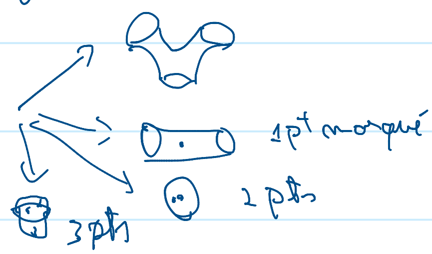
Compactification de l'espace de modules de courbes - Sylvain

Thursday, November 25, 2021 14:14

g, m toujours tq $\chi(S_{g,m}) < 0$

Décomposition en pantalons



$\parallel$

multicourbe maximale $(\gamma_1, \dots, \gamma_{3g-3+m})$

Fenchel - Nielsen :

$$\begin{tikzcd}
(\mathbb{R}_{>0})^{3g-3+m} \times \mathbb{R}^{3g-3+m} \arrow[r, "\sim"] \arrow[d] & \mathcal{T}_{g,m} \arrow[d] \\
\mathbb{R}_{>0}^{3g-3+m} \times (\mathbb{R}/2\pi\mathbb{Z})^{3g-3+m} \arrow[r, "\text{pas inj}"] & \mathcal{M}_{g,m}
\end{tikzcd}$$

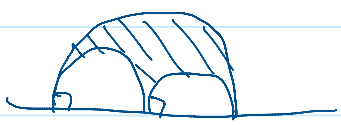
longueurs $l_i(x)$ coordonnées de torsion $t_i(x)$ compact

Exemples :

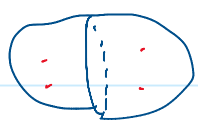
(Rappel : $\# \text{Courbes} = 3g-3+m$

$\# \pi_0 (X - \cup \gamma_i) = 2g-2+m$)

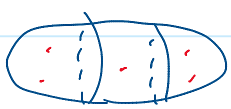
g	m	# Courbes	# comp
0	3	0	1



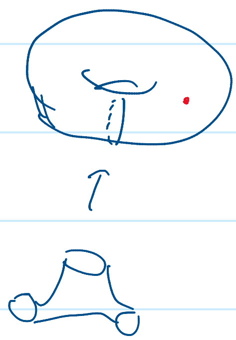
0	4	1	2



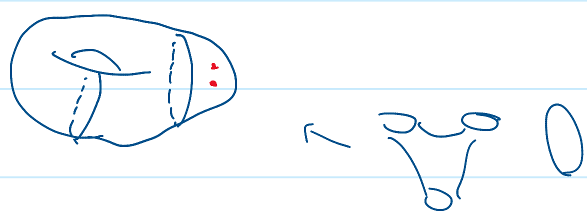
0	m	$m-3$	$m-2$



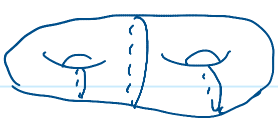
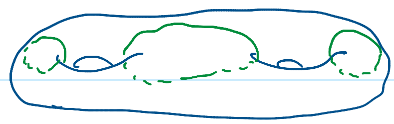
1	1	1	1



1 2 2 2



2 0 3 2

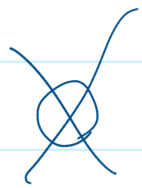
 ou 

Déf : Surface de Riemann avec noeuds de type (g,n) est

X compact, n pts marqués tq

atlas : carte $\rightsquigarrow$ $\varphi : \mathbb{D} \xrightarrow{\sim} X$ (loin des pts marqués) atlas : carte $\rightsquigarrow$ $\varphi : \mathbb{D} \xrightarrow{\sim} X$ (loin des pts sngs)

$\varphi : \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \begin{array}{l} |z_1| < 1 \\ |z_2| < 1 \\ z_1 z_2 = 0 \end{array} \right\} \to X$



finie de fois

noeud

tq $X - \{\text{noeuds}\}$: $\forall$ comp connexe est hyperbolique

$$g = \sum g_i + n - p + 1$$

où g_i : genre de comp. connexes de $X \setminus$ noeuds

n : # noeuds

p : # comp connexes de $X - \{\text{noeuds}\}$

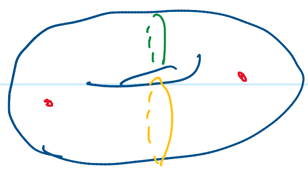
$\overline{\mathcal{M}}_{g,n} = \{\text{surfaces à noeuds}\}/_{\text{isom}}$

$\cup$

$\mathcal{M}_{g,n}$

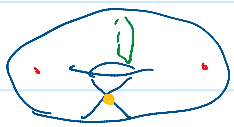
But : Munir $\overline{\mathcal{M}}_{g,n}$ d'une topologie

<u>But</u> : Munir $\overline{\mathcal{M}}_{g,n}$ d'une topologie qui étende celle de $\mathcal{M}_{g,n}$ t.q $\mathcal{M}_{g,n}$ dense dans $\overline{\mathcal{M}}_{g,n}$ et $\overline{\mathcal{M}}_{g,n}$ compact.



$g = g_1 + m - p + 1$

$1 = 0 + 1 - 1 + 1$



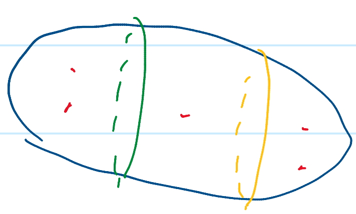
$g = 1$

$n = 2$

$m = 1$

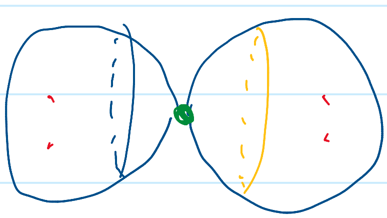
$p = 1$

$g_1 = 0$



$g = 0$

$n = 5$



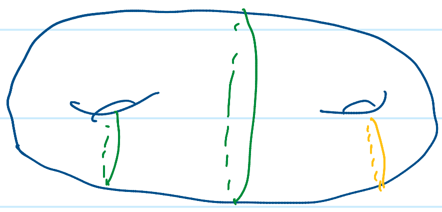
$m = 1$

$p = 2$

$1 + 2 + 1 = 0$

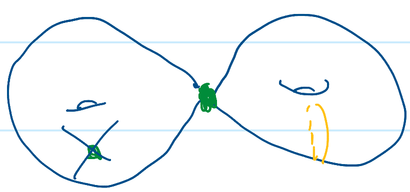
$g_1 = 0$

$g_2 = 0$



$g = 2$

$n = 0$ $\qquad$ $1 + 2 - 2 + 1 = 2$



(X)

$m = 2$

$p = 2$

$g_1 = 0$

$g_2 = 1$

(la déf est tq contracter ne change pas le genre)

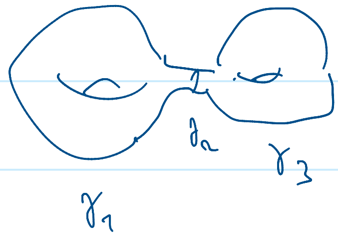
Extension de Fenchel-Nielsen :

$X_0 \in \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$

$\rightarrow$ désingulariser X_ε $\quad (\varepsilon > 0)$

$$\left.\begin{array}{l} |z_1| < 1 \\ |z_2| < 1 \\ z_1 z_2 = 0 \end{array}\right| \rightsquigarrow z_1 z_2 = \varepsilon$$

$(*)$ $\rightsquigarrow$



base de voisinages - multicourbe maximale

$(\underbrace{\gamma_1, \dots \gamma_m}, \dots \gamma_{3g-3+m})$ tq

m premières correspondent aux noeuds

coord FN $\longrightarrow \overset{l_i}{\mathbb{R}_{>0}^{3g-3+m}} \times \overset{t_i}{\mathbb{R}^{3g-3+m}}$

étendre à $\mathbb{R}_{\geq 0}^{3g-3+m} \times \mathbb{R}^{3g-3+m}$

noeud $\Longleftrightarrow l_i(X) = 0$ ← on fait ce choix

(et on peut prendre $t_i = 0$)

Ex : $\gamma_1\ \gamma_2\ \gamma_3$ $\qquad X_0 : (0, 0, \underset{t_0}{l(\gamma_3)}, 0, 0, t(\gamma_3))$

P

$(0, \text{———} \ 0)$

Rmq : $\varphi_P : \mathbb{R}_{\geq 0}^{3g-3+n} \times \mathbb{R}^{3g-3+3} \longrightarrow \overline{\mathcal{M}_{g,n}}$

pas surj

Lemme : Il n'y a qu'un nombre fini de P à Diff($\Sigma_{g,n}$) près.

$K_P = \varphi_P\left([0,c], \dots, [0,c], [0,2\pi], \dots [0,2\pi] \right)$

X_0 : $\mu, \tau > 0$

$$U_{\mu,\tau} = \left\{ X \in \overline{\mathcal{M}_{g,n}} \;\middle|\; \begin{array}{l} \forall i \; |l_i(X) - l_i(X_0)| < \mu \\ \text{et } \forall i > n \; |\tau_i(X) - \tau_i(X_0)| < \tau \end{array} \right\}$$

base de voisinages

$$\underbrace{\bigcup K_P}_{\text{compact}} \supset \mathcal{M}_{g,n}$$

Prop : $\forall g,n$ $\chi_{g,n} < 0$

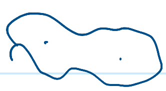
$\exists\, c = c(g,n)$ tq

$\forall X$ surface de Riem lisse de type (g,n)

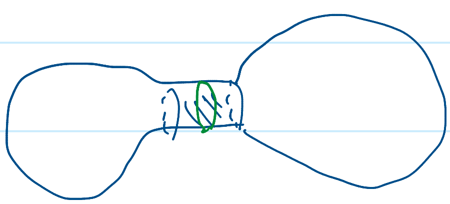
$\exists$ multicourbe max $P=(\gamma_1,\dots,\gamma_{3g-3+n})$
tq $l(\gamma_i) \leq C(g,n)$ $\forall i$.

Pour démontrer la Prop:

<u>Lemme de Margulis - décomposition mince/épaisse</u>

$X =$  surface compacte

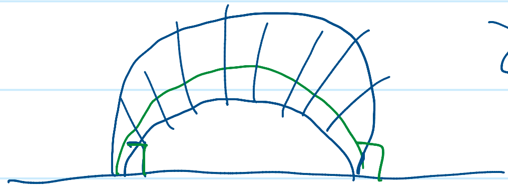
Def: <u>Tube de Margulis</u> γ géodésique fermée.



$N_r(\gamma)$ ↝ voisinage métrique

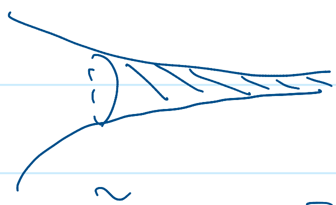
topologiquement un collier

Revêt univ : $\tilde{X} = \mathbb{H}^2$



$Z(\pi_1(X))$ engendré par un élément hyperbolique (la classe de γ)

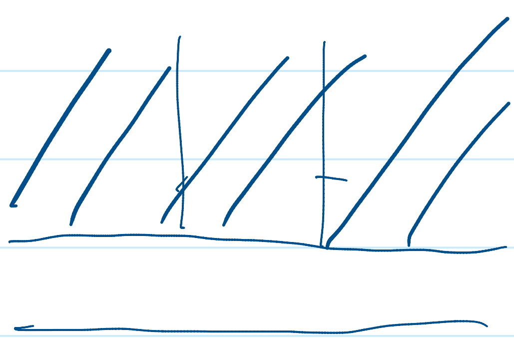
Def Voisinage de pointe ("cusp")



aire finie

$\simeq S \times [0, +\infty[$

$\tilde{X} = \mathbb{H}^2$



parabolique $z \mapsto z+1$

horocycle

Lemme de Margulis :

$\exists \mu > 0 \quad \forall X$

$\exists$ collection finie de tubes de Margulis $T_1, \dots, T_n$

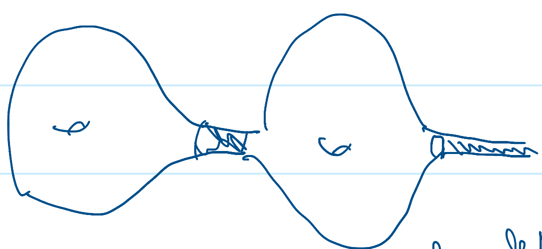
$\exists$ collection finie de tubes de Margulis $T_1, \dots, T_r$ et de voisinages de pointes $c_1, \dots, c_s$ disjoints et dont le complémentaire

$\forall x \in X \setminus \bigcup T_i \setminus \bigcup C_i$ partie μ-mince

inj $x \geq \mu$

$\underbrace{\text{inj}}_{\text{plus petit rayon tq } B(x,r) \text{ isométrique à une boule dans } \mathbb{H}^2}$

$X - X_{\mu\text{-mince}} =: X_{\mu\text{-épaisse}}$



On fixe μ $\exists C' = c'(g,n)$ tq $\forall X$ (lemme de μ)

Lemme : Soit α arc géo minimisant

$\alpha \subset X_{\mu\text{-épaisse}}$

$l(\alpha) \leq C'$

<u>Preuve</u> :

Idée $\rightsquigarrow$ aire $\leq 2\pi\,|\chi(S_{g,n})|$ //