Thesis overview

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Contents

If X is a geometric object, *e.g.* a scheme or a topological space, the Picard group of X is the group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles on X, in a sense given by the context. This abstract group often underlies a natural *algebraic* group. For instance, if X is a compact Riemann surface of genus g, it is well known that we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and that $\mathrm{Pic}^0(X)$ (the group of isomorphism classes of invertible sheaves of degree zero) has a natural structure of complex torus of dimension g. We even know that this "jacobian" $\mathrm{Pic}^0(X)$ is in fact a projective variety, thus algebraic.

Weil gave later on a purely algebraic definition of the jacobian of a curve over an arbitrary base field, but the construction was still rather unsatisfactory. Grothendieck clarified greatly the situation by defining the "Picard functor" of a scheme X over an arbitrary base scheme S, denoted $\mathrm{Pic}_{X/S}$. Then the Picard scheme of X over S is the scheme representing this functor, if it exists. For instance if X is a projective smooth curve over the field of complex numbers $\mathbb{C}$, the connected component of the identity element of the Picard scheme $\mathrm{Pic}_{X/S}$ is isomorphic to the jacobian $\mathrm{Pic}^0(X)$, and the whole Picard scheme is a disjoint union of copies of the jacobian, indexed by the degree of invertible sheaves. One of the advantages of Grothendieck's point of view is that the *geometric* structure on the Picard group is completely defined *a priori* and we can even study some properties of $\mathrm{Pic}_{X/S}$ (smoothness, properness...) without knowing if it is representable or not.

The Picard functor of a scheme has been studied extensively (by Grothendieck, Artin and others...). However, Giraud's thesis followed by a ground-breaking article by Deligne and Mumford gave birth in the 70's to the notion of an algebraic stack. Today the algebraic stack seems to be an essential object for the algebraic geometer, and recent works (see for instance the use of twisted curves by Abramovich and Vistoli) show that stacks are becoming the basic object of study in algebraic geometry, as schemes were. In my thesis I studied the Picard functor of an algebraic stack $\mathscr{X}$ over a base scheme S.

The definitions of the several Picard functors of an algebraic stack are the same as for schemes. We first define a functor $P_{\mathscr{X}/S}$ by

$$P_{\mathscr{X}/S}(U) := \frac{\mathrm{Pic}(\mathscr{X} \times_S U)}{\mathrm{Pic}(U)}.$$

We then denote $\mathrm{Pic}_{\mathscr{X}/S\,(\mathrm{Zar})}$ (resp. $\mathrm{Pic}_{\mathscr{X}/S\,(\text{Ét})}$, $\mathrm{Pic}_{\mathscr{X}/S\,(\mathrm{fppf})}$) the associated sheaf with respect to the Zariski (resp. étale, *fppf*) topology. *The* Picard functor is $\mathrm{Pic}_{\mathscr{X}/S\,(\mathrm{fppf})}$ and will simply be denoted $\mathrm{Pic}_{\mathscr{X}/S}$. We also define the Picard stack of $\mathscr{X}/S$. This is the stack, denoted $\mathscr{P}ic(\mathscr{X}/S)$, whose fiber category over an S-scheme U is the category of invertible sheaves on $\mathscr{X} \times_S U$.

The first chapter of my thesis is devoted to some general facts about these objects: cohomogical description of the Picard group and the Picard functors, comparison between the various Picard functors, and between the Picard functor and the Picard stack. In particular, in the case f is cohomologically flat in dimension zero, we prove that $\mathscr{P}ic(\mathscr{X}/S)$ is algebraic if and only if $\mathrm{Pic}_{\mathscr{X}/S}$ is an algebraic space (and, in addition, the natural morphism from $\mathscr{P}ic(\mathscr{X}/S)$ to $\mathrm{Pic}_{\mathscr{X}/S}$ is faithfully flat and locally of finite presentation). We also prove that the $\mathbb{G}_{\mathrm{m}}$-gerbe $\mathscr{P}ic(\mathscr{X}/S)$ over $\mathrm{Pic}_{\mathscr{X}/S}$ is a trivial gerbe if and only if there exists a universal invertible sheaf on $\mathscr{X} \times_S \mathrm{Pic}_{\mathscr{X}/S}$. For more details about this part, we refer to the thesis itself.

We then study the following points:

- deformation theory of invertible sheaves
- representability of $\mathrm{Pic}_{\mathscr{X}/S}$ by an algebraic space
- separation properties
- the connected component of the identity (definition, properness...)
- examples (gerbes, twisted curves...)

We were also led to review the lisse-étale cohomology of an algebraic (Artin) stack, and to prove a lot of technical details about it that were lacking to the literature.

1 Representability

1.1 Deformations of invertible sheaves

The study of deformations is the cornerstone of Artin's theorems for representability (see [7] and [8]). This is often the more delicate point to study.

Let us define what a deformation of an invertible sheaf is. Let $\mathscr{X}$ be an algebraic stack over a base scheme T and $\mathscr{L}$ an invertible sheaf on $\mathscr{X}$. Suppose we are given a closed immersion:

$$i : \mathscr{X} \longrightarrow \widetilde{\mathscr{X}}$$

defined by a nilpotent quasi-coherent ideal I on $\widetilde{\mathscr{X}}$. We then denote by $\mathrm{Defm}(\mathscr{L})$ the category of deformations of $\mathscr{L}$ to $\widetilde{\mathscr{X}}$ defined in the following way. An object of $\mathrm{Defm}(\mathscr{L})$ is a couple $(\widetilde{\mathscr{L}}, \lambda)$ where $\widetilde{\mathscr{L}}$ is an invertible sheaf on $\widetilde{\mathscr{X}}$ and λ is an isomorphism $\lambda : i^*\widetilde{\mathscr{L}} \xrightarrow{\sim} \mathscr{L}$. A morphism from $(\widetilde{\mathscr{L}}, \lambda)$ to $(\widetilde{\mathscr{M}}, \mu)$ is an isomorphism $\alpha : \widetilde{\mathscr{L}} \xrightarrow{\sim} \widetilde{\mathscr{M}}$ such that $\mu \circ i^*\alpha = \lambda$. The set of isomorphism classes of $\mathrm{Defm}(\mathscr{L})$ will be denoted by $\overline{\mathrm{Defm}(\mathscr{L})}$.

In my thesis I prove the following theorem.

Theorem 1.1.1 ([10] 3.2.5) *(1) There is an element ω in $H^2(\mathscr{X}, I)$ the vanishing of which is equivalent to the existence of a deformation of $\mathscr{L}$ to $\widetilde{\mathscr{X}}$.*
(2) If $\omega = 0$, then $\overline{\mathrm{Defm}(\mathscr{L})}$ is a torsor under $H^1(\mathscr{X}, I)$.
(3) If $(\widetilde{\mathscr{L}}, \lambda)$ is a deformation of $\mathscr{L}$, its group of automorphisms is isomorphic to $H^0(\mathscr{X}, I)$.

I give there two different proofs. In the first one we use a standard technique: we reduce to the case of algebraic spaces by taking a presentation of $\mathscr{X}$. The second one is shorter: we work directly with invertible sheaves on $\mathscr{X}$ and we use the exact sequence of abelian sheaves

$$0 \to I \to \mathscr{O}^\times_{\widetilde{\mathscr{X}}} \to i_*\mathscr{O}^\times_{\mathscr{X}} \to 1.$$

Nevertheless, we have to solve a small technical difficulty: it is necessary to relate the cohomology of abelian sheaves on $\mathscr{X}$ to that on $\widetilde{\mathscr{X}}$. This is done in the appendix of my thesis. This problem did not appear when working with étale cohomology because $\mathscr{X}$ and $\widetilde{\mathscr{X}}$ have the same étale site (if they are Deligne-Mumford stacks). This is no more true with the lisse-étale site.

Note that this theorem can also be obtained taking $\mathscr{Y} = \mathrm{B}\mathbb{G}_{\mathrm{m}}$ in the theorem 2.1.1 of [6] and calculating the cotangent complex $L_{\mathrm{B}\mathbb{G}_{\mathrm{m}}/T}$ and the groups $\mathrm{Ext}^i(Lf^*L_{\mathrm{B}\mathbb{G}_{\mathrm{m}}/T}, I)$ (this computation is done in my thesis [10]).

1.2 Consequences

The study of deformations of invertible sheaves allows us to give a new and direct proof of the following theorem:

Theorem 1.2.1 (Aoki, [6] 5.1 and [5]) *If $\mathscr{X}$ is proper and flat over S, then the stack $\mathscr{P}ic(\mathscr{X}/S)$ is an algebraic (Artin) stack.*

Aoki first proved that the stack $\mathscr{H}om(\mathscr{X}, \mathscr{Y})$ is algebraic and then specialised to $\mathscr{Y} = \mathrm{B}\mathbb{G}_{\mathrm{m}}$. It turns out that our proof is much shorter than Aoki's original proof. The reason is that the case of invertible sheaves is much simpler than that of morphisms.

As an other consequence of the study of deformations of invertible sheaves, we get the following generalization of a classical result:

Theorem 1.2.2 *Let k be a field, $S = \operatorname{Spec} k$, and $\mathscr{X}$ an algebraic stack over S. Let us denote by $\operatorname{Pic}_{\mathscr{X}/k}$ the relative Picard functor $\operatorname{Pic}_{\mathscr{X}/S\,(\text{Ét})}$ and assume that it is representable by an algebraic space locally of finite type over S.*

a) Then the tangent space at the origin is

$$T_0 \operatorname{Pic}_{\mathscr{X}/k} = H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}).$$

b) The algebraic space $\operatorname{Pic}_{\mathscr{X}/k}$ has the same dimension everywhere. Moreover this dimension is lower than $\dim_k H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$, and equality holds if and only if $\operatorname{Pic}_{\mathscr{X}/k}$ is smooth at the origin. In that case, $\operatorname{Pic}_{\mathscr{X}/k}$ is smooth of dimension $\dim_k H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ everywhere. This is the case if k is of characteristic zero.

1.3 Separation properties

If F is an algebraic space, we say F is locally separated if its diagonal is a quasi-compact immersion. In [7] Artin proved among other things that the Picard functor of a proper cohomologically flat scheme X is locally separated. The proof is based on a kind of valuative criterion for quasi-compact immersions. We generalize his techniques to prove that the same is true if we replace the scheme X with an algebraic stack.

For stacks, separation properties are not so good in general. Let us consider a base scheme S and an algebraic stack $\mathscr{P}$ over S. Assume that $\mathscr{P}$ is locally separated in the usual sense for schemes. Then its diagonal is by definition an immersion, thus a monomorphism. Yet it is well known that the diagonal is a monomorphism if and only if the stack $\mathscr{P}$ is in fact an algebraic space. So we see that a stack is never locally separated unless it is an algebraic space. A more reasonable condition is that of quasi-separatedness. An algebraic stack is said to be quasi-separated if its diagonal is quasi-compact. Many things do not work very well when stacks are not quasi-separated (see the remark II 1.9 in [13]). That is the reason why it is often part of the definition (see [14]). But there are also "nice" stacks that we might call "algebraic" but that are not quasi-separated: just have a look at the classifying stack $B\mathbb{Z}$. Sometimes it is not clear whether a stack is quasi-separated or not (see a remark of Aoki in [6]). For instance in [6], Masao Aoki proves that the stack $\mathscr{H}om(\mathscr{X}, \mathscr{Y})$ is quasi-separated in the very particular case where $\mathscr{X}$ is an algebraic space and $\mathscr{Y}$ has a proper covering. I proved in my thesis that the Picard stack $\mathscr{P}ic(\mathscr{X}/S)$ is quasi-separated.

Theorem 1.3.1 ([10] 2.3.1) *Let S be a noetherian scheme. Let $\mathscr{X}$ be a proper and cohomologically flat algebraic stack. Then the Picard stack $\mathscr{P}ic(\mathscr{X}/S)$ is quasi-separated (and so is algebraic in the sense of [14]).*

In order to demonstrate this, I was led to prove the following criterion. It was first inspired by Artin's criterion for proving that algebraic spaces are locally separated.

Proposition 1.3.2 *Let S be a locally noetherian scheme and $\mathscr{X}$ an algebraic (Artin) stack over S locally of finite presentation. Assume that the following two conditions are satisfied:*

(i) For all $U \in \operatorname{ob}(\mathrm{Aff}/S)$ and $x \in \operatorname{ob} \mathscr{X}_U$, the morphism $\underline{\operatorname{Aut}}(x) \longrightarrow U$ is quasi-compact.

(ii) Let $U \in \mathrm{ob}\,(\mathrm{Aff}/S)$ be an integral affine scheme, and let x, y be two objects of $\mathscr{X}_U$. We assume that there is a dense subset of points t in U, such that there exists an extension $L(t)$ of $\kappa(t)$ such that $x_{L(t)} \simeq y_{L(t)}$. Then x and y are isomorphic over a dense open subset of U.

Then $\mathscr{X}$ is quasi-separated (hence is algebraic in the sense of [14]).

With this tool in hand, we should be able to prove that some other stacks are quasi-separated.

2 The connected component of the identity

If X is a smooth projective curve over a field, its Picard scheme is an infinite disjoint sum of copies of the jacobian, indexed by the degree of invertible sheaves. So it is obvious that the Picard scheme itself is not proper, whereas the jacobian is. It turns out that the connected component of the identity has nice finiteness properties, while it contains in fact a large amount of the information contained in the whole Picard scheme.

On a general basis the definition of the neutral component is not so easy. Actually, there is no more *an* identity element but a whole identity section. The most natural idea to define the identity component is to look at the connected component of the identity element *in each fiber* and to consider the union of all these components.

Definition 2.1 *Let S be a scheme and $\mathscr{X}$ an algebraic stack over S, such that the Picard functor $\mathrm{Pic}_{\mathscr{X}/S}$ is an algebraic space locally of finite type. We define a subfunctor $\mathrm{Pic}^0_{\mathscr{X}/S}$ of $\mathrm{Pic}_{\mathscr{X}/S}$ in the following way. For each S', a point ξ in $\mathrm{Pic}_{\mathscr{X}/S}(S')$ is in $\mathrm{Pic}^0_{\mathscr{X}/S}(S')$ if and only if for each s' in S', the restriction $\xi_{s'}$ is in $\mathrm{Pic}^0_{\mathscr{X}_{s'}/\kappa(s')}(\kappa(s'))$.*

The matter is that it is not obvious that this construction gives an open subset of the Picard functor. When the Picard functor is a scheme, we can apply EGA IV (15.6.5) (about the connected component of the fibers of a morphism along a section) to prove that we do get an open group subscheme in nice cases. But if the Picard functor is an algebraic space we cannot apply this result.

I prove a generalization for algebraic spaces (this is not an immediate consequence of the result for schemes, see the proof of the lemma 4.2.8 in [9]). We then get the following.

Proposition 2.2 ([9] 4.2.10) *Let S be a locally noetherian scheme and $\mathscr{X}$ an algebraic stack over S. Assume that the Picard functor $\mathrm{Pic}_{\mathscr{X}/S}$ is an algebraic space locally of finite presentation over S and that the fibers $\mathrm{Pic}_{\mathscr{X}_s/\kappa(s)}$ are geometrically reduced. Assume moreover that one of the following holds:*

- *a) the morphism $\mathrm{Pic}_{\mathscr{X}/S} \to S$ is universally open at each point of $\mathrm{Pic}^0_{\mathscr{X}/S}$ (*e.g. *if it is flat);*
- *b) the function $s \mapsto \dim(\mathrm{Pic}_{\mathscr{X}_s/\kappa(s)})$ is locally constant on S (*e.g. *if $\mathrm{Pic}_{\mathscr{X}/S}$ is smooth along the unit section).*

Then the morphism $\mathrm{Pic}^0_{\mathscr{X}/S} \to \mathrm{Pic}_{\mathscr{X}/S}$ is an open immersion. Moreover, $\mathrm{Pic}^0_{\mathscr{X}/S}$ is of finite type over S.

Now we would like $\mathrm{Pic}^0_{\mathscr{X}/S}$ to be proper when $\mathscr{X}$ is smooth over S for instance. This is indeed true when $\mathscr{X}$ is a scheme (see [12] 5.20). We have done the first step in [10] by proving the following result:

Theorem 2.3 ([10] 4.2.2) *Assume that $\mathscr{X}$ is a proper, geometrically normal algebraic stack over the spectrum of a field k. Assume also that $\mathscr{X}$ is cohomologically flat in dimension zero. Then the neutral component $Pic^0_{\mathscr{X}/k}$ is proper over k.*

We conclude in [9]:

Theorem 2.4 ([9] 4.3) *Let $\mathscr{X}$ be an S algebraic stack such that $\mathrm{Pic}_{\mathscr{X}/S}$ is an algebraic space locally of finite presentation. Assume that $\mathrm{Pic}^0_{\mathscr{X}/S}$ is an open subspace of $\mathrm{Pic}_{\mathscr{X}/S}$, separated and of finite type over S, and that each fiber $\mathrm{Pic}^0_{\mathscr{X}_s/\kappa(s)}$ is proper over $\kappa(s)$. Then $\mathrm{Pic}^0_{\mathscr{X}/S}$ is proper over S.*

To prove 2.3 we reduce to proving the following. (This is done by proving that every $\mathbb{Z}$-torsor over $\mathscr{X}$ is trivial.)

Theorem 2.5 *Let $\mathscr{X}$ be a locally noetherian, normal algebraic stack. Then the group $H^1(\mathscr{X}, \mathbb{Z})$ is trivial.*

3 Examples

To illustrate the text, we calculate some concrete examples of Picard schemes.

3.1 Weighted projective spaces

Let $\mathscr{X}$ be a weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ over $S = \mathrm{Spec}\,\mathbb{Z}$ where $a_0, \ldots, a_n$ are positive integers. Using the deformation theory of line bundles, we can prove that the morphism $\mathrm{Pic}_{\mathscr{X}/S} \to S$ is unramified (*cf.* [9] 5.2.2). This allows us to give a new and very short proof of the following.

Theorem 3.1.1 ([15], main) *The Picard stack $\mathscr{P}ic(\mathscr{X}/S)$ is isomorphic to $\mathbb{Z} \times_S \mathrm{B}\mathbb{G}_{\mathrm{m}}$.*

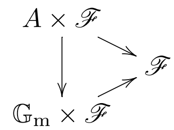
3.2 The n^{th} root of a line bundle

Let X be an S-scheme and $\mathscr{L}$ an invertible sheaf on X. Let n be a positive integer. We build a stack $[\mathscr{L}^{\frac{1}{n}}]$ in the following way. For each U over S, the fiber category $[\mathscr{L}^{\frac{1}{n}}]_U$ is the category of triples $(x, \mathscr{M}, \varphi)$ where

$$\begin{cases} x : U \to X \text{ is an element of } X(U) \\ \mathscr{M} \text{ is an invertible sheaf on U} \\ \varphi : \mathscr{M}^{\otimes n} \to x^*\mathscr{L} \text{ is an isomorphism of invertible sheaves.} \end{cases}$$

Then $[\mathscr{L}^{\frac{1}{n}}]$ is an algebraic stack, and it is a gerbe over X for the *fppf* (étale if n is invertible in S) topology, banded by μ_n.

Now, for any gerbe $\pi : \mathscr{X} \to X$ banded by a commutative group scheme A and for any invertible sheaf $\mathscr{F}$ on $\mathscr{X}$, we construct ([10] 5.2.7) a character $\chi_{\mathscr{F}} : A \to \mathbb{G}_{\mathrm{m}}$ associated with $\mathscr{F}$. It is defined by the property that the natural action of A on $\mathscr{F}$ (given by the A-gerbe structure of $\mathscr{X}$) is induced, *via* $\chi_{\mathscr{F}}$, by the natural action of $\mathbb{G}_{\mathrm{m}}$ on $\mathscr{F}$ (given by the structure of $\mathscr{O}_{\mathscr{X}}$-module of $\mathscr{F}$).



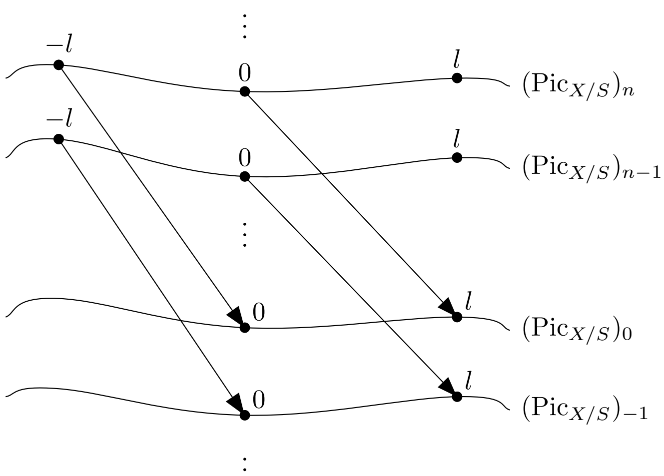
The character $\chi_{\mathscr{F}}$ has the following properties: it is compatible with base change and product, and $\chi_{\mathscr{F}}$ is trivial if and only if $\mathscr{F}$ is isomorphic to the inverse image of an invertible sheaf on X.

Coming back to the case of the stack $[\mathscr{L}^{\frac{1}{n}}]$, we get an exact sequence

$$1 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}([\mathscr{L}^{\frac{1}{n}}]) \longrightarrow \widehat{\mu_n} \longrightarrow 1.$$

Moreover, there is a “canonical” invertible sheaf Ω on $[\mathscr{L}^{\frac{1}{n}}]$ (to each object $(x, \mathscr{M}, \varphi)$ associate $\mathscr{M}$) which is an n^{th} root of $\mathscr{L}$. The associated character χ_Ω is a generator of $\widehat{\mu_n}$.

Now, all these constructions behave well under base change, so that we get a very concrete and complete description of the Picard functor $\mathrm{Pic}_{[\mathscr{L}^{\frac{1}{n}}]/S}$ in terms of $\mathrm{Pic}_{X/S}$. Assume that $\mathrm{Pic}_{X/S}$ is an algebraic space, then $\mathrm{Pic}_{[\mathscr{L}^{\frac{1}{n}}]/S}$ is the disjoint union of n copies of $\mathrm{Pic}_{X/S}$. The group law is given by the relation $\Omega^{\otimes n} = \mathscr{L}$ (see [10] or [9] for more details).



The Picard functor of $[\mathscr{L}^{\frac{1}{n}}]$: glue $\mathbb{Z} \times_S \mathrm{Pic}_{X/S}$ along the isomorphism drawn.

3.3 Abramovich-Vistoli curves

Using the results of Cadman ([11]), we describe the Picard scheme of some of the twisted curves introduced by Abramovich and Vistoli ([2], [3] et [4]). We show that the phenomenon that occurs is analogous to the phenomenon described in the preceding paragraph for $[\mathscr{L}^{\frac{1}{n}}]$: the ”stacky” structure added to the underlying coarse moduli scheme modifies the Picard functor by adding n^{th} roots of the classes of some natural invertible sheaves (more details in [10] or [9]).

4 A review of lisse-étale cohomology

As said before, the lack of references about lisse-étale cohomology led me to add to my thesis an appendix devoted to it. For instance once we have defined the *lisse-étale* cohomology groups of an algebraic stack, we should verify that they are equal to the *étale* cohomology groups when $\mathscr{X}$ is a Deligne-Mumford stack. There are also many properties of cohomology that usually derive from general results of SGA4 (spectral sequences, calculation of higher direct images...). But here it often happens that we cannot apply SGA4. In fact the lisse-étale topos is not functorial in a nice way. If f is a morphism of algebraic stacks, the morphism f^{-1} may not be exact, contrary to what is asserted in [14]. Thus we have to verify all the properties "by hand". Most of the subsequent work has been done by Olsson, however he just treats the case of quasi-coherent sheaves. As we often had to deal with $\mathbb{G}_{\mathrm{m}}$, which is not quasi-coherent, we could not apply the results of Olsson. Let us give a rough list of what you may find in this appendix.

- The proof that the lisse-étale cohomology of a Deligne-Mumford stack coincides with étale cohomology.
- A new site, the "stacky lisse-lisse site" that is more convenient when working with direct images.
- The spectral sequence associated with a smooth covering.
- Higher direct images of abelian sheaves.
- The Leray spectral sequence associated with a morphism of algebraic stacks.
- Cohomology and base change.
- Comparison between the cohomology on $\mathscr{X}$ and the cohomology on $\widetilde{\mathscr{X}}$ if $i : \mathscr{X} \to \widetilde{\mathscr{X}}$ is an infinitesimal extension of algebraic stacks.
- Flat cohomology and comparison with lisse-étale cohomology for smooth groups (generalizing a result of [1], exp. VI).

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