# The semicontinuity theorem for stacks

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**Abstract**

We prove the semicontinuity theorem for a coherent sheaf over a (non necessarily tame) proper algebraic stack. For schemes, this theorem (as well as many other "cohomology and base change" theorems) relies on the existence of a finite complex of flat modules computing the cohomology of a given coherent sheaf. We discuss the existence of such a complex for stacks. This note has more or less been included in the appendix of [4].

Nous démontrons le théorème de semi-continuité pour un faisceau cohérent sur un champ algébrique propre (non nécessairement modéré). Dans le cadre des schémas, ce théorème repose sur l'existence d'un complexe fini de modules plats calculant la cohomologie d'un module cohérent donné. Nous discutons l'existence d'un tel complexe dans le cadre des champs algébriques. L'essentiel de cette note a été inclus dans l'appendice de [4].

## 1 Introduction

In scheme theory, the key point to get some "base change theorems" for cohomology of a coherent sheaf $\mathscr{F}$ on a proper scheme (over a noetherian ring $A$), is the existence of a finite complex of finite free $A$-modules computing "universally" the cohomogy of $\mathscr{F}$ (see [12] §5). A significant difference between an algebraic stack and a scheme (or even an algebraic space) is that the former can have infinite cohomological dimension. Consequently, there does not always exist such a finite complex. However, we explain below that even without such a complex, the semicontinuity theorem still holds for stacks.

**Theorem 1.1 (see 4.2)** *Let $S$ be a scheme, $\mathscr{X}$ a proper algebraic stack of finite presentation over $S$, and $\mathscr{F}$ a coherent $\mathscr{O}_{\mathscr{X}}$-module that is flat over $S$. Then for any integer $i \geq 0$, the function*

$$d_i\colon \begin{cases} S \longrightarrow \mathbb{N} \\ s \longmapsto \dim_{\kappa(s)} H^i(\mathscr{X}_s, \mathscr{F}_s) \end{cases}$$

*is upper semicontinuous over $S$.*

Moreover, there are two particular cases in which it is worth mentionning that we can say much more :

a) if the stack is tame;

b) if the base ring has finite global cohomological dimension (*e.g.* if it is regular and finite dimensional).

Indeed, in these two cases, we can actually compute the cohomology of a given $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ (at least the first groups) with a finite complex of $A$-modules.

**Proposition 1.2 (see 2.4)** *Let $S$ be the spectrum of a ring $A$ (resp. a noetherian ring $A$) and let $\mathscr{X}$ be a quasi-compact and separated (resp. proper) tame stack on $S$. Let $\mathscr{F}$ be a quasi-coherent (resp. coherent) sheaf on $\mathscr{X}$ that is flat over $S$. Then there is a complex of flat $A$-modules (resp. of finite type)*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow 0$$

*with $M^i$ free over $A$ for $1 \leq i \leq n$, and isomorphisms*

$$H^i(M^\bullet \otimes_A A') \longrightarrow H^i(\mathscr{X} \otimes_A A', \mathscr{F} \otimes_A A')\,, \quad i \geq 0$$

*functorial in the $A$-algebra $A'$.*

**Proposition 1.3 (see 3.4)** *Let $A$ be a noetherian ring with finite global cohomological dimension, $S$ its spectrum, $\mathscr{X}$ a proper algebraic stack over $S$, and $\mathscr{F}$ a coherent $\mathscr{O}_{\mathscr{X}}$-module, flat over $S$. Let $n$ be a natural integer. Then there is a finite complex of flat and finite type $A$-modules*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow M^{n+1},$$

*and functorial isomorphisms*

$$H^i(M^\bullet \otimes_A B) \xrightarrow{\sim} H^i(\mathscr{X} \otimes_A B, \mathscr{F} \otimes_A B)\,, \quad 0 \leq i \leq n\,.$$

Consequently, in these cases, the cohomology of $\mathscr{F}$ really behaves under base change as if $\mathscr{X}$ were a scheme (see 2.5, 2.6, 3.5).

Since the proof of the theorem 1.1 relies on the case b), we will start with these particular cases.

**Notations and conventions.** Following [11], all algebraic stacks (*a fortiori* all schemes and algebraic spaces) are supposed to be quasi-separated. The cohomology groups on an algebraic stack $\mathscr{X}$ are computed with respect to the smooth-étale topology.

**Acknowledgments.** We warmly thank Angelo Vistoli, who gave us the idea of using rings with finite global cohomological dimension.

## 2 If the stack is tame

Before dealing with complexes, let us recall the definition of tame stacks and give a few facts about them. If $\mathscr{X}$ is an Artin stack locally of finite presentation over a scheme $S$, and if its inertia stack $\mathscr{I} = \mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}$ is finite over $\mathscr{X}$, it follows from [10] that there is a moduli space $\pi : \mathscr{X} \to X$ for $\mathscr{X}$ (see also [6]). Note by the way that the map $\pi$ is proper and quasi-finite. Moreover, the algebraic space $X$ is locally of finite type if $S$ is locally noetherian, and if $\mathscr{X}$ is separated then $X$ is separated as well.

**Definition 2.1 ([2] 3.1)** *Let $S$ be a scheme and let $\mathscr{X} \to S$ be a locally finitely presented algebraic stack over $S$ with finite inertia. Let $\pi : \mathscr{X} \to X$ be its Keel-Mori moduli space. We say that $\mathscr{X}$ is tame if the functor*

$$\pi_* : \mathfrak{Qcoh}(\mathscr{X}) \longrightarrow \mathfrak{Qcoh}(X)$$

*is exact.*

We also recall that the class of tame stacks is stable under arbitrary base change ([2] 3.4) and that if $\mathscr{X}$ is tame then forming its moduli space commutes with any base change ([2] 3.3).

**Lemma 2.2** *With the notations and assumptions of 2.1, let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathscr{X}$ and let $\mathscr{N}$ be a quasi-coherent sheaf on $X$. Then the natural morphism*

$$(\pi_*\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{N} \longrightarrow \pi_*(\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^*\mathscr{N})$$

*is an isomorphism. In particular, if $\mathscr{F}$ is flat over $S$, then so is $\pi_*\mathscr{F}$.*

**Proof.** We use more or less the same argument as in the proof of [2] 3.3 (b). Since the question is local on $X$ for the étale topology, we can assume that $X$ is an affine scheme. The statement is obvious if $\mathscr{N}$ is free. In the general case, let $\mathscr{Q}_1 \to \mathscr{Q}_0 \to \mathscr{N} \to 0$ be a free presentation of $\mathscr{N}$. Then we have a commutative diagram with exact rows (since $\pi_*$ is exact):

$$\begin{array}{ccccccc}
(\pi_*\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{Q}_1 & \longrightarrow & (\pi_*\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{Q}_0 & \longrightarrow & (\pi_*\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{N} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_*(\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^*\mathscr{Q}_1) & \longrightarrow & \pi_*(\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^*\mathscr{Q}_0) & \longrightarrow & \pi_*(\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^*\mathscr{N}) & \longrightarrow & 0.
\end{array}$$

The first two columns are isomorphisms, hence so is the third. The last assertion follows immediately, keeping in mind the fact that $\pi_*$ is exact. □

Let us recall the following lemma from [12] §5 (see also [8] chap. 0 (11.9.1)).

**Lemma 2.3 ([12] §5 lemma 1)**

*a) Let $A$ be a ring and let $C^\bullet$ be a complex of $A$-modules such that $C^p \neq 0$ only if $0 \leq p \leq n$. Then there exists a complex $K^\bullet$ of $A$-modules such that $K^p \neq 0$ only if $0 \leq p \leq n$ and $K^p$ is free if $1 \leq p \leq n$, and a quasi-isomorphism of complexes $K^\bullet \to C^\bullet$. Moreover, if the $C^p$ are flat, then $K^0$ will be $A$-flat too.*

*b) If $A$ is noetherian and if the $H^i(C^\bullet)$ are finitely generated $A$-modules, then the $K^p$'s can be choosen to be finitely generated.*

**Proof.** The assertion b) is exactly [12] §5 lemma 1. For a), the same proof works, erasing the words "finitely generated" everywhere. □

**Proposition 2.4** *Let $S$ be the spectrum of a ring $A$ (resp. a noetherian ring $A$) and let $\mathscr{X}$ be a quasi-compact and separated (resp. proper) tame stack on*

*$S$. Let $\mathscr{F}$ be a quasi-coherent (resp. coherent) sheaf on $\mathscr{X}$ that is flat over $S$. Then there is a complex of flat $A$-modules (resp. of finite type)*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \dots \longrightarrow M^n \longrightarrow 0$$

*with $M^i$ free over $A$ for $1 \leq i \leq n$, and isomorphisms*

$$H^i(M^\bullet \otimes_A A') \longrightarrow H^i(\mathscr{X} \otimes_A A', \mathscr{F} \otimes_A A')\,, \quad i \geq 0$$

*functorial in the $A$-algebra $A'$.*

**Proof.** Let $\pi : \mathscr{X} \to X$ be the moduli space of $X$. Note that $X$ is separated. Choose a finite affine covering $\mathfrak{U} = (U_i)_{i \in I}$ of $X$ by affine open subschemes. Then form the Čech complex $C^\bullet = C^\bullet(\mathfrak{U}, \pi_* \mathscr{F})$ of alternating Čech cochains. It is a finite complex of flat (2.2) $A$-modules and it computes the cohomology groups $H^i(X, \pi_* \mathscr{F})$. Since $X$ is separated, the elements of the covering $\mathfrak{U} \otimes_A A'$ obtained after a base change $A \to A'$ are still affines, so the cohomology of the complex $C^\bullet$ is universally isomorphic to the cohomology of $\pi_* \mathscr{F}$ on $X$. But this is also the cohomology of $\mathscr{F}$ on $\mathscr{X}$, since the functor $\pi_*$ is exact (use for instance the Leray spectral sequence for $\pi$, [5] A.2.8). Note that if $A$ is noetherian, $\mathscr{X}$ proper and $\mathscr{F}$ coherent, then the modules $H^i(\mathscr{X}, \mathscr{F})$ are finitely generated by [13] (1.2), so in this case the cohomology modules of the complex $C^\bullet$ are finitely generated. Now it is enough to apply 2.3 and [12] §5 lemma 2. □

**Remark 2.5** Because of the existence of this complex, all the corollaries that are in [12] §5 hold for tame stacks. In other words, the cohomology of such stacks behaves like that of schemes under base change.

To illustrate how this complex can be used, let us give an other application in the same spirit.

**Corollary 2.6** *Let $S$ be a scheme and let $f : \mathscr{X} \to S$ be a quasi-compact and separated tame stack on $S$. Let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathscr{X}$ that is flat over $S$. If all the sheaves $R^i f_* \mathscr{F}$ ($i \geq 0$) are flat over $S$ then forming them commutes with any base change,* i.e. *if $\varphi : S' \to S$ is a base change morphism and if $\varphi' : \mathscr{X} \times_S S' \to \mathscr{X}$ and $f' : \mathscr{X} \times_S S' \to S'$ denote the induced morphisms, then the natural morphisms*

$$\varphi^* R^i f_* \mathscr{F} \longrightarrow R^i f'_*(\varphi'^* \mathscr{F})\,, \quad i \geq 0$$

*are isomorphisms.*

**Proof.** Since forming the higher direct images commutes with any flat base change ([5] A.3.4), the assertion is local on both $S$ and $S'$ so that we can assume that they are both affine, say $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$. Let $M^\bullet$ be the complex given by 2.4. In view of 2.4, it is enough to prove that forming $H^i(M^\bullet)$ commutes with base change. But for any $A$-module $N$ there is a Künneth spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_{-p}(H^q(M^\bullet), N) \Longrightarrow H^{p+q}(M^\bullet \otimes_A N).$$

Since all the $H^q(M^\bullet)$ are flat, this spectral sequence degenerates and yields isomorphisms

$$H^i(M^\bullet) \otimes_A N \longrightarrow H^i(M^\bullet \otimes_A N).$$

□

# 3 If the base ring has finite global dimension

First, we prove that in the general case, there is always an *infinite* complex of flat modules computing universally the cohomology of $\mathscr{F}$.

**Lemma 3.1** *Let $S$ be the spectrum of a ring $A$ and $\mathscr{X}$ a quasi-compact algebraic stack over $S$. Let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathscr{X}$. Then there is a complex of $A$-modules*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \dots \longrightarrow M^n \longrightarrow \dots$$

*and isomorphisms*

$$H^i(M^\bullet \otimes_A A') \longrightarrow H^i(\mathscr{X}', \mathscr{F}')$$

*functorial in the $A$-algebra $A'$ (where $\mathscr{X}' = \mathscr{X} \otimes_A A'$ and $\mathscr{F}' = \mathscr{F} \otimes_A A'$). If moreover $\mathscr{F}$ is flat over $S$, then we can assume that all the $M^i$'s are flat $A$-modules.*

**Proof.** Let $U^0 \twoheadrightarrow \mathscr{X}$ be a presentation of $\mathscr{X}$, such that $U^0$ is an affine scheme. Let $V^1 = U^0 \times_{\mathscr{X}} U^0$. Let $W^1 \twoheadrightarrow V^1$ be a presentation of the algebraic space $V^1$, the source of which $W^1$ is an affine scheme, and let $U^1 = U^0 \coprod W^1$. We then get a truncated hypercover[1]

$$U^1 \rightleftarrows U^0 \longrightarrow \mathscr{X}.$$

Let $U^\bullet$ be the 1-coskeleton[2] of this diagram. Clearly this is a hypercover (of type 1) for $\mathscr{X}$. Moreover, we can see easily in the construction of the coskeleton (cf. [7] (0.8)) that for every $n \geq 0$, the algebraic stack $U^{n+2}$ can be expressed in terms of fiber products obtained from the diagram:

$$U^{n+1} \; \vdots \; U^n.$$

We deduce that for every $n \geq 0$, $U^n$ is an affine scheme. We denote by $\mathscr{F}^i$ the pullback of $\mathscr{F}$ on $U^i$. To $U^\bullet$ we can associate for every $q$ the alternating chain complex

$$H^q(U^0, \mathscr{F}^0) \longrightarrow H^q(U^1, \mathscr{F}^1) \longrightarrow \dots \longrightarrow H^q(U^p, \mathscr{F}^p) \longrightarrow \dots$$

and we denote by $\check{H}^p(H^q(U^\bullet, \mathscr{F}^\bullet))$ the $p$-th cohomology group of this complex. Applying [1] V (7.4.0.3) there is a spectral sequence:

$$E_2^{p,q} = \check{H}^p(H^q(U^\bullet, \mathscr{F}^\bullet)) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Since $\mathscr{F}$ is quasi-coherent, we have $H^q(U^i, \mathscr{F}^i) = 0$ for all $q > 0$ and for all $i$, thus $E_2^{p,q} = 0$ for all $q > 0$. In other words, the spectral sequence degenerates and induces for every $p$ an isomorphism:

$$\check{H}^p(H^0(U^\bullet, \mathscr{F}^\bullet)) \xrightarrow{\sim} H^p(\mathscr{X}, \mathscr{F}).$$

[1] for the definitions of hypercovers, we refer to [1] V (7.3.1.2)

[2] the 1-coskeleton functor is by definition the right-adjoint of the 1-truncation functor, which to any simplicial object associates its first order truncation

Now, if $A'$ is an $A$-algebra and $S' = \operatorname{Spec} A'$, the simplicial object $U^\bullet \times_S S'$ obtained by base change is an hypercover for $\mathscr{X}'$, and its objects are affine schemes. Thus we also have an isomorphism:

$$\check{H}^p(H^0(U^\bullet \times_S S', \mathscr{F}'^\bullet)) \xrightarrow{\sim} H^p(\mathscr{X}', \mathscr{F}').$$

Taking $M^i = H^0(U^i, \mathscr{F}^i)$, we get our complex. Now if $\mathscr{F}$ is flat over $S$, the $M^i$'s are obviously flat over $A$. $\square$

The complex given by 3.1 can be useful in some circumstances. For instance, the following is an immediate corollary (see [3] and [5]).

**Corollary 3.2 ([5] A.3.4)** *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-compact morphism of $S$-algebraic stacks, and let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathscr{X}$. Let $u : \mathscr{Y}' \to \mathscr{Y}$ be a flat base change morphism. Let us take the following notations.*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{v} & \mathscr{X} \\ {\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle f} \\ \mathscr{Y}' & \xrightarrow{u} & \mathscr{Y} \end{array}$$

*Then for every $q \geq 0$ the natural morphism*

$$u^* R^q f_* \mathscr{F} \longrightarrow (R^q g_*)(v^* \mathscr{F})$$

*is an isomorphism.*

To get deeper results (*e.g.* semicontinuity), we need a finite complex. For that, we would like to truncate the infinite complex given by 3.1. So, for a fixed $n$, we want to consider a complex $M'^\bullet$ with $M'^i = M^i$ if $i < n$ and $M'^i = 0$ for $i > n$. Now we have at least two possibilities for the choice of $M'^n$: either we keep $M'^n = M^n$, but in this case the last cohomology module is changed, or we take $M'^n = \operatorname{Ker}(M^n \to M^{n+1})$, but in this case $M'^n$ is not necessarily flat. In both cases, an assumption is missing when we want to replace this finite complex by a finite complex of *finite* modules with the same cohomology (in the first case the last cohomology module is not of finite type and in the second case the last module of the complex might not be flat). In the sequel we will choose the second option, and the whole point is to check that, when the base ring has finite global dimension, we can still replace the complex by a complex of *finite* modules without affecting the first cohomology groups (even after base change!). This is what we do in the following variation of Mumford's lemmas 1 and 2.

**Lemma 3.3** *Let $n$ be an integer and $A$ a noetherian ring with global cohomological dimension $k \leq n+1$. Let $M^\bullet$ be a complex of $A$-modules*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \dots \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow 0$$

*with $M^i \neq 0$ only if $0 \leq i \leq n+1$. Assume that $M^i$ is flat for $0 \leq i \leq n$ and that all the cohomology modules $H^i(M^\bullet)$ are of finite type. Then there exists a complex $K^\bullet$ of $A$-modules of finite type and a morphism of complexes $K^\bullet \to M^\bullet$ such that*

*a)* $K^i \neq 0$ *only if* $0 \leq i \leq n+1$*;*
*b)* $K^i$ *is free for* $1 \leq i \leq n+1$*;*
*c)* $K^0$ *is flat;*
*d)* $K^\bullet \longrightarrow M^\bullet$ *is a quasi-isomorphism;*
*e) for any A-algebra B and for every i,* $0 \leq i \leq n-k-1$*, the morphism*

$$H^i(K^\bullet \otimes_A B) \longrightarrow H^i(M^\bullet \otimes_A B)$$

*is an isomorphism.*

**Proof.** Using Mumford's lemma 1 ([12] §5), there is a complex $K^\bullet$ of finite type $A$-modules and a morphism $f : K^\bullet \longrightarrow M^\bullet$ satisfying the properties a), b) and d). The "mapping cylinder" $L^\bullet$ associated with this morphism of complexes is defined by

$$\begin{aligned} L^p &= K^p \oplus M^{p-1} \\ \delta_L(x,y) &= (\delta_K x, f(x) - \delta_M y) \quad \forall x \in K^p, y \in M^{p-1} \end{aligned}$$

There is a short exact sequence of complexes

$$0 \longrightarrow M'^\bullet \longrightarrow L^\bullet \longrightarrow K^\bullet \longrightarrow 0$$

where $M'^\bullet$ is defined by $M'^p = M^{p-1}$ and $\delta_{M'} = -\delta_M$. This short exact sequence induces a long exact sequence of cohomology

$$\begin{aligned} 0 \longrightarrow H^0(L^\bullet) \longrightarrow H^0(K^\bullet) \xrightarrow{\partial} H^0(M^\bullet) \longrightarrow \\ \longrightarrow H^1(L^\bullet) \longrightarrow H^1(K^\bullet) \xrightarrow{\partial} H^1(M^\bullet) \longrightarrow \\ \dots \end{aligned}$$

and the cobordism $\partial : H(K^\bullet) \longrightarrow H(M^\bullet)$ coincides with the morphism induced by $f : K^\bullet \longrightarrow M^\bullet$. Since $f$ is a quasi-isomorphism, this proves that all the $H^i(L^\bullet)$ vanish, in other words that the complex $L^\bullet$ is exact. But $L^\bullet$ is:

$$0 \longrightarrow K^0 \longrightarrow K^1 \oplus M^0 \longrightarrow \dots \longrightarrow K^{n+1} \oplus M^n \longrightarrow M^{n+1} \longrightarrow 0.$$

All the terms in the middle are flat, thus, splitting this exact sequence in short exact sequences, we get an isomorphism

$$\mathrm{Tor}\,^A_{n+2}(M^{n+1}, N) \xrightarrow{\sim} \mathrm{Tor}\,^A_1(K^0, N)$$

for any $A$-module $N$. But the global cohomological dimension of $A$ is $k \leq n+1$, so $\mathrm{Tor}\,^A_{n+2}(M^{n+1}, N) = 0$ and $K^0$ is therefore flat.

It remains to prove e). The complex $L^\bullet$ is a flat resolution of $M^{n+1}$, so for any $A$-module $N$ and for $i \leq n$, the module $H^i(L^\bullet \otimes N)$ is isomorphic to $\mathrm{Tor}\,^A_{n+1-i}(M^{n+1}, N)$. In particular, for $i \leq n-k$ (*i.e.* $n+1-i \geq k+1$) we have $H^i(L^\bullet \otimes N) = 0$. Using the "mapping cylinder" and the cohomology long exact sequence associated to

$$0 \longrightarrow M'^\bullet \otimes N \longrightarrow L^\bullet \otimes N \longrightarrow K^\bullet \otimes N \longrightarrow 0$$

we deduce that for any $A$-module $N$ and for every $0 \leq i \leq n-k-1$, the natural morphism
$$H^i(K^\bullet \otimes_A N) \longrightarrow H^i(M^\bullet \otimes_A N)$$
is an isomorphism. □

**Corollary 3.4** *Let $A$ be a noetherian ring with finite global cohomological dimension, $S$ its spectrum, $\mathscr{X}$ a proper algebraic stack over $S$, and $\mathscr{F}$ a coherent $\mathscr{O}_{\mathscr{X}}$-module, flat over $S$. Let $n$ be a natural integer. Then there is a finite complex of flat and finite type $A$-modules*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow M^{n+1},$$

*and functorial isomorphisms*

$$H^i(M^\bullet \otimes_A B) \xrightarrow{\sim} H^i(\mathscr{X} \otimes_A B, \mathscr{F} \otimes_A B)\,, \quad 0 \leq i \leq n\,.$$

**Proof.** Let $M^\bullet$ be the infinite complex given by the first lemma. Since $\mathscr{X}$ is proper and $\mathscr{F}$ is coherent, the $A$-modules $H^i(M^\bullet)$ are of finite type. Let $k$ be the global dimension of $A$ and let $N = n+k+1$. Let $M'^\bullet$ be the complex given by

$$\begin{aligned} M'^p &= M^p \text{ if } p \leq N, \\ M'^{N+1} &= \mathrm{Ker}\,(M^{N+1} \to M^{N+2}), \\ M'^p &= 0 \text{ if } p \geq N+2. \end{aligned}$$

Apply the previous lemma to $M'^\bullet$ and let $K^\bullet$ be the resulting complex. Then the $(n+1)$-th truncation of $K^\bullet$ is suitable. □

**Remark 3.5** As in the case of tame stacks, the existence of this complex implies that all the corollaries from [12] §5 hold for a stack over a noetherian ring with finite global dimension (*e.g.* a regular ring with finite dimension). There is also an analogue of 2.6 (see the appendix of [4]).

# 4 The semicontinuity theorem

In this section we will prove the semicontinuity theorem for an arbitrary base ring and for non necessarily tame algebraic stacks. First, let us recall the following fact:

**Lemma 4.1** *Let $S$ be a noetherian scheme and $f : S \to \mathbb{N}$ a function on $S$. Then $f$ is upper semicontinuous if and only if the two following conditions are satisfied:*

*a) For any discrete valuation ring $A$ and for any morphism $g : \mathrm{Spec}\,A \to S$, we have*
$$f(g(\eta)) \leq f(g(\xi))$$
*where $\eta$ (resp. $\xi$) denotes the generic (resp. special) point of $\mathrm{Spec}\,A$.*

*b) For any noetherian domain $A$ and for any morphism $g : \mathrm{Spec}\,A \to S$, there is a nonempty open subset of $\mathrm{Spec}\,A$ on which the function $f \circ g$ is constant.*

**Proof.** This is an easy consequence of EGA $0_{\mathrm{III}}$ ([8]) 9.3.3 and 9.3.4. □

**Theorem 4.2** *Let $S$ be a scheme, $\mathscr{X}$ a proper algebraic stack of finite presentation over $S$, and $\mathscr{F}$ a coherent $\mathscr{O}_{\mathscr{X}}$-module that is flat over $S$. Then for any integer $i \geq 0$, the function*

$$d_i \colon \begin{cases} S \longrightarrow \mathbb{N} \\ s \longmapsto \dim_{\kappa(s)} H^i(\mathscr{X}_s, \mathscr{F}_s) \end{cases}$$

*is upper semicontinuous over $S$.*

**Proof.** Obviously we can assume that $S$ is affine, say $S = \operatorname{Spec} A$. By standard limit arguments, we can also assume that $A$ is of finite type over $\mathbb{Z}$. Owing to the previous lemma, it is enough to prove that the theorem holds

a) when $A$ is a discrete valuation ring;

b) over a nonempty open subset of $\operatorname{Spec} A$, when $A$ is a domain.

But if $A$ is an integral, finite type $\mathbb{Z}$-algebra, there is a nonempty open subset of $\operatorname{Spec} A$ which is regular. Thus in both cases it is enough to prove the theorem when $A$ is a regular, integral $\mathbb{Z}$-algebra of finite type. Such a ring has finite global cohomological dimension (see *e.g.* [9] chap. 0 17.3.1). Hence, using the lemma 3.4, there is a finite complex of flat and finite type $A$-modules computing universally the cohomology modules of $\mathscr{F}$ over $\mathscr{X}$ at least up to the $i$-th rank. Now, to reach the conclusion that $d_i$ is upper semicontinuous, we can proceed exactly as in [12] §5 (see the corollary p. 50). □

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