A Derived and Homotopical View on Field Theories

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Homological technics have been widely used in physics for a very long time. It seems that their first appearance in quantum field theory goes back to the so-called *Faddaev-Popov ghosts* [16], which have later been mathematically identified as Chevalley generators. In more geometric terms one would nowadays justify their appearance as follows: the quotient space of the phase space by symmetries of the Lagrangian $\mathcal{L}$ might be singular and one shall rather deal with the *quotient stack* instead.

The usefulness of (higher) stacks in quantum field theory is argued in Chap. 6. Let me anyway emphasize that the quotient stack carries some relevant information (such as finite gauge symmetries) that can't be encoded by simply adding new fields.

Another crucial step is the introduction of anti-fields and anti-ghosts. A geometric explanation for anti-fields is that the quantities one wants to compute localize on the critical points of $\mathcal{L}$, which might be degenerate or non-isolated. A smart idea is to consider *the derived critical locus* of $\mathcal{L}$ instead, which one defines as the derived intersection of the graph of $\mathrm{d}_{dR}\mathcal{L}$ with the zero section inside the cotangent of the phase space. A derived intersection can be concretely computed by first applying a (homological) perturbation to one of the two factors and then taking the intersection: anti-fields then simply appear as Koszul generators.

The derived critical locus inherits a (-1)-shifted symplectic structure (see below) which is at the heart of the anti-bracket formalism (a.k.a. BV formalism) [5]. The symmetries of the Lagragian act in a Hamiltonian way on the derived critical locus, and anti-ghosts appear when one is taking the derived zeroes of the moments.

We refer to [25] for related considerations and a wonderful exposition of the homological nature of the BV formalism.

All this seems to be nowadays well-known, but we would like to emphasize two points:

- the usual homological approach to higher structures (see e.g. Chaps. 3 and 10) does not distinguish clearly the "derived" and "stacky" directions, while the rapidly emerging field of derived geometry takes care of it.

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- one **has to** make use of derived geometry in order to get symplectic structures: the use of non-derived stacks in Chap. 6 systematically destroys the non-degeneracy of Hamiltonian structures.

The second point is very much related to what happens with symplectic structures on moduli spaces (which are deeply studied in Chap. 11). For instance, the moduli stack of flat G-bundles (G being a compact Lie group) on a closed oriented surface **does not** carry any symplectic structure for very simple degree reasons: its tangent complex sits in cohomological degrees -1 and 0. It is only when restricted to a specific locus that the natural pre-sympectic form becomes non-degenerate. But there is a natural **derived stack** of flat G-bundles on a closed surface which is symplectic (its tangent complex sits in degrees -1, 0 and 1).

In this introductory chapter we provide an informal and partial discussion of the usefulness of derived and homotopical technics in field theories.

We begin with a description of field theories of AKSZ type [2] in the framework of derived (algebraic) geometry. The derived geometric approach makes very transparent the fact that this class of theories fits into the axiomatic framework of Atiyah–Segal [3, 26]. We refer to Chap. 9 for a detailed discussion of the compatibility between the BV formalism and the Atiyah–Segal framework.

We then discuss two mathematical formulations of the physical concept of locality: *factorization algebras* and *fully extended field theories*. We put a lot of emphasis on topological field theories and say a few words about conformal field theories. We also mention how these two approaches are related.

We finally end this Chapter with the example of $3d$ Chern–Simons theory with a finite gauge group and sketch how one could recover the results of [17, Sect. 4] from this approach.

1 Classical Fields and the AKSZ-PTVV Construction

Classical fields are usually described mathematically as sections of (infinite dimensional) fiber bundles. A large class of theories, called *σ-models*, actually describe fields as maps. In the seminal paper [2] the authors introduce the notion of *Q-manifolds*, that allow one to deal with many theories as σ-models. Moreover, the so-called *AKSZ-construction* make them fit into the framework of the *BV formalism* [5] (a.k.a. *anti-bracket formalism*).

A mathematical treatment of perturbative quantum field theory within the framework of the BV quantization (not only for AKSZ theories) can be found in [13] (see also Chap. 9 for examples).

1.1 Transgression

At the heart of the AKSZ formalism [2] and its modern reformulation in [22] (known as PTVV formalism, which is formulated in the language of *derived geometry*[1]) one finds the so-called *transgression procedure*. Let X, Y be generalized spaces (Q-manifolds in the AKSZ formalism, derived stacks in the PTVV formalism). Let ω be a symplectic form of cohomological degree n on Y and assume that X carries an integration theory of cohomological degree d. Then the formula

$$\int_X ev^*\omega\,,$$

where $ev : X \times \mathbf{Map}(X, Y) \longrightarrow Y$ is the evaluation map, defines a symplectic form of cohomological degree $n - d$ on the mapping space $\mathbf{Map}(X, Y)$.

1.1.1 AKSZ versus PTVV: Integration Theory

There are subtle but important differences between the AKSZ and the PTVV formalisms.

In the case of the AKSZ formalism, the integration theory one is referring to is nothing but the *Berezin integration* [7]. Here are three examples of Q-manifolds carrying an integration theory of cohomological degree d in this sense:

1. $(V[1], 0)$, where V is vector space of dimension d.
2. $\Sigma_{dR} := \big(T[1]\Sigma, \mathrm{d}_{dR}\big)$, where Σ is a compact oriented differentiable manifold of dimension d.
3. $\Sigma_{Dol} := \big(T^{0,1}[1]\Sigma, \overline{\partial}\big)$, where Σ is a compact complex manifold of dimension d equipped with a nowhere vanishing top degree holomorphic form η.

Within the PTVV formalism an integration theory of degree d on a derived stack X is a chain map $[X] : \mathbf{R}\Gamma(\mathcal{O}_\Sigma) \longrightarrow \mathbf{k}[-d]$, where $\mathbf{R}\Gamma(\mathcal{O}_\Sigma)$ denotes the complex of derived global functions on Σ, which satisfies a suitable non-degeneracy condition (the definition of non-degeneracy mimics the abstract formulation of Poincaré duality). Any integration theory of cohomological degree d on a Q-manifold in the AKSZ sense induces an integration theory on its associated derived stack in the PTVV sense. But:

different Q-manifolds might have equivalent associated derived stacks.

This is an important point. Derived stacks are model-independent: it doesn't matter how a derived stack is constructed. In the physics language one could view derived stacks as reduced phase space while Q-manifolds carry some information about

[1] We refer to [27] and references therein for an introduction to derived geometry.

the original phase space (e.g. the moduli stack of flat G-bundles, compared the Q-manifold of all G-connections).

Note that there is a stack with an integration theory that can't be described using Q-manifolds. Let Σ be a Poincaré duality d-space; there is stack Σ_B classifying local systems on Σ (it can be explicitely described using a combinatorial presentation of Σ, such as a triangulation or a cellular structure). Derived global functions on Σ_B are cochains on Σ and thus the fundamental class $[\Sigma]$ determines an integration theory of degree d on Σ_B.

1.1.2 AKSZ versus PTVV: Symplectic Structures

The model independence of derived stacks forces all definitions to be homotopy invariant and as such the required properties can't be strictly satisfied (i.e. they might only hold up to coherent homotopies). This is particularly visible when it comes to closed forms. Roughly speaking, the complex of forms on a derived stack (or a Q-manifold) has two "graduations": the weight (k-forms have weight k) and the cohomological degree. Similarly the differential has two components: the internal differential d_{int} (the Lie derivative with respect to the cohomological vector field Q) and the de Rham differential d_{dR}. In the PTVV formalism a k-form of degree n is a weight k d_{int}-cocycle ω_0 of cohomological degree n, and

being a closed form is an additional structure.

Namely, a closed k-form of degree n consists in a sequence $(\omega_0, \omega_1, \dots)$ where

- ω_0 is a k-form of degree n.
- ω_i has weight $k + i$ and cohomological degree n.
- $\mathrm{d}_{dR}(\omega_i) \pm \mathrm{d}_{int}(\omega_{i+1}) = 0$.

Somehow we are considering forms which are closed up to homotopy, while the AKSZ formalism only considers closed forms which are strictly closed.

Something similar happens for the non-degeneracy property when one defines symplectic structures. An *n-symplectic structure* is the data of a closed 2-form of degree n such that its underlying 2-form of degree n is *non-degenerate* (recall that in the AKSZ formalism the underlying form coincides with the closed one): non-degenerate means that the morphism it induces between the tangent and the cotangent complexes is a quasi-isomorphism (while it is required to be an isomorphism in the AKSZ formalism).

Remark 1 The AKSZ formalism also makes an extensive use of infinite dimensional differential geometry, while derived geometry is designed so that many derived mapping stacks are still locally representable by finite dimensional objects (it is often the case that the reduced phase space is a finite dimensional object even though the original phase space is not).

Example 1 Here are some nice examples of symplectic structures in the derived setting:

- if G is a compact Lie group then $BG = [*/G]$ carries a 2-symplectic structure (see [22]).
- if G is any Lie group then $[\mathfrak{g}^*/G] = T^*[1](BG)$ carries a 1-symplectic structure (see [8]).
- if G is a compact Lie group then $[G/G] = \mathbf{Map}(S^1_B, BG)$ carries a 1-symplectic structure (see [8, 23]).
- the derived critical locus of a function carries a (-1)-symplectic structure (see [22]).

1.2 Transgression with Boundary

In [10, 11] (see also Chap. 9) the AKSZ construction is extended to the case when the source of the σ-model has a boundary and the authors use it to produce field theories that satisfy the axiomatics of Atiyah–Segal [3, 26]. The analogous construction also exists for the PTVV formalism (see [8]).

1.2.1 AKSZ versus PTVV: Lagrangian Structures

Let $X \xrightarrow{f} Y$ be a morphism of generalized spaces and assume we have an n-symplectic structure ω on Y. As usual in derived geometry (and more generally in homotopy theory), being Lagrangian is not a property but rather an additional structure. Namely, a *Lagrangian structure* on f is a homotopy γ (inside the space of closed 2-forms of degree n on X) between $f^*\omega$ and 0 such that the underlying path γ_0 between $f^*\omega_0$ and 0 is non-degenerate. In more explicit terms:

- $\gamma = (\gamma_0, \gamma_1, \dots)$ is such that $f^*\omega_0 = \mathrm{d}_{int}(\gamma_0)$ and
$$f^*\omega_i = \mathrm{d}_{int}(\gamma_i) \pm \mathrm{d}_{dR}(\gamma_{i-1})\,.$$
- the identity satisfied by γ_0 ensures that the map $\mathbb{T}_X \longrightarrow f^*\mathbb{L}_Y[n]$ given by $f^*\omega_0$ lifts to $\mathbb{T}_X \longrightarrow \mathbb{L}_f[n+1]$, where $\mathbb{L}_f$ is the relative cotangent complex. The non-degeneracy condition says that it is a quasi-isomorphism.

Usual Lagrangian subspaces are Lagrangian in the above sense, but any kind of map can carry a Lagrangian structure. There are actually Lagrangian structures arising in a quite surprising way:

Example 2 (*See* [8, 9, 23]). (a) A Lagrangian structure on the morphism $X \longrightarrow *_{(n+1)}$, where $*_{(n+1)}$ is the point equipped with its canonical $(n+1)$-symplectic structure, is the same as an n-symplectic structure on X.

(b) A moment map $\mu : X \longrightarrow \mathfrak{g}^*$ induces a Lagrangian structure on the map $[\mu] : [X/G] \longrightarrow [\mathfrak{g}^*/G]$.
(c) A Lie group valued moment map (in the sense of [1]) $\mu : X \longrightarrow G$, where G is a compact Lie group, induces a Lagrangian structure on the map $[\mu] : [X/G] \longrightarrow [G/G]$.

1.2.2 Relative Integration Theory

A *relative integration theory* (a.k.a. non-degenerate boundary structure or relative orientation, see [8]) on a morphism $X \xrightarrow{f} Y$ is the data of an integration theory $[X]$ on X together with a homotopy η between $f_*[X]$ and 0 that is non-degenerate.[2]

Example 3 There are two important examples of relative integration theories on a morphism. Consider a compact oriented $(d+1)$-manifold Σ with oriented boundary $\partial\Sigma$. Then the morphisms $(\partial\Sigma)_{dR} \longrightarrow \Sigma_{dR}$ and $(\partial\Sigma)_B \longrightarrow \Sigma_B$ both carry a relative integration theory.

Let $X \xrightarrow{f} Y$ be a morphism together with a relative integration theory $([X], \eta)$, and let Z be equipped with an n-symplectic structure ω. It is shown in [8] that

$$\int_\eta ev^*\omega$$

defines a Lagrangian structure on the pull-back morphism $\mathbf{Map}(Y, Z) \longrightarrow \mathbf{Map}(X, Z)$.

1.2.3 Field Theories from Transgression with Boundary

Given a generalized space Y together with an n-symplectic structure, the process of transgression with boundary allows one to produce a functor $\mathbf{Map}(-, Y)$ from a category with

- objects being generalized spaces with an integration theory,
- morphisms from X_1 to X_2 being cospans $X_1 \coprod \overline{X_2} \to X_{12}$ equipped with a relative integration theory,[3]
- composition being given by gluing: $X_{12} \circ X_{23} := X_{12} \coprod_{X_2} X_{23}$.

[2] We won't detail what non-degeneracy means here, but simply say that its definition again mimics the main abstract feature of relative Poincaré duality.

[3] Here an below, $\overline{??}$ means that we consider the opposite integration theory or the opposite symplectic structure on ?? (it should be clear from the context).

to a category with

- objects being generalized spaces with a shifted symplectic structure,
- morphisms from Z_1 to Z_2 being Lagrangian correspondences $Z_{12} \to Z_1 \times \overline{Z_2}$,
- composition being given by fiber products: $Z_{12} \circ Z_{23} := Z_{12} \underset{Z_2}{\times} Z_{23}$.

If we restrict objects of the source category to those of the form described in Example 3, then we precisely get a topological field theory taking values in a category of Lagrangian correspondences. Note that usually, categories of Lagrangian correspondences are ill-defined (as some compositions might not be well-behaved), but working in the homotopy setting and considering derived fiber products resolves this problem.

Remark 2 The gadget one actually has to work with is called an $(\infty, 1)$*-category*, and one shall emphasize that *categroids* (which appear in the main references for Chaps. 5 and 12) are often shadows of an underlying $(\infty, 1)$-category (in other words, even though some compositions might seem to be ill-defined, they actually happen to be well-defined *up to homotopy*).

1.3 Examples

We now provide examples of classical topological field theories that can be treated using the above approach, even though some superconformal field theories (as described in Chap. 4) can be obtained as well.

1.3.1 Classical Chern–Simons Theory

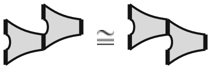
Classical Chern–Simons theory can be recovered if one starts with $Y = BG$ for a compact Lie group G. Details can be found in [23]. One can also include all kinds of boundary conditions (Lagrangian morphisms) or domain-walls (Lagrangian correspondences), which allow to recover all the symplectic moduli spaces of flat connections over quilted surfaces that are obtained *via* the quasi-Hamiltonian formalism in Chap. 11.

1.3.2 Moore–Tachikawa Theory

There is a $2d$ TFT that have been sketched by Moore and Tachikawa [21], of which the target category is a certain category of holomorphic symplectic varieties. This category is a particular case of our category of Lagrangian correspondences (see [9]) and it is very likely that their TFT can be obtained from mapping spaces.

1.3.3 Poisson σ-model

Let (X, π) be a Poisson manifold and consider its π-twisted 1-shifted cotangent $Y := T_X[-1]_\pi$. The derived stack Y, resp. the zero section morphism $X \longrightarrow Y$, can be shown to carry a 1-symplectic structure, resp. a Lagrangian structure. One can show that the mapping stack from ()$_B$ to Y with boundary condition in X, which happens to be the derived self-intersection $\mathcal{G} := X \times^h_Y X$ of X into Y, is 0-symplectic (see [8, 9, 27] for general statements about symplectic structures on relative derived mapping stacks). The cobordism with boundary  is sent to a Lagrangian correspondence between $\mathcal{G}\times\mathcal{G}$ and $\mathcal{G}$, which turns $\mathcal{G}$ into an algebra object within the $(\infty, 1)$-category of Lagrangian correspondences. For instance, associativity of composition is given by the following diffeomorphism:



In [12] Contreras and Scheimbauer show that $\mathcal{G}$ is actually a Calabi-Yau algebra (in the sense of [19]), which clarifies the mysterious axioms of a relational symplectic groupoid of Chap. 12.

2 Mathematical Formulations of Locality

The AKSZ-PTVV theories are expected to be local, in the sense that one can compute everything from local data that one would later glue. In this section we briefly sketch two mathematical approaches to the concept of locality.

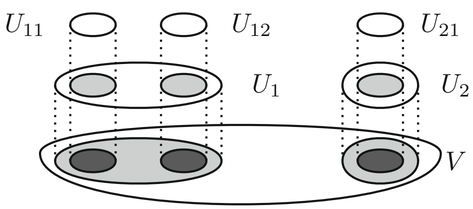
2.1 Factorization Algebras

A *factorization algebra* E over a topological space X consists of

- the data of a vector space (or a cochain complex) E_U for every open subset $U \subset X$.
- the data of a linear map (or a chain map) $\bigotimes_{i\in I} E_{U_i} \longrightarrow E_V$ for every inclusion $\coprod_{i\in I} U_i \subset V$ of pairwise disjoint open subsets.

satisfying the following properties:

- associativity, that can more or less be depicted as follows:



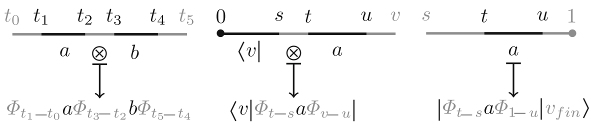
- gluing (one can reconstruct E_U from a nice open cover $\mathcal{U}$ of U and $E_{\mathcal{U}}$).

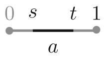
Remark 3 The gluing property is typically a locality property.

We refer to Chaps. 3 and 13 for precise definitions.

Example 4 (*Topological quantum mechanics, see* [14]). Let A be an associative algebra (e.g. $A = \mathbf{End}(V)$) and let $(\Phi_t)_t$ be a 1-parameter group of automorphisms of A (e.g. $\Phi_t = e^{-\frac{it}{\hbar}H}$ is the time evolution). We also give ourselves a right A-module M_r (e.g. V^*) and a left A-module M_ℓ (e.g. V), together with initial and final states $v_{init} \in M_r$ and $v_{fin} \in M_\ell$. From these data one can describe a factorization algebra E on the closed interval $X = [0, 1]$.

- on open intervals of X we set: $E_{[0,s[} = M_r$, $E_{]t,u[} = A$ and $E_{]v,1]} = M_\ell$.
- here are examples of the factorization product:



- one can show that $E_{[0,1]} = M_r \underset{A}{\otimes} M_\ell$ ($\mathbb{C}$ in our example).
- we finally interpret  $\longmapsto \langle v_{init}|\Phi_s a \Phi_{1-t}|v_{fin}\rangle$ as an expectation value.

2.1.1 Factorization Algebras in the BV Formalism

Producing factorization algebras from the local observables in the BV formalism is the main achievement of Costello–Gwilliam (see [14], and also Chap. 3). At the classical level they consider observables with compact support in order to get factorization algebras. It seems that for topological and conformal AKSZ (or PTVV) theories one can consider mapping spaces with compact support in order to get a factorization algebra structure on classical local observables. In particular it is expected that the transgression procedure (both for symplectic and Lagrangian structures) still makes sense locally and glues well.

The main difficult part in Costello–Gwilliam work is of course the quantization of these classical theories. One has to consider effective field theories in the sense of [13] and renormalize (when possible). In the $2d$ conformal case one gets in the end a structure which is very similar to the one of a vertex algebra (see Chap. 3 for a precise statement and Chap. 4 for the definition of a vertex algebra and its use in conformal field theory). In the topological case one obtains in the end a *locally constant* factorization algebra: on $\mathbb{R}^n$ this boils down to the datum of an algebra over the little disks operad.

Renormalization is actually trivial in the topological case, even though it is not so obvious in Costello's framework. We propose here a different approach to the quantization of classical topological BV theories. The first step is to discretize the theory one is working with, so that one can easily write a factorization algebra of classical discrete local observables that carries a bracket of degree 1. The main point is that on a finite region the algebra of local observable is finitely generated, so that BV quantization can be performed very easily (there is no need to apply any kind of energy cut-off as we have only finitely many states).

The final and very hard step is to make the mesh of the discretization tend to zero. There is some magic that happens for topological theories:

there is no need to make the mesh tend to zero.

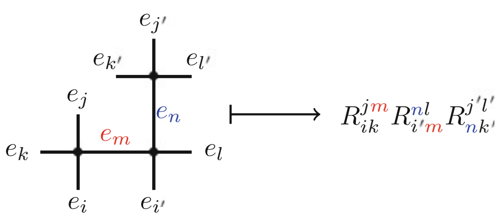
The reason is that, even though the factorization algebra of local observables is not locally constant, it becomes *locally constant at a sufficiently large scale* (the scale depending on the size of the mesh).

Remark 4 Understanding the renormalization procedure for lattice field theories in terms of factorization algebras could lead to a non-perturbative alternative to the constructions of QFTs proposed in [13, 14]. At the moment we[4] can only recover the Weyl algebra from a discrete $1d$ model. The next step would be to understand the renormalization of discrete models in 2 dimensions (with an emphasis on conformal ones).

2.1.2 Locally Constant Factorization Algebras from Discrete Models

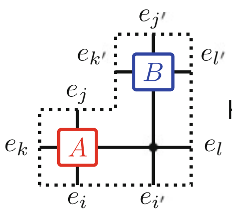
One can prove that any factorization algebra that is locally constant above a given scale gives rise to a locally constant factorization algebra that coincides with the original one above that scale. The idea is very simple: discard the badly behaved part (the one below the given scale) and replace it by a rescaled copy of what happens at large scale... note that implementing this idea actually requires the use of the higher categorical machinery.

Let us provide a potential application of this quite intuitive idea to lattice models. We will formulate things in dimension 2 but it works in arbitrary dimension. Let H, V be vector spaces of states (horizontal and vertical) and let $R \in GL(H \otimes V)$ be an interaction matrix: $R_{ik}^{jl} = \exp\left(-\frac{1}{kT}\epsilon_{ik}^{jl}\right)$. Computing a state sum is nothing but tensor calculus:



One can define a factorization algebra $\mathcal{F}_R$ which associates the space of its boundary states to a given open region of $\mathbb{R}^2$, and for which the factorization product can be depicted in the following way:

[4] This is a joint project with Giovanni Felder.



$$\longmapsto \qquad A \otimes B \mapsto \left(A^{jm}_{ik} R^{nl}_{i'm} B^{j'}_{nk'}\right)^{jj'll'}_{ii'kk'}$$

Note that the lattice $\mathbb{Z}^2$ acts on global sections $\mathcal{F}_R(\mathbb{R}^2)$ of $\mathcal{F}_R$.

Conjecture 1 (Kontsevich). $C^\bullet\big(\mathbb{Z}^2, \mathcal{F}_R(\mathbb{R}^2)\big)$ *has an action of the (chains on the) little disks operads in dimension* 2*.*

The idea to prove this conjecture is to define a new factorization algebra $\widetilde{\mathcal{F}}_R$, very similar to $\mathcal{F}_R$ but carrying an additional discretized de Rham differential,[5] such that

- $\widetilde{\mathcal{F}}_R$ is locally constant at scale > 2.
- $\widetilde{\mathcal{F}}_R(\mathbb{R}^2) = C^\bullet\big(\mathbb{Z}^2, \mathcal{F}_R(\mathbb{R}^2)\big)$.

This would imply Kontsevich's conjecture.

2.2 Fully Extended Field Theories

The axiomatics of fully extended field theories is a higher categorical analog of Atiyah–Segal axiomatics. Roughly speaking, it is a symmetric monoidal functor from a symmetric monoidal higher category of cobordisms to another symmetric monoidal higher category. Higher categories of cobordism can be informally described as follows (we refer to [19] for precise definitions in the topological setting):

- objects are 0 dimensional manifolds of a certain type.
- 1-morphisms are 1-cobordisms between these.
- 2-morphisms are 2-cobordisms,
- ...

It is only for topological field theories that the above has been formalized in a mathematically precise way (see [4, 19]). The *cobordism hypothesis* (which is now a Theorem thanks to the work of Lurie) states that fully extended topological field theories are completely determined by their value on the point. One can see this as a very strong locality property (everything can be reconstructued from the point!). Objects that are images of the point under fully extended TFTs are called *fully dualizable*: being fully dualizable is a very strong finiteness requirement.

We refer to [18] for a very nice review of the cobordism hypothesis (note that the cobordism hypothesis appears implicitly in Chaps. 6 and 9).

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[5] Roughly, $\mathcal{F}_R$ carries a discrete flat connection and $\widetilde{\mathcal{F}}_R$ is the factorization algebra of derived flat sections of $\mathcal{F}_R$.

2.2.1 Examples of Fully Extended TFTs

In dimension 1, fully dualizable objects are genuine dualizable objects (e.g. finite dimensional vector spaces).

Classical field theories of AKSZ-PTVV type are fully extended. This has been announced (without proof) in [11] and [8, 9]. The target category to work with is a suitable category of iterated Lagrangian correspondences, that is currently the subject of ongoing investigations.

It is expected that modular tensor categories are fully dualizable in the 4-category of braided monoidal categories, leading to a large class of fully extended 4d TFTs.

2.2.2 Chiral and Factorization Homologies

Locality in 2d conformal field theory can be formalized either using modular functors or vertex algebras. Chiral homology, that was invented by Beilinson–Drinfeld [6], allows one to produce a modular functor out of a (quasi-conformal) vertex algebra.

Factorization homology (a.k.a. topological chiral homology) achieves the same goal in the topological setting. If A is an algebra over the little disks operad and M is a framed manifold then factorization homology of M with coefficients in A, denoted

$$\int_M A\,,$$

is defined as the “integral”, over all open balls in M, of the value of A on them. Lurie proved [19, 20] that factorization homology is indeed a TFT, and conjectured that it is fully extended. Chapter 7 presents perturbative Chern–Simons theory in dimension 3 as a by-product of factorization homology.

The fact that factorization homology is a fully extended TFT was recently proved in [24].

2.2.3 Chern–Simons Theory with a Finite Gauge Group

Let G be a finite group.

Remark 5 The cotangent complex of BG reduces to $\{0\}$, so that BG is trivially n-symplectic for any $n \in \mathbb{Z}$. Therefore symplectic structures won’t play a significant rôle for this specific example. But they are essential when one deals with non-discrete compact Lie groups.

We have a 3d fully extended TFT with values in a higher category of iterated correspondences that is given by $\mathbf{Map}(-, BG)$.

It is very unlikely that the category of correspondences can provide numerical invariants. In order to get that we have to “linearize” our field theory.

Let us sketch how to do this in dimensions 1-2-3:

- we replace $\mathbf{Map}\big(S^1_B, BG\big) = [G/G]$ by its category of quasi-coherent sheaves $QCoh([G/G])$, which is nothing but the category $Rep\big(D(G)\big)$ of (complexes of) representation of the Drinfeld double of G.
- the correspondence given by $\mathbf{Map}(\Sigma, BG)$ for a 2-cobordism Σ can be used to produce a convolution functor $QCoh([G/G]^k) \longrightarrow QCoh([G/G]^l)$.
- mapping spaces from $3d$ manifolds produce natural transformations of functors.

Remark 6 One can even associate the monoidal category $QCoh(BG) = Rep(G)$ to the point. It is important to notice that not every object is fully dualizable in the 3-category of monoidal categories. But $Rep(G)$ surely is,[6] so that we have a nice and linear enough fully extended TFT.

It would be interesting to get back this fully extended Chern–Simons TFT with finite gauge group by means of factorization homology. In order to do so one shall construct a locally constant factorization algebra on $\mathbb{R}^3$ that is locally constant. We would suggest to use a discrete model.

Remark 7 Observe that $Rep(G)$ is a fusion category and is thus, after [15], a fully dualizable object in the symmetric monoidal 3-category of monoidal categories. It thus produces a fully extended $3d$ TFT. The fact that the partition function of this TFT can be computed *via* a state sum (see [28]) is a strong evidence in favor of our suggestion.

One must say that already for Yang-Mills theory in dimension 2 it is an interesting task to produce an E_2-algebra from the data of a Hopf algebra with an integral, by means of a discrete model.

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[6] This is very much related to the fact that the symplectic structure on BG is zero. In the case of compact groups, $Rep(G)$ isn't finite enough and must be deformed in order to get a rigid enough object... this is where the quantum group comes from.

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