in I} \varepsilon(i,I) u_i g_{I-i} = \sum_{i\in I} \varepsilon(i,I) u_i \sum_{k\in E_n} x_k a_{I-i}^k$$

Hence,

$$\sum_{i\in I} \varepsilon(i,I) u_i \sum_{k\in E_n} x_k a_{I-i}^k - (-1)^\ell \sum_{k=\ell+1}^{n} (-1)^k \Delta_I^{E_\ell\cup\{k\}} x_k m = 0\,.$$

By our assumption (ii), the coefficient of $x_{\ell+1}$ in the above equation belongs to $J_x M$. But this coefficient is

$$\sum_{i\in I} \varepsilon(i,I) u_i a_{I-i}^{\ell+1} + \Delta_I^{E_{\ell+1}} m\,.$$

This proves the statement $(A_{\ell+1})$. In particular, (A_n) holds and hence there are elements $a_I^n \in M$ for $I \in \mathcal{P}_{n-1}$ such that the element

$$g_{E_n} = \Delta_{E_n}^{E_n} m + \sum_{i\in E_n} \varepsilon(i,E_n) u_i a_{E_n-i}^n$$

belongs to $J_x M$. Since $\Delta_{E_n}^{E_n} = 1$, this proves that $m \in J_u M$. □

Before proceeding to the proof of Theorem 1.1, let us recall a result from [4]. It will provide a useful characterization for the flatness of M over B.

Definition 2.2 [4, 3.1]. *Let R be a local ring and M be an R-module. We say that M is weakly torsion-free if, for every $\lambda \in R$ and every $m \in M$, the relation $\lambda m = 0$ implies that $\lambda = 0$ or $m \in \mathfrak{m}_R M$.*

Proposition 2.3 [4, 3.7]. *Let R be an Artin local ring. Assume that R is Gorenstein and contains a field. Then an R-module M is flat over R if and only if it is weakly torsion-free.*

Proof of Theorem 1.1. Let M be a nonzero A-flat B-module. We will apply Lemma 2.1 with $(x_1,\dots,x_n)$ the image in B of a minimal system of generators of $\mathfrak{m}_A$, and $(u_1,\dots,u_n)$ a system of generators of $\mathfrak{m}_B$, so that $J_x = \mathfrak{m}_A B$ and $J_u = \mathfrak{m}_B$. The assumption (i) holds because the morphism $A \to B$ is local, and the assumption (ii) holds by [4, 4.2 and 4.3] because M is flat over A. Then $\Delta \notin \mathfrak{m}_A B$, otherwise we would have $\Delta M \subset \mathfrak{m}_A M$, which would imply that $M = \mathfrak{m}_B M$ by Lemma 2.1, hence $M = 0$, which is a contradiction. In particular, for any choice of the matrix W and the generators $u_1,\dots,u_n$ of $\mathfrak{m}_B$ such that $x = Wu$, we have $\det W \neq 0$.

This proves that $\mathrm{edim}(B) = \mathrm{edim}(A)$, because if $\mathrm{edim}(B) < \mathrm{edim}(A)$ we can choose W with a zero column. Moreover, in the ring $B/\mathfrak{m}_A B$, the matrix $\overline{W}$ satisfies $\overline{W}u = 0$ and $\det \overline{W} \neq 0$. In the language of [19, Definition 2.6], this means that $\overline{W}$ is (the transpose of) a u-Wiebe matrix for the ring $B/\mathfrak{m}_A B$. By [19, 2.7], this implies that the ring $B/\mathfrak{m}_A B$ is a complete intersection of dimension zero, and proves (2).

Let us prove that M is B-flat. By [4, 2.3], it suffices to prove that $M/\mathfrak{m}_A M$ is flat over $B/\mathfrak{m}_A B$. We have seen that $B/\mathfrak{m}_A B$ is a complete intersection of dimension zero and hence a Gorenstein Artin local ring. Moreover, it contains the field $\kappa(A)$. Hence, by Proposition 2.3, it suffices to prove that $M/\mathfrak{m}_A M$ is weakly torsion-free over $B/\mathfrak{m}_A B$, i.e. that for any $\lambda \in B$ and any $m \in M$, the relation $\lambda m \in \mathfrak{m}_A M$ implies that $\lambda \in \mathfrak{m}_A B$ or $m \in \mathfrak{m}_B M$. So, let $\lambda \in B$ and $m \in M$ be such that $\lambda m \in \mathfrak{m}_A M$ and $\lambda \notin \mathfrak{m}_A B$. Let us prove that $m \in \mathfrak{m}_B M$. By [19, 2.7], we know that the image $\overline{\Delta}$ of Δ is a generator of the socle of the Gorenstein ring $B/\mathfrak{m}_A B$. Hence, $(\overline{\Delta}) \subset (\overline{\lambda})$ in $B/\mathfrak{m}_A B$. Since $\lambda m \in \mathfrak{m}_A M$, this implies that $\Delta m \in \mathfrak{m}_A M$. By Lemma 2.1, we get that $m \in \mathfrak{m}_B M$, as required.

Lastly let us prove that $A \to B$ is flat. Let $E \to F$ be an injection of A-modules. Let K be the kernel of $E \otimes_A B \to F \otimes_A B$. Since M is B-flat, $K \otimes_B M$ is the kernel of $E \otimes_A M \to F \otimes_A M$, which is zero because M is A-flat. But M is faithfully flat over B (because it is free and nonzero) and hence $K = 0$. □

REMARK 2.4. If we replace the assumption (iii) of Theorem 1.1 with "A and B are Artin or M is of finite type over A", then we do not need to assume that the morphism $\varphi : A \to B$ is local: it is then a consequence of the other hypotheses. Indeed, if there is an element $a \in \mathfrak{m}_A$ such that $\varphi(a) \in B^\times$, then $M = \varphi(a)M$ and hence $M = \mathfrak{m}_A M$. Then $M = 0$ by Nakayama's lemma (or because $\mathfrak{m}_A$ is nilpotent if A is Artin), which is a contradiction. On the other hand, if A is Artin, I am not sure that we really need the assumption that B is Artin too.

The following is a global version of Theorem 1.1.

THEOREM 2.5. *Let $f : X \to S$ be a morphism of locally Noetherian schemes. Assume that for any $x \in X$, the dimensions of the tangent spaces T_x and $T_{f(x)}$ satisfy*

$$\dim T_x \leqslant \dim T_{f(x)}$$

Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, flat over S. Then:

(1) $\mathcal{M}$ is flat over X.

(2) For each point $x \in \mathrm{Supp}(\mathcal{M})$, f is flat at x, the fiber $f^{-1}(f(x))$ is a complete intersection of dimension zero at x (i.e. its local ring at x is a complete intersection of dimension zero) and $\dim T_x = \dim T_{f(x)}$.

In particular, if there exists a coherent $\mathcal{O}_X$-module $\mathcal{M}$ flat over S whose support is X, then f is a complete intersection morphism of relative dimension zero (i.e. f is flat and its fibers are zero-dimensional complete intersections).

Proof. Note that the local ring of X at a point x is unaltered through the base change along $\mathrm{Spec}\, \mathcal{O}_{S,f(x)} \to S$. By [2, Ch. II, §3, no 4, Prop. 14], the module $\mathcal{M}_x$ is flat over $\mathcal{O}_{S,f(x)}$. Since $\dim T_x = \mathrm{edim}(\mathcal{O}_{X,x})$ and $\dim(T_{f(x)}) = \mathrm{edim}(\mathcal{O}_{S,f(x)})$, we can apply Theorem 1.1 and $\mathcal{M}_x$ is flat over $\mathcal{O}_{X,x}$. This proves (1). If $x \in \mathrm{Supp}(\mathcal{M})$ then $\mathcal{M}_x \neq 0$ and by Theorem 1.1 we deduce that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{S,f(x)}$, that these rings have the same embedding dimension and that $\mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$ is a zero-dimensional complete intersection (where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{S,f(x)}$). Since the latter ring $\mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$ identifies with the local ring at x of the fiber $X_{f(x)}$, this proves (2). □

3. Application to the proof of Fermat's Last Theorem

To conclude his proof, Wiles had to prove that a certain morphism of $\mathcal{O}$-algebras $\Phi_\Sigma : R_\Sigma \to T_\Sigma$ is an isomorphism and that T_Σ is a complete intersection over $\mathcal{O}$. Here $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_\ell$, R_Σ is the universal deformation ring for a Galois representation $\overline{\rho} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ (where k is the residue field of $\mathcal{O}$) and T_Σ is a certain Hecke algebra. The proof proceeds in two steps: first it is proved for $\Sigma = \varnothing$ (the "minimal case"), then the general case is deduced from this one using some numerical criterion for complete intersections. To handle the minimal case, Wiles constructs a system of sets Q_n, $n \geqslant 1$, where Q_n is a set of primes congruent to 1 mod ℓ^n with some other ad hoc properties. Then, by considering a kind of "patching" of the morphisms $\Phi_{Q_n} : R_{Q_n} \to T_{Q_n}$, Wiles manages to prove the desired result for $\Sigma = \varnothing$. Among other things, the proof relied on the fact (due to Mazur and Ribet) that the homology of the modular curve is a free module of rank 2 over $T_\varnothing$ (known as a "multiplicity one" result). Diamond improved the method in [8]: by patching the modules as well as the algebras, he managed to remove multiplicity one as an ingredient of the proof of Fermat's Last Theorem. His proof relies on his freeness criterion [8, 2.1]. Since we only want to illustrate how our Theorem 1.1 can be used to avoid the patching method and the use of Taylor-Wiles systems, we will work in the rather restrictive setting of [7] and [8]. We use their notations and statements in the sequel.

Let us prove that $\Phi_\varnothing$ is an isomorphism. Let λ be a uniformizer of $\mathcal{O}$. By [7, 2.49] (note that we only use it for $n = 1$, so here also the proof can be simplified), there exists a finite set of prime numbers Q such that:

- $\mathrm{edim}(\overline{R_Q}) \leqslant \#Q =: r$, where $\overline{R_Q} = R_Q \otimes_{\mathcal{O}} k = R_Q/\lambda R_Q$
- For any $q \in Q$, $q \equiv 1$ (ℓ), $\overline{\rho}$ is unramified at q and $\overline{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues.

Let G be the ℓ-Sylow subgroup of $\prod_{q\in Q}(\mathbb{Z}/q\mathbb{Z})^\times$. We endow R_Q with the structure of an $\mathcal{O}[G]$-algebra as in [7, §2.8]. Let N be the integer $N_\varnothing$ of [7, (4.2.1)] and $M = p^2\prod_{q\in Q} q$, where p is a well-chosen auxiliary prime (see [7, §4.3]). Consider the group

$$\Gamma = \Gamma_0(N) \cap \Gamma_1(M)$$

and let X_Γ be the associated modular curve. By [7, 4.10], there is an isomorphism $T_Q \to T'_{\mathfrak{m}}$, where T' is the algebra of Hecke operators acting on $H^1(X_\Gamma, \mathcal{O})$ and $\mathfrak{m}$ is a certain maximal ideal of T'. Consider the $T'_{\mathfrak{m}}$-module

$$H := H^1(X_\Gamma, \mathcal{O})^-_{\mathfrak{m}}$$

of elements on which complex conjugation acts by -1. We can view H as a module over $\mathcal{O}[G]$, R_Q, or T_Q via the $\mathcal{O}$-algebra morphisms

$$\mathcal{O}[G] \to R_Q \to T_Q \to T'_{\mathfrak{m}}\,.$$

Theorem 3.1. *With the above notation:*

(i) H *is free over* T_Q*,*

(ii) T_Q *is free over* $\mathcal{O}[G]$*,*

(iii) T_Q *is a relative complete intersection over* $\mathcal{O}[G]$ *(hence also over* $\mathcal{O}$*),*

(iv) $\Phi_Q : R_Q \to T_Q$ *is an isomorphism.*

Proof. By [8, Lemma 3.2], $\overline{H} = H/\lambda H$ is free over $k[G]$. Note that there is an isomorphism $\mathcal{O}[G] \simeq \frac{\mathcal{O}[[S_1,\dots,S_r]]}{((1+S_1)^{\alpha_1}-1,\dots,(1+S_r)^{\alpha_r}-1)}$, where the α_i are the cardinalities of the l-Sylow subgroups of the $(\mathbb{Z}/q\mathbb{Z})^\times$ for $q \in Q$. In particular, $\alpha_i \geqslant \ell \geqslant 2$ and it follows that $\mathrm{edim}(k[G]) = r$. We can apply Theorem 1.1 to the morphism $k[G] \to \overline{R_Q}$ and the module $\overline{H}$. We get that $\overline{H}$ is free over $\overline{R_Q}$, and that $\overline{R_Q}$ is free and is a relative complete intersection over $k[G]$. In particular, $\overline{R_Q} \to \overline{T_Q}$ must be injective (otherwise $\overline{H}$ would have torsion over $\overline{R_Q}$). Since it is already known to be surjective, it is an isomorphism. We have proved all the statements after $\otimes_{\mathcal{O}} k$. Since T_Q is free over $\mathcal{O}$, the theorem now follows from Lemma 3.2 below. □

LEMMA 3.2. *Let $\mathcal{O}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k.*

(i) Let $f : M \to N$ be a morphism of finite-type $\mathcal{O}$-modules. Assume that N is free over $\mathcal{O}$ and that $f \otimes_{\mathcal{O}} k$ is an isomorphism. Then f is an isomorphism.

(ii) Let A be a finite $\mathcal{O}$-algebra and M be an A-module which is free of finite rank over $\mathcal{O}$. If $M \otimes_{\mathcal{O}} k$ is free over $A \otimes_{\mathcal{O}} k$, then M is free over A.

(iii) Assume that $\mathcal{O}$ is complete. Let $A \to B$ be a morphism of finite local $\mathcal{O}$-algebras, with B free over $\mathcal{O}$. If $B \otimes_{\mathcal{O}} k$ is a relative complete intersection over $A \otimes_{\mathcal{O}} k$, then B is a relative complete intersection over A (i.e. the ring $B/\mathfrak{m}_A B$ is a complete intersection).

Proof. These are standard consequences of Nakayama's Lemma. □

REMARK 3.3. By [7, Cor. 2.45 and 3.32], the canonical morphisms $R_Q \to R_\varnothing$ and $T_Q \to T_\varnothing$ induce isomorphisms $R_Q/\mathfrak{a}_Q R_Q \simeq R_\varnothing$ and $T_Q/\mathfrak{a}_Q T_Q \simeq T_\varnothing$, where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathcal{O}[G]$ (i.e. the ideal generated by $S_1, \dots, S_R$). Hence, from Theorem 3.1, it follows immediately that $\Phi_\varnothing : R_\varnothing \to T_\varnothing$ is an isomorphism and $T_\varnothing$ is a complete intersection over $\mathcal{O}$.

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