User's guide for the Maple code about De Smit's conjecture on flatness.

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1 Introduction

In the paper [1], we studied the following question from Bart De Smit. Let $\varphi : A \to B$ be a flat morphism of Artin local rings, with the same embedding dimension[1]. Is it true that any A-flat B-module is B-flat? We proved that the answer is positive if the embedding dimension is 2, or under an additional assumption on the ring $B_0 = B/\mathfrak{m}_A B$ (see [1] thm. 1.6). If the embedding dimension is 3, this additional assumption simply says that there is a minimal generator ν of the maximal ideal $\mathfrak{m}_{B_0}$ of B_0, such that $B_0/(\nu)$ is a complete intersection. If φ is flat, it is also equivalent to the following (see [1] 5.6):

$$(*) \qquad \exists u \in \mathfrak{m}_B \setminus \mathfrak{m}_B^2, \quad \exists x \in \mathfrak{m}_A B \setminus \mathfrak{m}_B \mathfrak{m}_A B \ \text{ such that } u \text{ divides } x \text{ in } B.$$

The Maple programs described in this note are designed to help you to check (in some particular cases) if, given a morphism φ explicitly, the condition (*) holds for φ. Here is a list of what you can do with our programs:

- check if the morphism φ is flat, and display a basis of B over A when this is the case (`IsFlat` and `A_Basis`);
- display a basis of B over $\mathbb{C}$ (`C_Basis`);
- check if (*) holds for φ (`IsKind`);
- if (*) holds for φ, compute explicitly some elements u and x satisfying (*) (`GiveGoodGenerators`).

Our programs only work with $\mathbb{C}$-algebras of embedding dimension 3, with residue field equal to $\mathbb{C}$. Moreover, we decided to stick to the case $\mathfrak{m}_A^2 = 0$ (by [1] 2.4 it does not lose generality), so that the ring A will always be

$$A = \frac{\mathbb{C}[x, y, z]}{(x, y, z)^2} \ .$$

Though, if you want to work with an other ring A, you only need to change the first line of the program (after `restart`).

2 Getting started

a) First method:

- Open a terminal and execute the command `maple`
- Download the file `bdsconj_prog.txt` on the webpage of the author.
- Copy/paste its content in the terminal mentionned above.

b) Second method:

- Download the file `bdsconj_prog.mws` on the webpage of the author and open it with Maple.
- Validate all the input lines (`Edit -> Execute -> Woksheet`).

3 The input data

All our programs have the same input data, so to speak the morphism φ. The ring B is a quotient of $\mathbb{C}[u, v, w]$, say $B = \frac{\mathbb{C}[u,v,w]}{I}$. The input data (describing φ) will consist of two lists of polynomials, `morph` and `eqB`. The polynomials in the lists are polynomials in the three variables u, v and w. The first list

[1] The embedding dimension of a local ring A with maximal ideal $\mathfrak{m}_A$ and residue field $\kappa(A)$ is $\dim_{\kappa(A)}(\mathfrak{m}_A/\mathfrak{m}_A^2)$.

of polynomials, `morph`, is $[\varphi(x), \varphi(y), \varphi(z)]$. The second one, `eqB`, is a list of of generators for the ideal I. Note that at the beginning of the algorithm, the polynomials $\varphi(x)^2, \varphi(y)^2, \varphi(z)^2, \varphi(x)\varphi(y), \varphi(x)\varphi(z)$ and $\varphi(y)\varphi(z)$ will automatically be added to `eqB` so that you don't need to add them by yourself and in many cases `eqB` can be left empty.

Caution Please make sure that the ring B you define with `morph` and `eqB` is Artin. Otherwise Maple will probably return an error or an inconsistent result. A good way to do that is to add in `eqB` some "huge" powers of u, v and w (*e.g.* u^{20}, v^{20} and w^{20}).

Exemple 3.1 Let us give the input data for some morphisms that will be used in the examples below. In all these examples, the ring A is $\frac{\mathbb{C}[x,y,z]}{(x,y,z)^2}$ and we define a morphism $\varphi_i : A \to B_i$.

$B_1 = \dfrac{\mathbb{C}[u,v,w]}{(u^3,v^2,w^3)^2}$ $\varphi_1(x) = u^3$ $\varphi_1(y) = v^2$ $\varphi_1(z) = w^3$	`> morph1:=[u^3,v^2,w^3]:` `> eqB1:=[]:`
$B_2 = \dfrac{\mathbb{C}[u,v,w]}{(u^2+vw,v^2+uw,w^2+uv)^2}$ $\varphi_2(x) = u^2+vw$ $\varphi_2(y) = v^2+uw$ $\varphi_2(z) = w^2+uv$	`> morph2:=[u^2+v*w,v^2+u*w,w^2+u*v]:` `> eqB2:=[]:`
$B_3 = \dfrac{\mathbb{C}[u,v,w]}{(u^2,v^2,w^2)^2}$ $\varphi_3(x) = u^2$ $\varphi_3(y) = v^2$ $\varphi_3(z) = u^2+v^2$	`> morph3:=[u^2,v^2,u^2+v^2]:` `> eqB3:=[w^4,u^2*w^2,v^2*w^2]:`
$B_4 = \dfrac{\mathbb{C}[u,v,w]}{(u^3+vw,v^2+uw^2,w^3+u^2v)^2}$ $\varphi_4(x) = u^3+vw$ $\varphi_4(y) = v^2+uw^2$ $\varphi_4(z) = w^3+u^2v$	`> morph4:=[u^3+v*w,v^2+u*w^2,w^3+u^2*v]:` `> eqB4:=[]:`

4 The function IsFlat

```
IsFlat(morph,eqB) --> Boolean
```

This procedure takes as entries `morph` and `eqB` described above, and returns `true` if φ is flat and `false` otherwise.

Exemple 4.1

```
> IsFlat(morph1,eqB1);
                                        true
> IsFlat(morph2,eqB2);
                                        true
> IsFlat(morph3,eqB3);
                                        false
> IsFlat(morph4,eqB4);
                                        true
```

5 Display a basis of B over $\mathbb{C}$ or over A

If you are trying to handle with some (counter-?) examples for De Smit's conjecture, you may want to know a basis of B over $\mathbb{C}$ or over A (if φ is flat). This is already implemented in Maple, with the function

`SetBasis`, but to avoid you reading Maple's help, we have written two functions `C_Basis` and `A_Basis` (they actually just call `SetBasis` in the relevant way).

```
C_Basis(morph,eqB) --> List of Polynomials
```

This procedure returns a list of polynomials which is a basis of the ring B over the field $\mathbb{C}$.

Exemple 5.1

```
> C_Basis(morph3,eqB3);
        2   3             2     3   2   2     3   3              2     3
[1, w, w , w , v, v w, v w , v w , v , v  w, v , v  w, u, u w, u w , u w , u v,

                2       3     2     2       3     3     2   2     2     2
    u v w, u v w , u v w , u v , u v  w, u v , u v  w, u , u  w, u  v, u  v w,

     3   3     3     3
    u , u  w, u  v, u  v w]
```

```
A_Basis(morph,eqB) --> List of Polynomials
```

This procedure returns a list of polynomials which is a basis of the ring $B/\mathfrak{m}_A B$ over the field $\mathbb{C}$, *even if B is not flat over A*. Of course, this will be an A-basis of B if B is flat over A. If you ask for an A-basis while φ is not flat, the procedure will not return an error message (which would be the behavior that you might have expected) but as we just told, it will return a $\mathbb{C}$-basis for the ring $B/\mathfrak{m}_A B$.

Exemple 5.2

```
> A_Basis(morph1,eqB1);
        2             2             2                   2   2   2     2  2   2
[1, w, w , v, v w, v w , u, u w, u w , u v, u v w, u v w , u , u  w, w  u , u  v,

     2       2    2
    u  v w, u  v w ]

> A_Basis(morph2,eqB2);
                                   2   3           2
                          [1, w, w , w , v, v w, v , u]

> A_Basis(morph3,eqB3);
        2   3           2       3             2     3                   2       3
[1, w, w , w , v, v w, w  v, v w , u, w u, u w , u w , v u, u v w, u v w , u v w ]
```

6 The functions IsKind and GiveGoodGenerator

```
IsKind(morph,eqB) --> Boolean
```

This procedure returns `true` if φ satisfies the condition (*) above and `false` otherwise.

Exemple 6.1

```
> IsKind(morph1,eqB1);
                                                  true
> IsKind(morph2,eqB2);
                                                  true
> IsKind(morph4,eqB4);
                                                 false
```

```
GiveGoodGenerators(morph,eqB) --> Couple of Polynomials
```

Work in progress... this function does not exist yet.

References

[1] Sylvain Brochard and Ariane Mézard. About De Smit's conjecture on flatness. À paraître dans *Math. Zeit.*, 2008.