Hopf (co)monads, tensor functors and exact sequences of tensor categories

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based on joint works
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Conference of the ANR project GALOISINT Quantum Groups : Galois and integration techniques
Motivation: Tannaka theory

Over $k$ field: $H$ Hopf algebra $\rightarrow$ a tensor category $C = \text{comod}$ $H +$ a fiber functor $C \rightarrow \text{vect}$

Reconstruction: given $C$ tensor category + $\omega$: $C \rightarrow \text{vect}$ fiber functor $\Rightarrow H = \text{Coend} (\omega) = \int_{X \in C} \omega(X) \otimes \omega(X)^*$

Hopf algebra with commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\omega} & C \\
\downarrow & \cong & \downarrow \\
\text{vect} & \otimes & \text{comod} H \\
\uparrow & \uparrow & \\
C & \xrightarrow{\omega} & \text{vect} \\
\end{array}
\]

A fiber functor is encoded by a Hopf algebra (in $\text{Vect}$)
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with commutative diagram:

$$C \xrightarrow{\omega} \text{vect} \quad \text{comod}H$$
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**Introduction**

Affine group scheme $G$ over a commutative Hopf algebra $H = \text{O}(G)$. Then $C = \text{comod}_H = \text{rep}_G$ and the fiber functor $C \to \text{vect}$ are both symmetric.

**Converse:** $C$ symmetric tensor category $\Rightarrow$ symmetric fiber functor $\Rightarrow H = \text{Coend}(\omega)$ commutative Hopf algebra, $G = \text{Spec} H$ affine group scheme and $C \cong \text{rep}_G$ as symmetric tensor categories.

Then there exists a commutative algebra $A$ in $C$ (or its Ind-completion) satisfying $\forall X$ in $C$, $A \otimes X \sim A$ as left $A$-modules, $\text{Hom}(1, A) = k$ and we have $\omega(X) = \text{Hom}(1, A \otimes X)$.

The proof of Deligne's internal characterization of tannaka categories consists in constructing such a trivializing algebra. Can we give similar encodings for arbitrary tensor functors?
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$G$ affine group scheme over $\mathbb{k} = \text{commutative Hopf algebra } H = O(G)$. Then $C = \text{comod } H = \text{rep } G$ and the fiber functor $C \to \text{vect}$ are both symmetric.

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1. \(\forall X\) in \(C\), \(A \otimes X \xrightarrow{\sim} A^n\) as left \(A\)-modules
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Can we give similar encodings for arbitrary tensor functors?
Let $\mathbb{k}$ be a field.

**Definition**

In this talk a *tensor category* is a $\mathbb{k}$-linear abelian category with a structure of rigid category (=monoidal with duals) such that:

- $C$ is locally finite (Hom’s are finite dim’l and objects have finite length)
- $\otimes$ is $\mathbb{k}$-bilinear and $\text{End}(1) = \mathbb{k}$

$C$ is *finite* if $C \cong_{R} mod$ for some finite dimensional $\mathbb{k}$-algebra $R$.

**Definition**

A *tensor functor* $F : C \to D$ is a $\mathbb{k}$-linear exact strong monoidal functor between tensor categories.

A tensor functor $F$ is faithful. It has a right adjoint iff it has a left adjoint; in that case we say that $F$ is *finite*.
Introduction

Examples

1. \textit{vect} is the initial tensor category.
Introduction

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Tannaka duality asserts that we have an equivalence of categories

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**Introduction**

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Let $F: C \to \mathcal{D}$ be a tensor functor.
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**Question 1**

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Yes, if $F$ is dominant.
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Yes, if $F$ is *dominant*.
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**Question 2**

Can one encode $F$ by an algebraic data in $C$ (or $\text{Ind}C$)?

Yes, if $F$ is *dominant*.

This data is a commutative algebra in the center of $C$ (or $\text{Ind}C$).
Outline of the talk

1. Introduction
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2. Hopf Monads - a sketchy survey
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4. Exact sequences of tensor categories
1 Introduction

2 Hopf Monads - a sketchy survey
   • Definition
   • Examples
   • Some aspects of the general theory

3 Hopf (co)-monads applied to tensor functors

4 Exact sequences of tensor categories
Monads

Let $C$ be a category. The category $\text{EndoFun}(C)$ is strict monoidal ($\otimes=\text{composition}, \mathbb{1} = 1_C$)
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A monad on $C$ is an algebra (=monoid) in $\text{EndoFun}(C):$

$$T: C \to C, \quad \mu: T^2 \to T \ (\text{product}), \quad \eta: 1_C \to T \ (\text{unit})$$
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A $T$-module is a pair $(M, r)$, $M \in \text{Ob}(C)$, $r : T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \text{id}_M.$$
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Example

A algebra in a monoidal category $C$

$\Rightarrow T =? \otimes A : X \mapsto X \otimes A \text{ is a monad on } C \text{ and } C^T = \text{Mod- } A$
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$T' = A \otimes ?$ is a monad on $C$ and $C^{T'} = A \text{-} \text{Mod}$
Monads and adjunctions

A monad $T$ on a category $C \rightsquigarrow$ an adjunction $F^T \leftrightarrow U^T$

where $U^T(M, r) = M$ and $F^T(X) = (T(X), \mu_X)$.
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An adjunction $\xymatrix{F \ar@<0.5ex>[r]^(0.4){\theta} & U \\ C \ar@<0.5ex>[u]_{\phi} \ar@<0.5ex>[l]^-{\epsilon}} \rightsquigarrow$ a monad $T = (UF, \mu := U(\epsilon_F), \eta)$ on $C$

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The adjunction $(F, U)$ is monadic if $K$ equivalence.
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An adjunction $\begin{array}{c} D \\ \downarrow \\ C \end{array} \begin{array}{c} \overset{F}{\leftarrow} \\ \downarrow \\ \overset{U}{\rightarrow} \end{array}$ $\begin{array}{c} \overset{D}{\downarrow} \\ \downarrow \\ \overset{C}{\rightarrow} \end{array}$ $\begin{array}{c} \overset{K}{\downarrow} \\ \downarrow \\ \overset{K}{\rightarrow} \end{array}$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $C$ where $\eta : 1_C \to UF$ and $\varepsilon : FU \to 1_D$ are the adjunction morphisms

$K : D \mapsto (U(D), U(\varepsilon_D))$ (the comparison functor)

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Bimonads [Moerdijk]

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- \(T\) is comonoidal endofunctor
  (with \(\Delta_{X,Y} : T(X \otimes Y) \rightarrow TX \otimes TY\) and \(\varepsilon : T1 \rightarrow 1\))
- \(\mu\) and \(\eta\) are comonoidal natural transformations.
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\[
T^2(X \otimes Y) \xrightarrow{T\Delta_{X,Y}} T(TX \otimes TY) \xrightarrow{\Delta_{TX,TY}} T^2X \otimes T^2Y
\]

\[
\mu_{X \otimes Y} \downarrow \quad \Delta_{X,Y} \quad \mu_{X \otimes \mu Y}
\]

\[
T(X \otimes Y) \quad \Delta_{X,Y} \quad TX \otimes TY
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Axioms similar to those of a bialgebra except the compatibility between \(\mu\) and \(\Delta\):

\[
\begin{align*}
T^2(X \otimes Y) & \xrightarrow{T\Delta_{X,Y}} T(TX \otimes TY) \xrightarrow{\Delta_{TX,TY}} T^2X \otimes T^2Y \\
T(X \otimes Y) & \xrightarrow{\Delta_{X,Y}} TX \otimes TY
\end{align*}
\]

No braiding involved!
Hopf monads

For a bimonad $T$ define the (left and right) fusion morphisms

- $H^l(X, Y) = (\text{id}_{TX} \otimes \mu_Y)\Delta_{X, TY} : T(X \otimes TY) \to TX \otimes TY$,
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For $T$ bimonad on $C$ rigid, equivalence:

(i) $C^T$ is rigid;
(ii) $T$ is a Hopf monad;
(iii) (older definition) $T$ admits a left and a right (unary) antipode $s^l_X : T(\triangleright TX) \rightarrow \triangleright X$ and $s^r : T(TX^\triangleright) \rightarrow X^\triangleright$. 

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Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a \textit{Hopf comonad}. A Hopf comonad is a monoidal comonad such that the cofusison operators are invertible.
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All results about Hopf monads translate into results about Hopf comonads. In particular, if $T$ is a Hopf comonad on $C$,

1. the category $C_T$ of comodules over $T$ is monoidal,

2. we have a Hopf monoidal adjunction: $\mathcal{D} \xleftarrow{U_T} C \xrightarrow{F_T} \mathcal{D}$

where $U_T$ is the forgetful functor and $F_T$ is its right adjoint, the cofree comodule functor.
Hopf monads from adjunctions

Let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and $U$ is strong monoidal)
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There are canonical morphisms:

- $F(c \otimes Ud) \rightarrowFc \otimes d$
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If the adjunction is Hopf, $T$ is a Hopf monad. Such is the case if either of the following hold:
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A bimonad is Hopf iff its adjunction is Hopf!
Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories
Hopf monads from Hopf algebras

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$H$ Hopf algebra in $\mathcal{B}$ braided category with braiding $\tau$
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The monad structure of $T$ comes from the algebra structure of $H$
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The comonoidal structure of $T$ is

$$\Delta_{X,Y} = (H \otimes \tau_{H,X} \otimes Y)(\Delta \otimes X \otimes Y): H \otimes X \otimes Y \to H \otimes X \otimes H \otimes Y$$

$\varepsilon = \text{counit of } H: H \to 1$
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Can we extend this construction to non-braided categories?
The Joyal-Street Center

C monoidal category → Z(C) braided category

Objects of Z(C) = half-braidings of C: pair (X,σ) with σ: X ⊗ Y → Y ⊗ X natural in Y s. t. σ Y ⊗ Z = (id Y ⊗ σ Z)(σ Y ⊗ id Z)

Morphisms f: (X,σ) → (X',σ') in Z(C) are morphisms f: X → X' in C s. t. σ'(f ⊗ id) = (id ⊗ f)σ

Braiding: c(X,σ), (X',σ') = σ'X

Hopf Monads - a sketchy survey

Examples
The Joyal-Street Center

\[ C \text{ monoidal category} \xrightarrow{\text{Joyal-Street Center}} \mathcal{Z}(C) \text{ braided category} \]
Objects of $\mathcal{Z}(C) =$ half-braidings of $C$:

pair $(X, \sigma)$ with $\sigma_Y : X \otimes Y \rightarrow Y \otimes X$ natural in $Y$ s. t.

$$\sigma_{Y \otimes Z} = (\text{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \text{id}_Z)$$
The Joyal-Street Center

Objects of $\mathcal{Z}(C) = \text{half-braidings}$ of $C$:

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Morphisms $f : (X, \sigma) \to (X', \sigma')$ in $\mathcal{Z}(C)$ are morphisms $f : X \to X'$ in $C$ s. t. $\sigma'(f \otimes \text{id}) = (\text{id} \otimes f)\sigma$
The Joyal-Street Center

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- **Objects of** \( \mathcal{Z}(C) = \text{half-braidings of } C \):
  
  pair \((X, \sigma)\) with \(\sigma_Y : X \otimes Y \overset{\sim}{\to} Y \otimes X\) natural in \(Y\) s. t.
  
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  \sigma_{Y \otimes Z} = (\text{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \text{id}_Z)
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- \(\mathcal{Z}(C)\):
  
  \[ (X, \sigma) \otimes_{\mathcal{Z}(C)} (X', \sigma') = (X \otimes X', (\sigma_Y \otimes \text{id})(\text{id} \otimes \sigma')) \]
Objects of $\mathcal{Z}(C)$ = **half-braidings** of $C$:

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Representable Hopf monads

A monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is equipped with a Hopf algebra $(H, \Delta, \epsilon)$ in $\mathcal{Z}(\mathcal{C})$ (which is braided) $\Rightarrow$ a Hopf monad $T = H \otimes \sigma$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$.

The comonoidal structure of $T$ is $\Delta X, Y = (H \otimes \sigma X \otimes Y)(\Delta \otimes X \otimes Y)$.

$\epsilon$ is the counit of $H$.

Moreover, $T$ is equipped with a Hopf monad morphism $e = (\epsilon \otimes ?) : T \rightarrow \text{id}_\mathcal{C}$.

Theorem (BVL)

This construction defines an equivalence of categories $\{\text{Hopf algebras in } \mathcal{Z}(\mathcal{C})\} \equiv \{\text{Hopf monads on } \mathcal{C}\} / \text{id}_\mathcal{C}$.

If $H$ is a Hopf algebra and $T = H \otimes \mathbf{1}$, we recover Sweedler's Theorem.
Representable Hopf monads

$C$ monoidal category, $(H, \sigma)$ a Hopf algebra in $\mathcal{Z}(C)$ (which is braided)
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Representable Hopf monads

Let $C$ be a monoidal category, $(H, \sigma)$ a Hopf algebra in $\mathcal{Z}(C)$ (which is braided). This corresponds to a Hopf monad $T = H \otimes \sigma$ on $C$, defined by $X \mapsto H \otimes X$. The comonoidal structure of $T$ is

$$\Delta_{X,Y} = (H \otimes \sigma_X \otimes Y)(\Delta \otimes X \otimes Y)$$

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Let $C$ be a monoidal category, $(H, \sigma)$ a Hopf algebra in $\mathcal{Z}(C)$ (which is braided), and $T = H \otimes \sigma ?$ a Hopf monad on $C$, defined by $X \mapsto H \otimes X$. The comonoidal structure of $T$ is

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Monadicity of the center

Let $C$ be a rigid category, with center $Z(C)$. 
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Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation $\forall Y \otimes X \otimes Y \to X$
Monadicity of the center

Let $C$ be a rigid category, with center $\mathcal{Z}(C)$. Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation $\mathcal{Y} \otimes X \otimes Y \to X$

We say that $C$ is *centralizable* if $\mathcal{Z}(X) = \int_{Y \in C} \mathcal{Y} \otimes X \otimes Y$ exists for all $X \in C$
Monadicity of the center

Let $C$ be a rigid category, with center $Z(C)$. Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \rightarrow Y \otimes X$ as a dinatural transformation $\bigvee Y \otimes X \otimes Y \rightarrow X$.

We say that $C$ is centralizable if $Z(X) = \int_{Y \in C} \bigvee Y \otimes X \otimes Y$ exists for all $X \in C$ (note that $Z(1)$ is the coend of $C$). Then a half braiding $\sigma$ corresponds with $\tilde{\sigma} : Z(X) \rightarrow X$. 

Remark: In general the Hopf monad $Z$ is not augmented, i.e. not representable by a Hopf algebra: e.g. $C = \{G\text{-graded vector spaces}\}$, for $G$ non abelian finite group.
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**Theorem (BV)**

If $C$ is centralizable, then $Z : X \mapsto Z(X)$ is a quasitriangular Hopf monad on $C$ and we have a braided isomorphism of categories

$$\mathcal{Z}(C) \to C^Z$$

$$(X, \sigma) \mapsto (X, \tilde{\sigma})$$
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The centralizer of a Hopf monad

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A Hopf monad $T : C \to C$ is \textit{centralizable} if

$$Z_T(X) = \int_{Y \in C} T(Y) \otimes X \otimes Y \text{ exists for all } X \in \text{Ob}(X)$$

\textbf{Proposition (BV)}

If $T$ is a centralizable Hopf monad, $Z_T : X \mapsto Z_T(X)$ is a Hopf monad called the \textit{centralizer} of $T$. 
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If $T$ is a centralizable Hopf monad, $Z_T: X \mapsto Z_T(X)$ is a Hopf monad called the **centralizer** of $T$.

In particular the monad $Z$ of the previous slide is the centralizer of $1_C$. 
The centralizer of a Hopf monad

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In particular the monad $Z$ of the previous slide is the centralizer of $1_C$.
In a sense the centralizer plays the role of the dual of the Hopf monad $T$. 
Hopf monads as ‘quantum groupoids’

Let $R$ be a unitary ring $\Rightarrow$ a monoidal category $(\mathcal{R}\text{Mod}_R, \otimes_R, R_R)$. 
Hopf monads as ‘quantum groupoids’

Let $R$ be a unitary ring $\rightsquigarrow$ a monoidal category $(R\text{Mod}_R, \otimes_R, R_R)$.

Facts

- linear bimonads on $R\text{Mod}_R$ with a right adjoint are bialgebroids in the sense of Takeuchi [Szlacháni]
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Hopf algebroids are non-commutative avatars of groupoids.
Hopf Monads - a sketchy survey

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Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf monad (much easier to manipulate).
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Theorem (BVL)

A finite tensor category $C$ over a field $k$ is tensor equivalent to the category of $A$-modules for some bialgebroid $A$. 
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Given a $k$- equivalence $C \overset{k}{\simeq}_R mod$ for some finite dimensional $k$- algebra $R$, one constructs a canonical Hopf algebroid $A$ over $R$. 
Tannaka dictionary
Outlook of General Theory of Hopf monads

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- Hopf modules and Sweedler decomposition theorem
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- Cross-products
- Bosonization for Hopf monads
- Applications to construction and comparison of quantum invariants (non-braided setting)
Hopf monad on $C \xrightarrow{T} T\mathbb{1}$ is a coalgebra in $C$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit $\varepsilon$)
Hopf modules and Sweedler’s Theorem for Hopf Monads

$T$ Hopf monad on $C \xrightarrow{\sim} T1$ is a coalgebra in $C$ (coproduct $\Delta_{1,1}$, counit $\varepsilon$) \newline lifts to a coalgebra $\hat{C} = F^T(1)$ in $C^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \rightarrow ? \otimes \hat{C}.$$
Hopf modules and Sweedler’s Theorem for Hopf Monads

\( T \) Hopf monad on \( C \xrightarrow{\sim} T \mathbb{1} \) is a coalgebra in \( C \) (coproduct \( \Delta_{\mathbb{1},\mathbb{1}} \), counit \( \varepsilon \)) \( \xrightarrow{\sim} \) lifts to a coalgebra \( \hat{C} = F^T(1) \) in \( C^T \). Moreover we have a natural isomorphism

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\]

**Proposition (BVL)**

\( \sigma \) is a half-braiding
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$T$ Hopf monad on $C \xrightarrow{\sim} T \mathbf{1}$ is a coalgebra in $C$ (coproduct $\Delta_{\mathbf{1}, \mathbf{1}}$, counit $\varepsilon$) $\xrightarrow{\sim}$ lifts to a coalgebra $\hat{C} = F^T(\mathbf{1})$ in $C^T$. Moreover we have a natural isomorphism

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**Proposition (BVL)**

$\sigma$ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(C^T)$ called the *induced central coalgebra* of $T$. 
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$T$ Hopf monad on $C \xrightarrow{\sim} T\mathbb{1}$ is a coalgebra in $C$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit $\varepsilon$) $\xrightarrow{\sim}$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $C^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes \hat{C} \rightarrow \hat{C} \otimes \hat{C}.$$

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$\sigma$ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(C^T)$ called the induced central coalgebra of $T$.

A (right) $T$-Hopf module is a (right) $\hat{C}$-comodule in $C^T$. 

Hopf modules and Sweedler’s Theorem for Hopf Monads

A (right) $T$-Hopf module is a (right) $\hat{C}$-comodule in $C^T$, i.e. a data $(M, r, \partial)$ with $(M, r)$ a $T$-module, $(M, \partial)$ a $T\mathbb{1}$-comodule + $T$-linearity of $\partial$. 

Proposition (BVL)

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Under suitable exactness conditions ($T$ is conservative, $C$ has coequalizers and $T$ preserves them):
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**Theorem (BVL)**

The assignment \(X \mapsto (TX, \mu_X, \Delta_X, 1)\) is an equivalence of categories

\[
Q : C \xrightarrow{\simeq} \{\{T\text{-Hopf modules}\}\}
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with quasi-inverse the functor *coinvariant part*.
Under suitable exactness conditions (\(T\) is conservative, \(C\) has coequalizers and \(T\) preserves them):

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with quasi-inverse the functor *coinvariant part*. Moreover if \(C\) has equalizers and \(T\) preserves them, \(Q\) is a monoidal equivalence, the category of Hopf modules (i.e. \(\hat{C}\)-comodules) being endowed with the cotensor product over \(\hat{C}\).
Proof of Sweedler’s theorem for Hopf monads

An adjunction \( F \xleftarrow{\eta} U \xrightarrow{\epsilon} C \) \( \sim \) a comonad \( \hat{T} = (FU, F(\eta_U), \epsilon) \) on \( D \).
Proof of Sweedler’s theorem for Hopf monads

An adjunction $F \overset{\eta}{\Rightarrow} U \leadsto \text{a comonad } \hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Denoting $\mathcal{D}_{\hat{T}}$ the category of $\hat{T}$-comodules we have a cocomparison functor $\hat{K}$:

\[\mathcal{D}_{\hat{T}} \rightarrow \{\text{right } \hat{T}-\text{Hopf modules}\}\]
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The adjunction $(F, U)$ is comonadic if $\hat{K}$ equivalence.
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\[ C \rightarrow \rightarrow D \]

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Denoting \( D_{\hat{T}} \) the category of \( \hat{T} \)-comodules we have a cocomparison functor \( \hat{K} \):

\[ D \rightarrow \rightarrow D_{\hat{T}} \]
\[ C \rightarrow \rightarrow \hat{K} \]

The adjunction \( (F, U) \) is \textit{comonadic} if \( \hat{K} \) equivalence.

If \( T \) is a monad on \( C \), its adjunction is comonadic under \textit{suitable exactness assumptions} (descent), i. e. \( \hat{K} : C \rightarrow (C^T)_{\hat{T}} \) is an equivalence.
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An adjunction $F \xRightarrow{\sim} U \xrightarrow{\sim} \text{comonad } \hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

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The adjunction $(F, U)$ is comonadic if $\hat{K}$ equivalence.

If $T$ is a monad on $C$, its adjunction is comonadic under suitable exactness assumptions (descent), i.e. $\hat{K}: C \xrightarrow{\sim} (C^T)_{\hat{T}}$ is an equivalence.

For $T$ Hopf monad, we have an isomorphism of comonads on $C^T$

$$\phi : \hat{T} \xrightarrow{\sim} ? \otimes \hat{C}$$

defined by $\phi(M, r) = (r \otimes \text{id}_{T(1)}) T_{M, 1}: TM \to M \otimes T1$.

Hence $C^T_{\hat{T}} \xrightarrow{\sim} \{\text{right } T\text{-Hopf modules}\}$
Introduction

Hopf Monads - a sketchy survey

Hopf (co)-monads applied to tensor functors

Exact sequences of tensor categories
We now consider tensor categories over a field $k$. 

Let $F: C \to D$ be a tensor functor. There exists a $k$-linear left exact comonad on $\text{Ind} C$ such that we have a commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{F} & C_T \\
\downarrow & & \downarrow \\
\text{ind vect} & \xrightarrow{\otimes} & \text{vect} \\
\end{array}
\]

where $C_T$ is the category of $T$-comodule whose underlying object is in $C$. 

Note that these are no longer rigid.
We now consider tensor categories over a field $k$.

If $C$ is a tensor category, its Ind-completion $\text{Ind}C$ is a monoidal abelian category containing $C$ as a full subcategory and whose objects are formal filtering colimits of objects of $C$. 
We now consider tensor categories over a field $\mathbb{k}$.

If $C$ is a tensor category, its Ind-completion $\text{Ind}C$ is a monoidal abelian category containing $C$ as a full subcategory and whose objects are formal filtering colimits of objects of $C$. For instance $\text{Ind} \text{vect} = \text{Vect}$, and $\text{Ind} \text{comod}H = \text{Comod}H$. 
We now consider tensor categories over a field $k$.

If $C$ is a tensor category, its Ind-completion $\text{Ind} C$ is a monoidal abelian category containing $C$ as a full subcategory and whose objects are formal filtering colimits of objects of $C$. For instance $\text{Ind \ vect} = \text{Vect}$, and $\text{Ind \ comod} H = \text{Comod} H$. Note that these are no longer rigid.
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**Theorem**

Let \( F : C \to \mathcal{D} \) be a tensor functor. There exists a \( \mathbb{k} \)-linear left exact comonad on \( \text{Ind} C \) such that we have a commutative diagram:
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**Theorem**

Let $F : C \to \mathcal{D}$ be a tensor functor. There exists a $k$-linear left exact comonad on $\text{Ind}C$ such that we have a commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{F} & \text{vect} \\
\downarrow{\cong} & & \downarrow{\otimes} \\
\mathcal{D}_T & \xleftarrow{\cong} & \mathcal{D}
\end{array}
\]

where $C_T$ is the category of $T$-comodule whose underlying object is in $C$. 
The functor $F : C \rightarrow \mathcal{D}$ extends to a linear faithful exact functor $\text{Ind}F : \text{Ind}C \rightarrow \text{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
The functor $F : C \to D$ extends to a linear faithful exact functor $\text{Ind}F : \text{Ind}C \to \text{Ind}D$ which preserves colimits and is strong monoidal. $\text{Ind}F$ has a right adjoint, denoted by $R$. 

**Example**

If $D = \text{vect}$, a linear Hopf comonad on $\text{Vect}$ is of the form $H \otimes ?$ for some Hopf algebra $H$, so we recover the classical tannakian result.
The functor $F : C \to D$ extends to a linear faithful exact functor $\text{Ind}F : \text{Ind}C \to \text{Ind}D$ which preserves colimits and is strong monoidal. $\text{Ind}F$ has a right adjoint, denoted by $R$. It is also a monoidal adjunction, which is Hopf.
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Example

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Proof

The functor $F : C \to D$ extends to a linear faithful exact functor $\text{Ind}F : \text{Ind}C \to \text{Ind}D$ which preserves colimits and is strong monoidal. $\text{Ind}F$ has a right adjoint, denoted by $R$. It is also a monoidal adjunction, which is Hopf. Its comonad $T = \text{Ind}FR$ is a Hopf comonad on $\text{Ind}C$. $\text{Ind}F$ being faithful exact, the adjunction $(\text{Ind}F, R)$ is comonadic by Beck, hence the theorem.

Example

If $D = \text{vect}$, a linear Hopf comonad on $\text{Vect}$ is of the form $H \otimes ?$ for some Hopf algebra $H$, so we recover the classical tannakian result.
Let $F : C \to \mathcal{D}$ be a tensor functor. We say that $F$ is *dominant* if the right adjoint $R$ of $\text{Ind} F$ is faithful exact. Then applying the classification theorem for Hopf modules in its dual form we obtain:

**Theorem**

*If $F$ is dominant, there exists a commutative algebra $(A, \sigma)$ in $\mathcal{Z}(\text{Ind} C)$ - the induced central algebra of $T$ - such that we have a commutative diagram*

$$
\begin{array}{c}
C & \xrightarrow{F_A} & A - \text{ mod } C \\
\downarrow{F} & & \downarrow{\simeq} \\
\mathcal{D} & \simeq \otimes & \\
\end{array}
$$

*where $A - \text{ mod}$ is the category of ‘finite type’ $A$-modules in $\text{Ind} C$ (=quotients of $A \otimes X$, $X \in C$), with tensor product $\otimes_{A, \sigma}$, and $F_A$ is the tensor functor $X \mapsto A \otimes X$.*

If $\mathcal{D} = \text{vect}_k$ and $C, F$ are symmetric, then $A$ is Deligne’s trivializing algebra.
An exact sequence of Hopf algebras in the sense of Schneider is a sequence

\[ K \overset{i}{\rightarrow} H \overset{p}{\rightarrow} H' \]

of Hopf algebras such that

1. \( p^{-1}(0) \) is a normal Hopf ideal of \( H \);
2. \( H \) is right faithfully coflat over \( H' \);
3. \( i \) is a categorical kernel of \( p \).
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We extend this notion to tensor categories.

Let $F : C \to D$ be a tensor functor. We denote by $\mathcal{k}_F$ the full tensor subcategory of $C$

$$\mathcal{k}_F = \{ X \in C \mid F(X) \text{is trivial} \}$$
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Let $F : C \to D$ be a tensor functor. We denote by $\mathcal{K}_F$ the full tensor subcategory of $C$

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Note that $F$ induces a fiber functor $\mathcal{K}_F \to \text{vect}$, $X \mapsto \text{Hom}(\mathbb{1}, F(X)$.

We say that $F$ is normal if the right adjoint $R$ of $\text{Ind}F$ satisfies $R(\mathbb{1}) \in \text{Ind}(\mathcal{K}_F)$.

This means that the subcategory $<\mathbb{1}>$ of $\text{Ind}C$ generated by $\mathbb{1}$ is stable under the Hopf comonad $T = UR$ which encodes $F$. 
An exact sequence of tensor categories is a sequence

\[ C' \xrightarrow{f} C \xrightarrow{F} C'' \]

of tensor categories such that:

1. \( F \) is normal and dominant;
2. \( f \) induces a tensor equivalence \( C' \to \mathcal{K}_F \).
An exact sequence of tensor categories is a sequence

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of tensor categories such that:
1. \( F \) is normal and dominant;
2. \( f \) induces a tensor equivalence \( C' \rightarrow \mathcal{K}_F \).

If \( H' \rightarrow H \rightarrow H'' \) is an exact sequence of Hopf algebras, then

\[ \text{comod}H' \rightarrow \text{comod}H \rightarrow \text{comod}H'' \]

is an exact sequence of tensor categories, and, if \( H \) is finite dimensional,

\[ \text{mod } H'' \rightarrow \text{mod } H \rightarrow \text{mod } H' \]

is also an exact sequence of tensor categories.
Exact sequences of tensor categories are classified by certain Hopf (co)-monads.
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A linear exact Hopf comonad $T$ on tensor category $C$ is normal if $T(\mathbb{1}) \in \langle \mathbb{1} \rangle$. We have $\langle \mathbb{1} \rangle \simeq \text{Vect}$, so if $T$ is normal it restricts to a Hopf algebra $H$ on $\text{Vect}$. If in addition $T$ is faithful, we have an exact sequence of tensor categories

$$\text{comod}H \to C_T \to C$$

and ‘all extensions of $C$ by $\text{comod}H$’ are of this form up to tensor equivalence [one has to be more precise].
Equivariantization

Let $G$ be a finite group acting on a tensor category $C$ by tensor automorphisms $(T_g)_{g \in G}$. Then we have an exact sequence

$$\text{rep}_G \rightarrow C_G \rightarrow C$$

where $C_G$ is the equivariantization functor. The endofunctor $T = \bigoplus T_g$ admits a structure of Hopf comonad, and $C_G$ is just $C_T$. The Hopf comonad $T_G$ is normal faithful exact, and its associated Hopf algebra is $k_G$. It has a certain commutativity property. These conditions characterize Hopf comonads corresponding with equivariantizations (at least over $C$).
Equivariantization
Let $G$ be a finite group acting on a tensor category $C$ by tensor automorphisms $(T_g)_{g \in G}$. Then we have an exact sequence

$$\text{rep} G \rightarrow C^G \rightarrow C$$

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Equivariantization

Let $G$ be a finite group acting on a tensor category $C$ by tensor automorphisms $(T_g)_{g \in G}$. Then we have an exact sequence

$$\text{rep} G \to C^G \to C$$

where $C^G \to C$ is the equivariantization functor.

The endofunctor $T = \bigoplus T_g$ admits a structure of Hopf comonad $T^G$ (it admits also a structure of Hopf monad), and $C^G$ is just $C^{T^G}$. The Hopf comonad $T^G$ is normal faithful exact, and its associated Hopf algebra is $k^G$. It has a certain commutativity property. These conditions characterize Hopf comonads corresponding with equivariantizations (at least over $\mathbb{C}$).
24. More on Hopf monads

Where Hopf diagrams are introduced as a means for computing the Reshetikhin-Turaev invariant in terms of the coend of a ribbon category and its structural morphisms.

Where the notion of Hopf monad is introduced, and several fundamental results of the theory of finite dimensional Hopf algebras are extended thereto.

**BV3.** *Categorical Centers and Reshetikhin-Turaev Invariants*, Acta Mathematica Vietnamica 33 3, 255-279
Where the coend of the center of a fusion spherical category over a ring is described, the modularity of the center, proven, and the corresponding Reshetikhin-Turaev invariant, constructed.

Where the general theory of centralizers and doubles of Hopf monads is expounded.

Where Hopf monads are defined anew in the monoidal world

**BN.** *Exact sequences of tensor categories*, arXiv:1006.0569.

See also: http://www.math.univ-montp2.fr/~bruguieres/recherche.html