# The Cohomology of the Moduli Space of Curves

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The purpose of these notes is to give an exposition of recent work of several people on the topology and geometry of the moduli space of curves. Moduli space may be approached in many different ways. For $g > 1$, it is simultaneously the space of isometry classes of hyperbolic metrics on a surface of genus $g$, the space of conformal equivalence classes of Riemann surfaces of genus $g$ and the space of algebraic curves of genus $g$ up to isomorphism. This means that there is an elaborate interplay between hyperbolic geometry, complex analysis and algebraic geometry going on. The mapping class group acts properly discontinuously on Teichmüller space with quotient moduli space, so the rational cohomology of moduli space may be identified with that of the mapping class group. This adds a topological and algebraic perspective to things.

The main emphasis in these notes will be on this topological side and we will discuss primarily work of our own. We will, however, spend some time on analysis as we discuss work of Scott Wolpert on the Weil-Petersson geometry of Moduli space. Our main theme then will be the question of how much of the topology and formal geometry of a symmetric space can be found for Teichmüller space and how many of the properties of an arithmetic group can be found for the mapping class group. This will lead us through a discussion of work by Charney and Lee, Harris, Miller, Morita, Mumford, Thurston, Zagier and many others.

We would like to thank Scott Wolpert for a great deal of help with these notes.

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## Chapter 1. Introduction

Let $F$ be a closed, oriented surface of genus $g$. The primary object we will be looking at in these notes is the space which parametrizes all the conformal structures carried by the surface $F$; it is called the moduli space of Riemann surfaces and is denoted $M_g$. This space can also be defined as the space of hyperbolic metrics on $F$ or the space of algebraic curves of genus $g$, up to appropriate notions of equivalence. We will not use this third point of view here, but we will constantly be switching between the other two.

### §1. First Definitions

We begin with the conformal point of view. Define a marked Riemann surface to be a pair $(R,[f])$ where $R$ is a Riemann surface (= complex 1-manifold), $f: R \to F$ is a homeomorphism and $[f]$ denotes the homotopy class of $f$. Two marked Riemann surfaces $(R_1,[f_1])$ and $(R_2,[f_2])$ are called equivalent if there is a conformal homeomorphism $h: R_1 \to R_2$ such that $[f_2 \circ h] = [f_1]$. The collection of equivalence classes is denoted $T_g$; it has a natural topology and is called the Teichmüller space of genus $g$. (We will discuss this topology later when we introduce Fenchel-Nielsen coordinates.)

The moduli space is obtained by forgetting the marking $[f]$. To make this precise we introduce the mapping class group $\Gamma_g$; it is the group of homotopy classes (or, equivalently, isotopy classes) of orientation preserving homeomorphisms of $F$. The formula

$$[g]\cdot(R,[f]) = (R,[gf])$$

defines an action of $\Gamma_g$ on $T_g$; the quotient space is denoted $M_g$ and is called the moduli space of conformal structures on $F$.

The second way to define $T_g$ and $M_g$ is using hyperbolic geometry $(g>1)$. By a hyperbolic surface we will mean a smooth surface $X$ equipped with a complete Riemannian metric of constant curvature $-1$. A marked hyperbolic surface is a pair $(X,[f])$ where $X$ is hyperbolic and $f: X \to F$ is a homeomorphism. We say that $(X_1,[f_1])$ is equivalent to $(X_2,[f_2])$ if there is an isometry $h: X_1 \to X_2$ with $[f_2 \circ h] = [f_1]$, and we denote the collection of equivalence classes by the same letter $T_g$. The justification for this is the uniformization theorem which states that every Riemann surface $R$ is conformally equivalent to one which admits a hyperbolic metric, and this metric is uniquely determined up to isometry by the conformal equivalence class of $R$.

More generally, one defines the spaces $T_g^s$, $M_g^s$ and the group $\Gamma_g^s$

as follows. Fix $s$ distinct points, ordered $p_1,\ldots,p_s$ on the surface $F_g$ and consider triples $(R,(q_1,\ldots,q_s),[f])$ where $R$ is a Riemann surface, $q_1,\ldots,q_s$ are distinct, ordered points on $R$ and $f: R \to F$ is a homeomorphism with $f(q_i) = p_i$ for each $i$. In this case $[f]$ denotes the homotopy class of $f$ $\mathrm{rel}\{q_i\}$. The definition of equivalence is the same as before: $(R_1,(q_1^1,\ldots,q_s^1), [f_1]) \sim (R_2,(q_1^2,\ldots,q_s^2), [f_2])$ if and only if there exists a conformal homeomorphism $h: R_1 \to R_2$ with $h(q_i^1) = q_i^2$ for each $i$ such that $[f_2 \circ h] = [f_1]$. The space of equivalence classes is denoted $\mathcal{T}_g^s$ and is again called Teichmüller space.

The mapping class group $\Gamma_g^s$ is the group of all orientation preserving homeomorphisms $\phi: F \to F$ such that $\phi(p_i) = p_i$ for all $i$, up to isotopies fixing each $p_i$. It acts on $\mathcal{T}_g^s$ as before and the quotient is moduli space $\mathcal{M}_g^s$.

To define $\mathcal{T}_g^s$ using hyperbolic surfaces we remove the $s$ points; set $F_g^s = F_g - \{p_1,\ldots,p_s\}$ and assume $\chi(F_g^s) < 0$. Consider pairs $(X,[f])$ where $X$ is complete hyperbolic of finite area and $f: X \to F_g^s$ is a homeomorphism. Since $X$ is complete and finite area its structure near a puncture is modeled on the pseudosphere. The definition of the equivalence is exactly as in the case $s = 0$.

§2. Fenchel-Nielsen Coordinates

To understand $\mathcal{T}_g^s$ better it is necessary to introduce Fenchel-Nielsen coordinates. These will be defined using the description of $\mathcal{T}_g^s$ as hyperbolic metrics.

The starting point is the observation that a right hexagon in the hyperbolic plane is determined up to isometry by the lengths of three alternating sides, and these lengths may be chosen to be arbitrary positive numbers. It also makes sense to allow these lengths to go to $0$ or $\infty$; for example, if $\ell_1$ goes to $0$ in Figure 1.1 we obtain ideal right pentagons, for which there are two parameters, and if $\ell_1$ and $\ell_2$ both go to $0$ we obtain ideal right quadralaterals with one parameter.

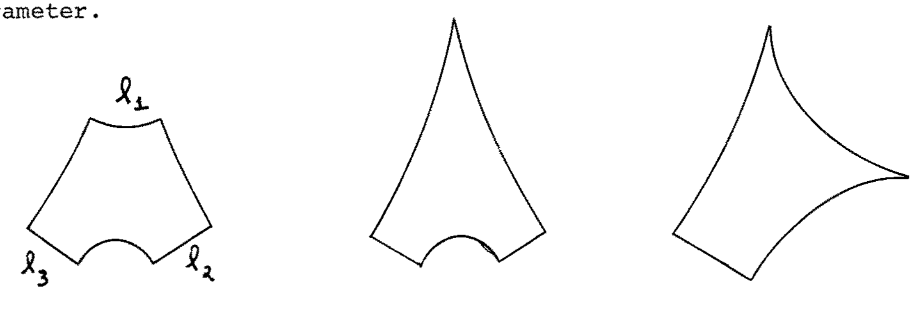



Figure 1.1

Given such a hexagon, form its double across the remaining sides to obtain the basic building block P, a pair of pants (Figure 1.2) with goedesic boundary. The metric on P is now determined by the lengths of its three boundary components and these also are arbitrary. Allowing one or two of these to have length 0 we have the ideal pairs of pants of Figure 1.2.

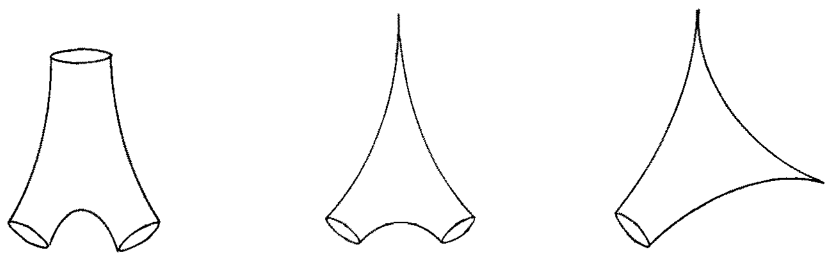


Figure 1.2

Next, fix a partition on $F_g^s$; this is a collection $C_1, \ldots, C_{3g-3+s}$ of disjoint simple closed curves such that $F_g^s - \{C_i\}$ is the disjoint union of pairs of pants, punctured annuli and twice punctured disks (if $g = 0$ assume $s > 3$, $T_0^3$ is a single point). We may build a marked hyperbolic surface by glueing hyperbolic pairs of pants (and ideal pairs of pants) together according to the pattern determined by $\{C_i\}$. The Fenchel-Nielsen coordinates are the free parameters for this construction; there are two for each $C_i$. The first is $\ell_i$, the length of $C_i$; two pairs of pants may be metrically glued along boundary curves to obtain a hyperbolic surface as long as these curves have the same length. The second, the twist parameter $\tau_i$, measures the displacement of the boundary curves along which we glue. The parameter $\tau_i$ is the hyperbolic distance between the feet of perpendiculars dropped from fixed boundaries (Figure 1.3). The parameters $\ell_i$ vary freely in $\mathbb{R}^+$ and the $\tau_i$ vary in $\mathbb{R}$. Fenchel and Nielsen proved

Theorem 1.1: The map $(\mathbb{R}^+ \times \mathbb{R})^{3g-3+s} \to T_g^s$ described above is a homeomorphism.

As we have not described a topology on $T_g^s$ we will think of this

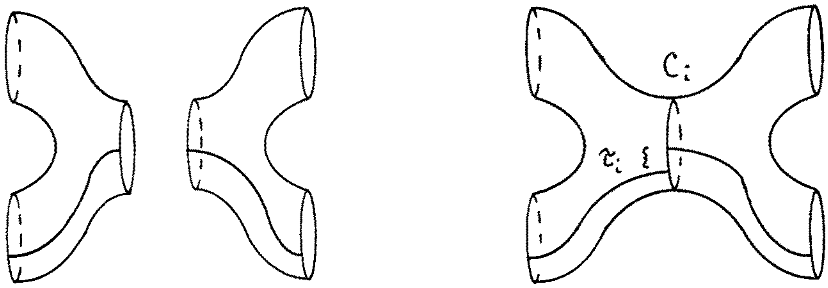



Figure 1.3

result as defining the topology. There is still content, however, to Theorem 1.1 since it is true independent of the choice of partition. Furthermore, we now know that $T_g^s$ is a cell.

The Fenchel-Nielsen coordinates may also be used to describe the Deligne-Mumford compactification $\overline{M}_g^s$ of $M_g^s$ as follows. If $\{C_i\}$ is a partition of $F_g^s$, allowing some of the $\ell_i$ to be 0 we obtain a <u>Riemann surface with nodes</u> at those $C_i$; $\overline{M}_g^s$ is obtained from $M_g^s$ by adjoining these singular surfaces. The complement then has irreducible components $D_0, D_1, \ldots, D_{[g/2]}$ of real codimension 2, where $D_0$ is the collection of surfaces with a node at a nonseparating curve (and perhaps other nodes) and $D_i$, $i > 0$, consists of the surfaces with a node at a curve which separates $F$ into surfaces of genus $i$ and $g - i$.

## §3. Homology of $M_g^s$ and $\Gamma_g^s$

A well-known result about hyperbolic surfaces is that their length spectrum (the collection of real numbers which occur as lengths of closed geodesics) is discrete. It is not difficult to use this to prove that the action of $\Gamma_g^s$ on $T_g^s$ is properly discontinuous; i.e. for every compact set $K \subset T_g^s$ the collection of $\phi \in \Gamma_g^s$ such that $\phi(K) \cap K \neq \emptyset$ is finite. This means that $M_g^s$ is a V-manifold or orbifold: each point has a neighborhood modeled on $\mathbb{R}^N$ modulo a finite group. Furthermore, $M_g^s$ is a "rational $K(\Gamma_g^s,1)$"; that is,

$$H_*(M_g^s;\mathbb{Q}) \cong H_*(\Gamma_g^s;\mathbb{Q}).$$

One can say more than this; actually we claim that $\Gamma_g^s$ is virtually torsion free. To see this, let $\mu : \Gamma_g^s \to Sp(2g;\mathbb{Z})$ be the map obtained by allowing a homeomorphism $\phi$ of $F_g^s$ to act on $H_1(F_g;\mathbb{Z})$. This gives an element of $Sp$ because $\phi$ preserves the intersection form on $F_g$. The map $\mu$ fits into the exact sequence

$$1 \to T_g^s \to \Gamma_g^s \xrightarrow{\mu} Sp(2g;\mathbb{Z}) \to 1 \qquad (S_0)$$

where $T_g^s$ is the Torelli group. Now look at $G_n$, the full congruence subgroup of level $n$ in $Sp(2g;\mathbb{Z})$ which is defined as the subgroup of matrices congruent to the identity mod n. For $n \geq 3$, $G_n$ is torsion free and it is also well-known that $T_g^s$ is torsion free. Therefore, the congruence subgroup $\Gamma_g^s[n] = \mu^{-1}(G_n)$ will also be torsion free, $n \geq 3$. Its index is the order of the finite group $Sp(2g;\mathbb{Z}/n\mathbb{Z})$ so the claim is established.

The quotient $\mathcal{T}_g^s/\Gamma_g^s[n] = \mathcal{M}_g^s[n]$ is called the moduli space of curves with <u>level</u> <u>n</u> <u>structure</u>. It is a manifold and we have

$$H_*(\Gamma_g^s[n];\mathbb{Z}) \cong H_*(\mathcal{M}_g^s[n];\mathbb{Z}).$$

At times it will be necessary to compare the homology groups of $\Gamma_g^s$ as we vary $g$ and $s$. For $s$ we have the exact sequence

$$1 \to \pi_1(F_g^s) \to \Gamma_g^{s+1} \xrightarrow{\eta} \Gamma_g^s \to 1, \qquad (S_1)$$

defined as follows. Let $\eta$ be the map obtained by forgetting $p_{s+1}$. If $\phi$ lies in $\mathrm{Ker}(\eta)$, then $\phi$ is isotopic to the identity fixing $p_1,\ldots,p_s$. Following $p_{s+1}$ under this isotopy determines an element of $\pi_1(F_g^s)$; the sequence $S_1$ is derived from this. The Lyndon-Hockshield-Serre spectral sequence may then be used to relate $H_*(\Gamma_g^s)$ to $H_*(\Gamma_g^{s+1})$. When we vary $g$, however, there is no natural way of mapping $\Gamma_g^s$ to $\Gamma_{g+1}^{s'}$ so it becomes necessary to introduce mapping class groups of surfaces with boundary. In Chapter 6 we will describe these and show how to use them to prove that for $g \gg k$ $H_k(\Gamma_g^s)$ is independent of $g$.

## §4. Algebraic Structure of the Mapping Class Group

Because $H_*(\mathcal{M}_g^s)$ and $H_*(\Gamma_g^s)$ are so intimately related, we will need to know some facts about the algebraic structure of $\Gamma_g^s$.

We first discuss a finite presentation for $\Gamma_g^s$. Let $C \subset F_g^s$ be a simple closed curve; the Dehn twist of $C$, denoted $\tau_C$, is (the isotopy class of) the homeomorphism of $F_g^s$ obtained by splitting along $C$, rotating one side 360° to the right and reglueing (Figure 1.4).

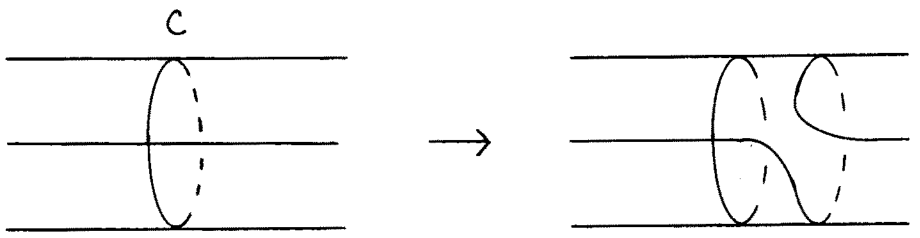



Figure 1.4

Dehn proved that $\Gamma_g^s$ is generated by Dehn twists on a finite number of curves ([D]) and Humphries determined the minimal number of twist generators necessary (the $2g+1$ curves of Figure 1.5 when $s = 0$).

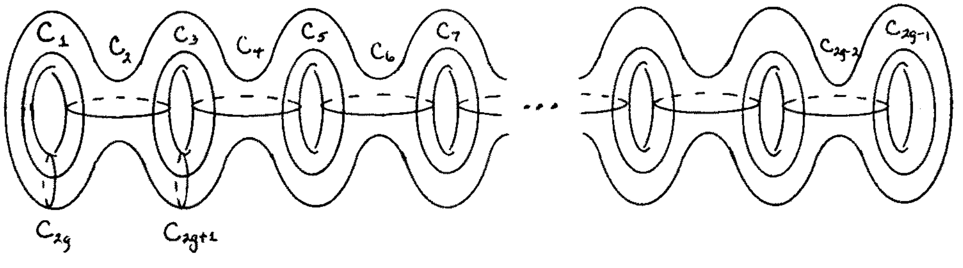



Figure 1.5

McCool([Mc]) gave an indirect proof that $\Gamma_g$ is finitely presented, but the first explicit presentation was provided by Hatcher and Thurston ([HT]). This was simplified by Wajnryb ([Wa]) whose presentation uses as generators $\tau_i = \tau_{C_i}$, $1 \le i \le 2g+1$, and has $\binom{2g+1}{2} + 3$ relations $(g > 2)$. The first of these are the braid relations:

$$\tau_i\,\tau_j = \tau_j\,\tau_i \qquad \text{if} \quad C_i \cap C_j = \emptyset,$$

$$\tau_i\,\tau_j\,\tau_i = \tau_j\,\tau_i\,\tau_j \qquad \text{if} \quad C_i \cap C_j \neq \emptyset.$$

At this point one has a group with $H_1 \cong \mathbf{Z}$ (the group is normally generated by any $\tau_i$), whereas in actual fact $H_1 = 0$ ([P], [M]). The "lantern relation" (see [Hl]) is added next, giving a group with $H_1 = 0$. Now the result has $H_2 = 0$ whereas $H_2$ should be $\mathbf{Z}$([Hl]). To fix this we add the "Chinese lantern relation" to get $H_2 \cong \mathbf{Z}$. It turns out that this is a presentation of the mapping class group of a surface with 1 boundary component. To get $\Gamma_g^0$ we add one more relation, called the "boundary relation". We refer the reader to [Wa] for explicit forms of these relations. Presentations for the groups $\Gamma_g^s$ can easily be obtained using $(S_1)$.

Next we briefly describe Thurston's classification of the elements of $\Gamma_g^s$. This classification is modeled on the decomposition of $PSL_2\mathbb{R}$ into elliptic, hyperbolic and parabolic elements. The analogue of the elliptics are the elements of finite order in $\Gamma_g^s$; each such may be realized as an isometry of some hyperbolic metric. Corresponding to hyperbolics are the pseudo-Anosovs; these are represented by maps which preserve a pair of transverse, measured foliations (with singularities) and they are distinguished by the fact that no element of $\pi_1(F)$ is brought back to a conjugate of itself by a positive power of the map. Finally, parabolics have as counterpart the reducible elements. Each such is represented by a map which fixes (setwise) a collection of disjoint, nontrivial, nonisotopic simple closed curves in $F$.

Thurston proves this result by constructing a spherical compactification $\bar{T}_g^s$ of $T_g^s$ and an extension of the action on $\Gamma_g^s$ to $\bar{T}_g^s$ ([T3]). The boundary sphere is the space of all equivalence classes of projective measured foliations on $F$. The theorem is proven by applying the Brouwer fixed point theorem.

Using this decomposition it is possible to say a great deal about the subgroup structure of the mapping class group. McCarthy has computed the centralizers and normalizers of the elements of $\Gamma_g^s$ ([McCa]). For finite order elements they are extensions of a finite cyclic group by a mapping class group, for pseudo-Anosovs they are finite-by-infinite cyclic and for reducibles they are a mixture of the two. Long ([L1]) showed that if $H < \Gamma_g$ is finitely generated and contains a free group of rank 2 generated by 2 pseudo-Anosovs (for example, $\Gamma_g$ itself), then $H$ contains uncountably many maximal subgroups of infinite index and the Frattini subgroup of $H$, which is the intersection of all maximal subgroups of $H$, is a torsion group. Finally, Birman, Lubotsky and McCarthy showed that every solvable subgroup of $\Gamma_g$ is virtually abelian ([BLM]).

All of these properties are similar to those of discrete subgroups of linear algebraic groups. In fact, slightly weaker forms of them would follow immediately if we had a discrete, faithful linear representation of $\Gamma_g$. Thus we are led naturally to the question: is $\Gamma_g$ actually linear? Or even more: is $\Gamma_g$ arithmetic?

## §5. The Analogy with Symmetric Spaces

The problem which will motivate us in this notes is this: how close is Teichmüller space to being a symmetric space and how close is $\Gamma_g^s$ to being arithmetic? When $G$ is a linear algebraic group defined over $\mathbb{Q}$ and $\Gamma < G_{\mathbb{Q}}$ is arithmetic (see Chapter 4 for definitions), then $\Gamma$ acts properly discontinuously on the symmetric space $X = G/K$, $K$ maximal compact in $G$. The space $X$ is diffeomorphic to Euclidean space and the quotient $\Gamma\backslash X$ is a V-manifold; it follows that $H_*(\Gamma;\mathbb{Q}) \cong H_*(\Gamma\backslash X;\mathbb{Q})$. This suggests an analogy between $T_g^s$ and $X$ and between $\Gamma_g^s$ and $\Gamma$ and most of what we will do springs from this analogy.

The first question which then arises is whether there exists some $G$ such that $T_g^s = G/K$ and $\Gamma_g^s = \Gamma$. Ivanov was the first to announce a proof that this is not the case; in fact he shows that $\Gamma_g^s$ is not arithmetic in any linear algebraic group. In Chapter 4 we will give a proof of this due to Bill Goldman.

Even though $\Gamma_g^s$ is not arithmetic, we can still ask which properties it shares with the arithmetic groups. This is the theme of Chapters 2, 3, 4, 6 and 8. Among other things we will see that Teichmüller space admits a Borel-Serre bordification (Chapter 3) and that the mapping class group is a virtual duality group (Chapter 4), satisfies homological stability as $g$ goes to infinity (Chapter 6) and admits a formula for its Euler-characteristic which involves the Riemann zeta-function (Chapter 8). All of these are properties of arithmetic groups.

We can also ask how much of the formal geometry of a symmetric space may be found for $T_g^s$. There are two well-known metrics for $T_g^s$. The first is the Teichmüller metric; it is only a Finsler metric and the geometry it provides is quite distorted. In fact, Royden ([R]) showed that in this metric the group of isometries of $T_g^s$ is exactly the mapping class group, so the situation is very unlike that of a symmetric space. The second is the Weil-Petersson metric. This metric is Kähler and is much more suited to our purposes. It has strictly negative Ricci curvature and holomorphic sectional curvatures. In Chapter 5 we will discuss Scott Wolpert's striking work on the symplectic and Hermitian geometry this metric gives for Teichmüller space. This geometry is intimately tied to the complex structure on; in fact, in 3 we will outline how Wolpert uses the Weil-Petersson Kähler form to give an analytic proof that $\overline{M}_g^s$ is projective.

# Chapter 2: Triangulating Teichmüller Space

The purpose of this chapter is to describe an ideal triangulation of Teichmüller space which is compatible with the action of the mapping class group. The construction works for any $s \geq 1$ but we will restrict to the case where $s = 1$ to keep the exposition simple. The original idea for this triangulation is due to Thurston and uses hyperbolic geometry; the details for this approach were provided by Bowditch and Epstein and will be given in §3. The first complete proof, however, was given by Mumford using the conformal point of view and was based on results of Strebel. (Yet another proof was provided more recently by Epstein and Penner using the interpretation of Teichmüller space as conjugacy classes of discrete, faithful representations of the fundamental group of a surface in $SO(2,1)$). We will present Mumford's proof first, in §2, after giving the combinatorial structure of the triangulation in §1.

## §1 The Simplicial Complex

Let $F$ be a closed, oriented surface of genus $g \geq 1$ and let $*$ be a basepoint in $F$. The isotopy class (rel $*$) of a family $\alpha_0,\ldots,\alpha_k$ of simple closed curves in $F$ through $*$ will be called a rank-$k$ arc-system if $\alpha_i$ intersects $\alpha_j$ only at $*$ when $i \neq j$ and the family satisfies the nontriviality condition that no $\alpha_i$ is null-homotopic and no distinct $\alpha_i$ and $\alpha_j$ are homotopic (rel $*$). The maximum rank an arc-system can have is $6g-4$ since $6g-3$ curves will decompose $F$ into triangles so that no more curves can be added without violating nontriviality.

### i) Definition of A

Form a simplicial complex $A$ by taking a $k$-simplex $\langle\alpha_0,\ldots,\alpha_k\rangle$ for each rank-$k$ arc-system in $F$ and identifying $\langle\beta_0,\ldots,\beta_\ell\rangle$ as a face of $\langle\alpha_0,\ldots,\alpha_k\rangle$ if $\{\beta_i\} \subset \{\alpha_j\}$. By the remarks above, $A$ has dimension $6g-4$. Points in $A$ correspond to pairs $(\alpha,w)$ where $\alpha$ is an arc-system represented by curves $\alpha_0,\ldots,\alpha_k$ and $w$ is a collection of non-negative weights $w_0,\ldots,w_k$ on the $\alpha_i$ such that $w_0+\ldots+w_k = 1$.

ii) Definition of $A_\infty$

A family of curves is said to fill the surface $F$ if each component of its complement is simply connected. Define $A_\infty$ to be the subcomplex of $A$ consisting of all simplices $\langle\alpha_0,\ldots,\alpha_k\rangle$ such that $\{\alpha_i\}$ does not fill $F$. There is a natural action of $\Gamma^1_g$ on $A$ given by $[f]\cdot\langle\alpha_0,\ldots,\alpha_k\rangle = \langle f(\alpha_0),\ldots,f(\alpha_k)\rangle$ and since this action preserves $A_\infty$ it restricts to an action on $A - A_\infty$. The main theorem of this chapter is:

Theorem 2.1: There is a homeomorphism $\omega\colon T^1_g \to A - A_\infty$ which commutes with the action of the mapping class group $\Gamma^1_g$.

Sections 2 and 3 of this chapter are devoted to the proof of this theorem. Before we go on, however, we will discuss in some detail the case where $g = 1$.

iii) Example, $g = 1$

A single simple closed curve cannot fill the torus, but any arc-system with 2 or more curves (rank = 1 or 2) must do so. This means that $A_\infty$ contains only the vertices of $A$; these in turn may be identified with $\mathbb{Q} \cup \{\infty\}$: if $\{m,\ell\}$ is a basis for $\pi_1F$ corresponding to two non-homotopic simple closed curves meeting only at $*$, then any other simple closed curve $\alpha$ through $*$ represents $a_1m + a_2\ell$ in $\pi_1F$ with $a_1$ prime to $a_2$. Associating $a_1/a_2$ to $\alpha$ gives the bijection between $A_\infty$ and $\mathbb{Q} \cup \{\alpha\}$. It is easy to see that two curves with parameters $(a_1,a_2)$ and $(b_1,b_2)$ are isotopic to ones which meet only at $*$ exactly when $a_1b_2 - a_2b_1 = \pm 1$ We may therefore identify $A$ with $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, where $\mathbb{H}$ is the hyperbolic plane as in figure 2.1 (upper half plane model) or figure 2.2 (Poincare model).

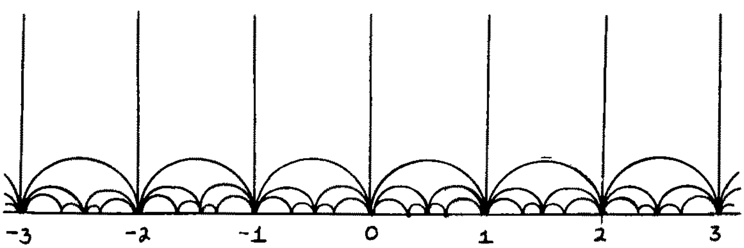
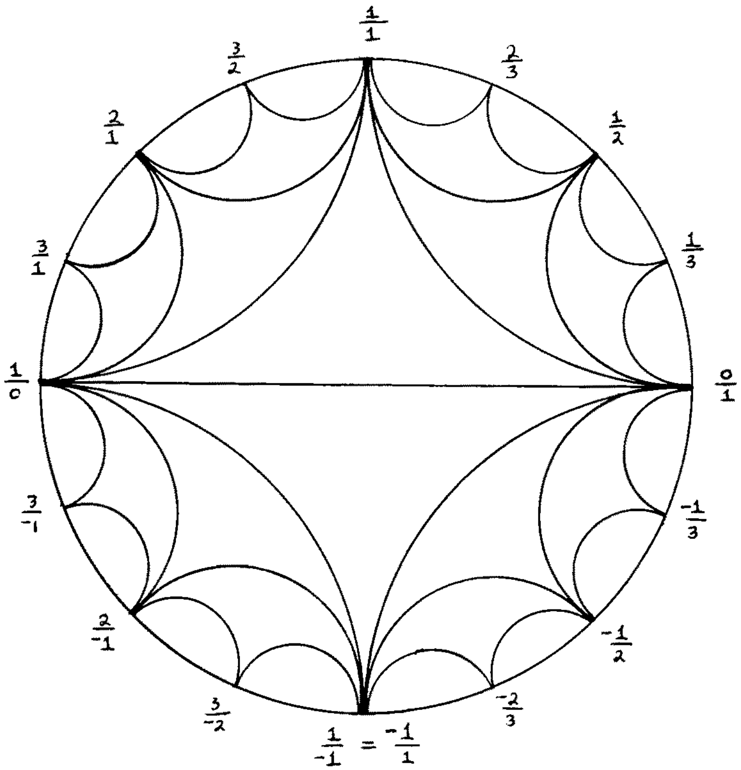



Figure 2.1




<u>Figure 2.2</u>

Since $T_1^1$ may be identified with $\mathbb{H}$, the picture gives an illustration of theorem 2.1.

It should be pointed out that the situation is much more complicated in higher genus. Since it takes $2g$ curves to fill the surface, $A_\infty$ contains the $2g-2$ skelton of $A$; however, it also contains pieces of the skeleta of $A$ up to codimension 2. The existence of these higher dimensional cells will turn out to be a red herring, however, because we will see in chapter 4 that $A_\infty$ has the homotopy type of a wedge of spheres of dimension $2g-2$.

## §2 The Conformal Point of View: Strebel Quadratic Differentials.

In this section we will give the Mumford-Strebel proof of Theorem 2.1. The main ingredient is the theory of quadratic differentials.

i) Definitions

Let $R$ be a Riemann surface with conformal structure $\{(U_i, z_i)\}$. An analytic (meromorphic) quadratic differential $\phi$ is a collection $\{\phi_i\}$ of analytic (meromorphic) functions in the $z_i$ which transform according to the rule

$$\phi_i(z_i)dz_i^2 = \phi_j(z_j)dz_j^2$$

whenever $U_i$ and $U_j$ intersect. The zeros and poles of $\phi$ are clearly independent of local representative, they are called the critical points of $\phi$. A pole of order $\geq 2$ will be called an infinite critical point while any other is called finite. Any non-critical point is of course called regular.

If $\gamma \subset R$ is any rectifiable curve, its $\phi$-length will be defined to be

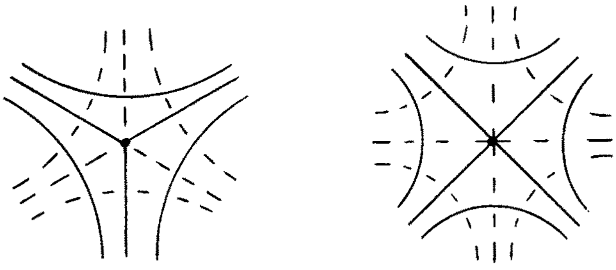
$$|\gamma|_\phi = \int_\gamma |\phi(z)|^{1/2} \; |dz|$$

and this quantity is easily seen to be finite unless $\gamma$ passes through an infinite critical point. The motivation for this definition comes from the fact that near any regular point $p$ of $\phi$ with conformal coordinate $z$ centered at $p$ the expression $w = \int \sqrt{\phi(z)}dz$ makes sense and gives a new parameter $w$ in which $\phi$ is identically equal to 1. Then $|\gamma|_\phi = \int_{\gamma'} |dw|$ is the ordinary Euclidean length of $\gamma'$ where $\gamma'$ is the image of $\gamma$ in the w-plane.

With the metric defined by $|\,|_\phi$ it is possible to talk about geodesics in R. The two types important to us are the horizontal and vertical trajectories of $\phi$: A smooth curve $\gamma$ is called horizontal if $\arg \phi(z)dz = 0$ along $\gamma$ and vertical if $\arg \phi(z) = \pi$ along $\gamma$. The horizontal trajectories of $\phi$ are the (unique) maximal horizontal curves through every regular point of $\phi$, with a similar definition for vertical trajectories. These trajectories give two perpendicular foliations of $R$ - critical points ; if we add the critical points (by taking the closure of the leaves) we obtain singular foliations $F_h$ and $F_v$ of R.

ii) Horocyclic Quadratic Differentials

Suppose $\phi$ is a quadratic differential on $R$ with exactly one pole of order 2 at $p \in R$ and no other poles. Suppose further that all the horizontal trajectories of $\phi$ which consist only of regular points are closed curves. In a neighborhood of the pole p there is a distinguished parameter $\zeta$ so that $\phi$ has the representation $\frac{c\, d\zeta^2}{\zeta^2}$. When $c < 0$ we will call such a quadratic differential horocyclic (this terminology will make sense after §3). The trajectory structure of $\phi$ near a zero is an n-pronged singularity, $n \geq 3$, illustrated in figure 2.3 for $n = 3, 4$. Around the pole the structure is as shown in figure 2.4. The horizontal trajectories are concentric circles around p, while the vertical ones are wheel-spokes emanating from p.



$n = 3$ $n = 4$

Figure 2.3

horizontal vertical

Figure 2.4

The reason we are interested in quadratic differentials is the following result of Strebel [S].

Theorem 2.2: Let R be a closed Riemann surface of genus g and p a point of R. Then there exists a horocyclic quadratic differential $\phi$ on R with its pole at p. The differential $\phi$ is unique up to multiplication by positive scalars.

Strebel proves this theorem by solving the following extremal mapping problem. Let $z$ be a coordinate at $p$ and consider the family of all conformal embeddings $\lambda: D\rho \to R$ where $D_\rho$ is a disk of radius $\rho$ in the $w$ plane, $\lambda(0) = p$ and $\left|\frac{dz}{dw}(0)\right| = 1$. Using a normal families argument he shows that a $\lambda$ exists which maximizes $\rho$ and it is unique up to multiplication by a constant. The inverse of $\lambda$ is then the distinguished parameter for a horocyclis quadratic differential $\phi$ whose nonsingular horizontal trajectories are the image under $\lambda$ of the circles $w$ = constant.

The map $\lambda$ has the added property that it extends to a map of the closed disk $\bar{\lambda}: \bar{D}_\rho \to R$, exhibiting $R$ as $\bar{D}\rho/\sim$ where $\sim$ is an identification on $\partial\bar{D}\rho$. More specifically, if $\{v_i\}$ is the inverse image under $\bar{\lambda}$ of the zeros of $\phi$, $\bar{D}\rho$ becomes a polygon with vertices $v_i$ on $\partial\bar{D}\rho$ and $\sim$ is an identification of the edges of this polygon (figure 2.5).

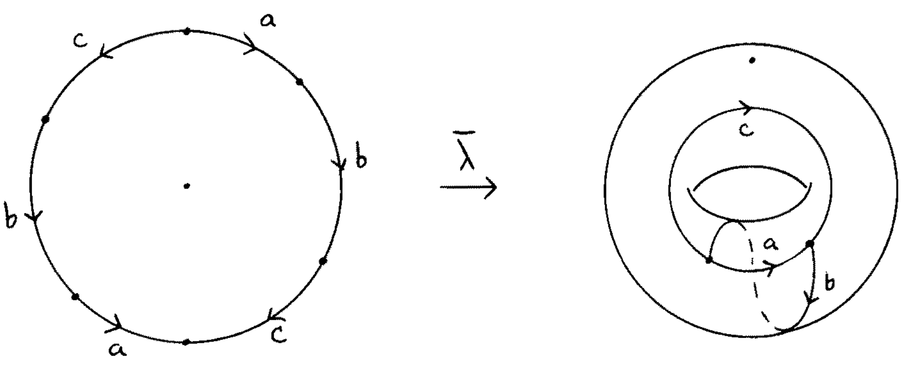



Figure 2.5

iii) Definition of $\omega: T_g^1 \to A - A_\infty$

We are finally ready to define the map $\omega: T_g^1 \to A - A_\infty$. Let $R$ be a Riemann surface, $p \in R$, and let $f: (R,p) \to (F,*)$ represent a marking of $R$; the triple $(R,p,[f])$ determines a point of $T_g^1$. By applying Theorem 2.2 we obtain a horocyclic quadratic differential on $R$ centered at $p$. Let $q_1,\ldots,q_n$ be the zeros of $\phi$ and let $\gamma_1,\ldots,\gamma_m$ be the singular leaves of the horizontal foliation $F_h$ determined by $\phi$. The $\gamma_i$ are closed horizontal arcs whose interiors consist of regular points and whose endpoints lie in $\{q_j\}$. Each nonsingular leaf of the vertical foliation is a loop based at $p$ which is perpendicular to $F_h$ and meets exactly one of the $\gamma_i$. Two such are homotopic (rel p) in $R - \{q_i\}$ if and only if they intersect the same $\gamma_i$; select one for each $\gamma_i$ and call it $\overline{\alpha}_i$. Let $\alpha_i = f(\overline{\alpha}_i)$; the collection $\{\alpha_i\}$ is then an arc-system in $F$. (An example is given in figure 2.6 .) If $\gamma_i$ has $\phi$-length $\ell_i$ and $\ell$ is the sum of the $\ell_i$, we let $w_i = \ell_i/\ell$ to obtain positive weights on the $\alpha_i$. The map $\omega$ is now defined by

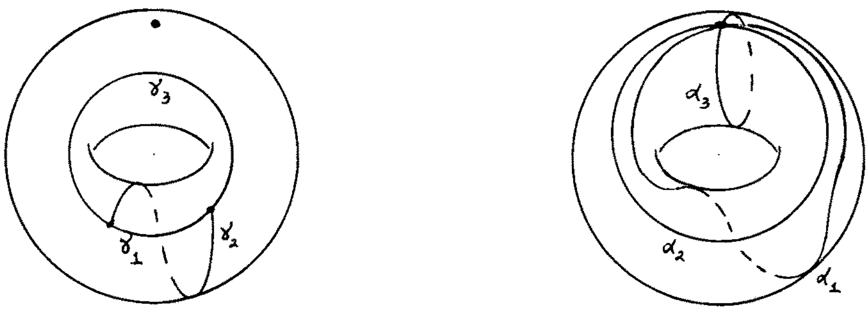



Figure 2.6

$\omega((R,p,[f])) = (\{\alpha_i\}, \{w_i\})$.

iv) Definition of $\eta = \omega^{-1}: A - A_\infty \to T_g^1$

To define the inverse $\eta$ of $\omega$, let $(\{\alpha_i\}, \{w_i\})$ represent a point of $A - A_\infty$ where the arc-system $\{\alpha_i\}$ has rank $k - 1$. The complement of the dual graph $\Omega$ to $\{\alpha_i\}$ is a 2-disk. Split $F$ along $\Omega$ to obtain a polygon $P$ with 2k-sides, an identification $\sim$ of the edges of $P$ and a surjective map $f_0: P \to F$ taking the center point $*$ of $P$ to $p$, the boundary of $P$ onto $\Omega$ and commuting with $\sim$ so that it induces a homeomorphism $P/\sim \to F$. Let the edges of $P$ have labels $\gamma_i^+, \gamma_i^-$ where $\gamma_i^+$ is paired to $\gamma_i^-$ by $\sim$ and $f_0(\gamma_i^+) = f_0(\gamma_i^-)$ is the edge of $\Omega$ which meets $\alpha_i$. We will now use the combinatorial data $(P,\{\gamma_i^\pm\},\{w_i\})$ to build a marked Riemann surface.

Begin with the closed unit disk $D = \{z \in \mathbb{C}: \|z\| \leq 1\}$ and choose a homeomorphism $f_1: P \to D$ taking $*$ to $0$ such that $f_1(\gamma_i^+)$ and $f_1(\gamma_i^-)$ have Euclidean length $\pi w_i$ for each $i$. The edges $\gamma_i^+$ and $\gamma_i^-$ map to arcs in $\partial D$ which we denote with the same symbols. Also let $\{v_i\}$ be the image under $f_1$ of the vertices of $P$. By identifying each $\gamma_i^+$ with $\gamma_i^-$ via the composition of the inversion $z \to 1/z$ with a rotation we obtain a Riemann surface $R_0$ with singularities $\{q_j\} = f_2(\{v_i\})$ where $f_2: D \to R_0$ is the quotient map (see figure 2.7).

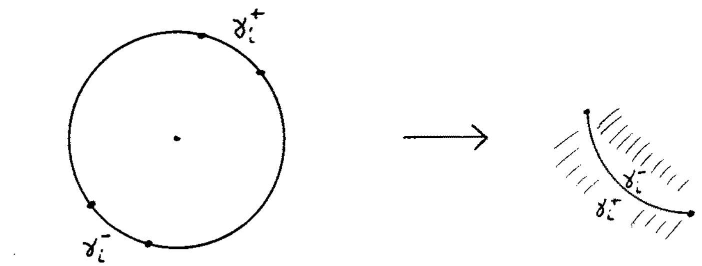



Figure 2.7

The total angle at each $q_j$ is $n_j\pi$ where $n_j$ is the number of $v_i$ which map to $q_j$. Since these angles are each commensurable with $2\pi$, a standard argument using branched coverings extends the complex structure to the $q_j$ and gives a closed Riemann surface $R$.

To define the marking $f: R \to F$ we merely complete the diagram

$$\begin{tikzcd} D \arrow[d, "f_2"'] & & P \arrow[ll, "f_1"'] \arrow[d, "f_0"] \\ R_0 \cong R \arrow[rr] & & F \end{tikzcd}$$

this can be done because the quotient maps $f_0$ and $f_2$ are compatible.

The expression $\phi = \frac{-dz^2}{z^2}$ defines a quadratic differential on $D$ with a double pole at the origin such that the $\phi$-lengths of $\gamma_i^+$ and $\gamma_i^-$ are their Euclidean lengths $\pi w_i$. This means $\phi$ is compatible with the identifications and therefore descends to $R$. The data which $\phi$ determines is clearly the original weighted arc-system $(\{\alpha_i\},\{w_i\})$, so the map

$$\eta((\{\alpha_i\},\{w_i\})) = (R,p,[f])$$

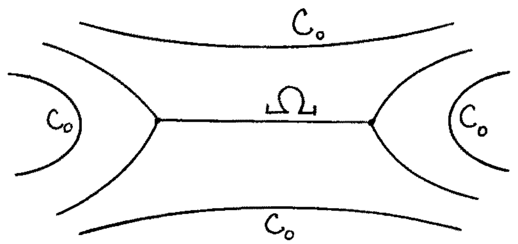
is the inverse of $\omega$ as required. □

## §3 The Hyperbolic Point of View

Because of the importance of theorem 2.1 we will present an alternate proof of it in this section. The original idea for this proof is due to Thurston and was explored by Mosher in [Mo]; the details were worked out by Bowditch and Epstein in [BE]. Throughout this section Teichmüller space will be treated as the space of marked hyperbolic structures (complete, finite area) on a surface of genus $g$ with 1 puncture.

i) <u>Definition of</u> $\omega$

First we define the map $\omega\colon T_g^1 \to A - A_\infty$. Let $X$ be a complete hyperbolic surface of finite area with one puncture and let $f\colon X \to F - \{*\}$ be a homeomorphism; $(X,[f])$ represents a point of $T_g^1$. Begin with an embedded horocycle $C_0$ around the puncture and let $X_0$ be $X$ with the open punctured disk enclosed by $C_0$ removed. The function $\rho\colon X_0 \to \{t \geq 0\}$ which associates to a point of $X_0$ its minimum distance from $C_0$ has level set $C_t$ at time $t$; $C_t$ is called the <u>quasi-horocycle</u> at distance $t$. For small $t$ $C_t$ is smooth but in general $C_t$ has singularities at those points of $X_0$ which are equidistant from 2 or more points of $C_0$. These singularities form a 1-dimensional connected graph $\Omega \subset X$ whose complement is a once-punctured disk (compare $\Omega$ in §2 and see figure 2.8).




<u>Figure 2.8</u>

The construction of the map $\omega$ is now similar to the construction in §2: the geodesics perpendicular to $C_0$ form a non-smooth singular foliation $F_v$ of $X_0$, where each nonsingular leaf of $F_v$ is the union of two geodesic segments joining $\Omega$ to $C_0$ (figure 2.9). For each edge $\gamma_i$ of $\Omega$ there is a

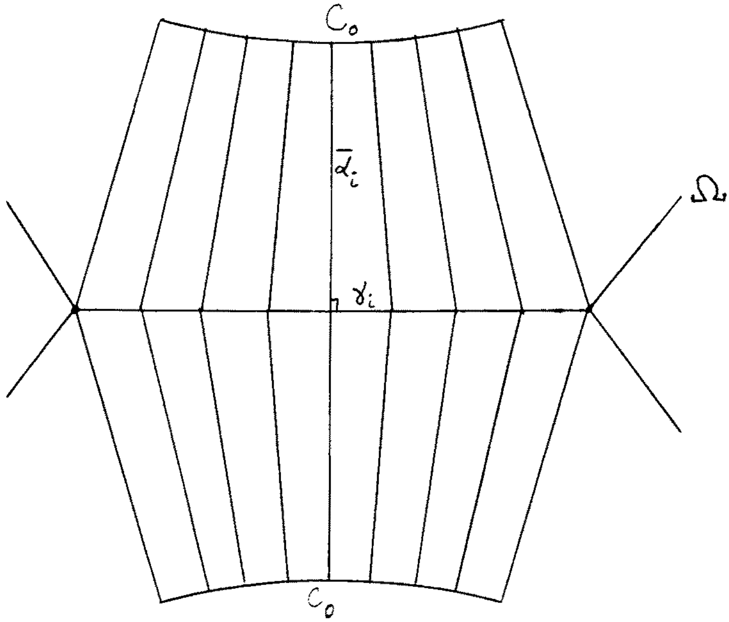



Figure 2.9

unique leaf $\overline{\alpha}_i$ of $F_v$ perpendicular to $\gamma_i$; $\overline{\alpha}_i$ is a smooth geogesic in $X_0$ and has a completion to a bi-infinite geodesic in X. Apply the map f and add on the point * to get an embedded arc $\alpha_i$ in F based at *; then $\{\alpha_i\}$ is the arc-system associated to (X,[f]). To determine the weights $w_i$ look at the leaves of $F_v$ which meet the singular points of $\Omega$. The intersection of these leaves with $C_0$ divides $C_0$ into segments each of which meets exactly one $\overline{\alpha}_i$. Some elementary hyperbolic geometry shows that the two segments that $\overline{\alpha}_i$ meets have the same length $\ell_i$; set $w_i = \ell_i/\ell$ where $\ell$ is the sum of the $\ell_i$ (figure 2.10). It is easy to check that $(\{\ell_i\},\{w_i\})$ does not depend on $C_0$ or the equivalence class of (X,[f]), so $\omega$ is well-defined by the formula

$$\omega((X,[f])) = (\{\alpha_i\},\{w_i\}).$$

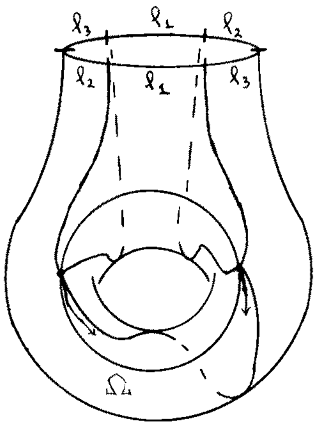



Figure 2.10

ii) Definition of $\eta = \omega^{-1}: A-A_\infty \to T_g^1$

The inverse map $\eta : A-A_\infty \to T_g^1$ is defined by constructing the strips between the singular leaves of the foliation $F_v$ explicitly (figure 2.10 ), and then glueing them together to build $X_0$. The problem will be to match the angles around the singular points of $F_v$ (= the vertices of $\Omega$).

We begin with a model for the combinatiorial structure of $X_0$, namely take $F_0 = F$ -(small open disk around *), $C_0 = \partial F_0$ and let $\bar{\alpha}_i = \alpha_i \cap F_0$. Also take the dual graph $\Omega$ with edges $\gamma_i$ and vertices $q_j$ and add to the picture embedded edges $e_j^k$ from $q_j$ to $C_0$, one for each homotopy class of paths from $q_j$ to $C_0$ in $F_0 - \{\bar{\alpha}_i\}$ (these correspond to the singular leaves of the foliation $F_v$). Splitting along $\{e_j^k\}$ divides $F_0$ into strips, each containing a single pair $\alpha_i, \gamma_i$.

It is clear that strips as illustrated in figure 2.11 exist in the hyperbolic plane with geodesic sides and horocyclic tops and bottoms (curvature $\equiv 1$). We choose $\{w_i\}$ very small, but in the correct projective class, and restrict to those strips which have length $w_i$ on top and bottom and are symmetric with respect to reflection through $\gamma$ (this was the case for the strips on $X$ between the singular leaves of $F_v$). There is then a 2-parameter family of such strips where the

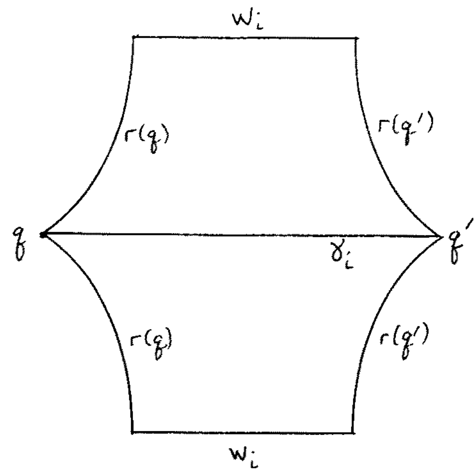



Figure 2.11

parameters are the distances $r(q)$ and $r(q')$ from the endpoints $q$, $q'$ of $\gamma_i$ to the top (or bottom). To glue the strips back together we only need to have constructed them so that the lengths of the edges $e_j^k$ from the vertex $q_j$ to $C_0$ are the same in each strip. By symmetry this distance can only depend on $q_j$, so any collection of $v$ positive numbers $r_1,\ldots,r_v (r_i = r(q_i)$ and $v =$ the number of vertices of $\Omega)$ gives a singular hyperbolic structure on $F_0$ with horocyclic boundry and with singularities at the vertices of $\Omega$. The total angle around $q_i$ will be denoted $\Theta_i$; the surface constructed above will be nonsingular when each $\Theta_i = 2\pi$. The problem is to show this occurs for an appropriate choice of $r_1,\ldots,r_v$.

Define $\mu : (0,\infty)^v \to (0,\infty)^v$ by setting $\mu(r_1,\ldots,r_v) = (\Theta_1,\ldots,\Theta_v)$. If $q_i$ has valence $n_i$ in $\Omega$, then since $n_i \geq 3$ the point $P = (2\pi,\ldots,2\pi)$ lies well in the interior of the convex hull $H$ of the $2^v$ points $(\varepsilon_1 n_1\pi,\ldots,\varepsilon_v n_v\pi)$, where $\varepsilon_i$ equals $0$ or $1$. We claim that if the $w_j$ are chosen small enough, then $\mathrm{im}(\mu)$ will contain enough of $H$ to contain $P$. In fact, as the $w_i \to 0$, $\mathrm{im}(\mu)$ will converge to an open set containing all of the interior of $H$. To see this, look at the upper portion (which is isometric to the lower portion) of a strip as in Figure 2.11. Label the top edge $\lambda_i$ and let $\psi(q)$, $\psi(q')$ be the interior angles at $q$, $q'$ respectively. There is a bound to how small $r(q)$ and $r(q')$ can be before $\gamma_i$ becomes tangent to $\lambda_i$; however, as we shorten $\lambda_i$ this bound goes to $0$. Therefore if we let $r(q)$ and $r(q')$ be near their minimum values, as $w_i$ goes to $0$ $\psi(q)$ and $\psi(q')$ approach $\pi/2$.

If follows that $(n_1\pi,\ldots,n_v\pi)$ lies in the limit of the closure of $\mathrm{im}(\mu)$ as all $w_i$ goes to $0$. Now suppose all the $r_i$ are short; pick $i_1,\ldots,i_s$ and keep $r_{i_1},\ldots,r_{i_s}$ short while the others are allowed to vary. It is clear that as $r_j \to \infty$, $\theta_j \to 0$. Furthermore, if $r(q)$ is kept short while $r(q')$ grows long in our strip, then $\psi(q)$ increases to a maximum value greater than $\pi/2$. From this it is not hard to see that the wall of $H$: $(t_1n_1\pi,\ldots,t_vn_v\pi)$ where $t_{i_1} = \ldots = t_{i_s} = 1$ and all other $t_j$ vary between $0$ and $1$, lies in the closure of $\mathrm{im}(\mu)$ as the $w_i$ go to $0$. On the other hand, any point in the dual wall where $t_{i_1} = \ldots = t_{i_s} = 0$ can be reached by letting $r_{i_1},\ldots,r_{i_s}$ go to $\infty$.

By now the picture is clear. Bowditch and Epstein complete the argument by showing that $\mu$ is proper and $1-1$. The intermediate value theorem then guarantees that for $w_i$ small enough, $(2\pi,\ldots,2\pi)$ lies in $\mathrm{im}(\mu)$.

## Chapter 3: The Borel-Serre Bordification of Teichmüller Space

In this chapter we begin the study of properties shared by Teichmüller space and Symmetric spaces by proving the existence of a Borel-Serre bordification for $\mathcal{T}_g^s$.

Let $G$ be a linear algebraic group defined over $\mathbb{Q}$ and let $X$ be the symmetric space $G/K$ where $K$ is maximal compact in $G$. In this situation Borel and Serre ([BS]) construct a manifold-with-corners $\overline{X}$ with $\overline{X} - \partial\overline{X} = X$ such that if $\Gamma \subset G_{\mathbb{Q}}$ is any arithmetic group, then the action of $\Gamma$ on $X$ extends to a properly discontinuous action of $\Gamma$ on $\overline{X}$ with $\Gamma\backslash\overline{X}$ compact. The manifold $\Gamma\backslash\overline{X}$ is then used to make cohomological computations for $\Gamma$.

Our goal will be the proof of:

Theorem 3.1: There exists a piecewise-linear manifold (or a smooth manifold with corners) $W$ which is contractible and has interior $\mathcal{T}_g^s$ such that the action of the mapping class group $\Gamma_g^s$ on $\mathcal{T}_g^s$ extends to a properly discontinuous action of $\Gamma_g^s$ on $W$ with $W/\Gamma_g^s$ compact.

Notice that $W/\Gamma_g^s$ is then a compactification of moduli space $\mathcal{M}_g^s$. In chapter 4 we will use $W$ to study the cohomology of $\Gamma_g^s$ and $\mathcal{M}_g^s$.

The first Borel-Serre bordification of $\mathcal{T}_g^s$ was discovered by Harvey ([Har]) who constructed $W$ by adding on copies of lower genus Teichmüller spaces (crossed with Euclidean spaces of the appropriate dimension) at infinity. We will give a description of Harvey's bordification in §2; however, it will be constructed inside Teichmüller space, not externally as Harvey did originally. This way it will be easy to see how $\Gamma_g^s$ act on $W$; the only difficult part will be identifying $\mathcal{T}_g^s$ with the interior of $W$.

Before we give Harvey's construction, however, we will use the triangulation of chapter 2 to find another description of $W$. The advantage of this second point of view is that $W$ is constructed combinatorially and this will allow us to analyze the homotopy type of $\partial W$ (chapter 4, Theorem 4.1). A second advantage of the combinatorial construction of $W$ is that when $s > 0$ $W$ retracts $\Gamma_g^s$-equivariantly onto a spine $Y \subset W$ of dimension $4g-3$. This implies:

Theorem 3.2: The moduli space $\mathcal{M}_g^s$ has the homotopy type of a finite cell-complex of dimension $4g-4+s$, $s > 0$. In particular,

$H_k(M_g^s) = 0, \qquad k > 4g-4+s, \quad s>0$ and

$H_k(M_g;\mathbb{Q}) = 0, \qquad k > 4g-5.$

It would be nice if we had an equivariant spine $Y \subset T_g$ of dimension $4g-5$ so that we could remove the $\mathbb{Q}$-coefficients in the theorem. Thurston has constructed a geometric candidate for $Y$ and shown how to retract $T_g$ onto it; but there is no combinatorial description of $Y$ available that would allow us to decide if it has the best possible dimension. We will describe this spine in §3.

## §1. The Combinatorial Construction

In this section we give the first construction of the Borel-Serre bordification $W$ of Teichmüller space. Just as in chapter 2 we will only consider the case $s = 1$ to keep the exposition simple (see [H3] for the general case). Recall that we have identified $A-A_\infty$ with $T_g^1$, we will work directly with $A$ in building $W$ and $Y$.

### i ) Definition of $\underline{W}$ and $\underline{Y}$

Let $A^o$ be the first barycentric subdivision of $A$. The complex $A^o$ has a vertex of _weight_ $\underline{k}$ for each rank-k arc-system $\langle\alpha_o,\ldots,\alpha_k\rangle$ in $F$ and an r-cell for each chain of $r + 1$ inclusions of arc systems. The subcomplex $A^o_\infty$ is defined similarly; we set $Y^o$ equal to the union of all the simplices of $A^o$ which have no face in $A^o_\infty$. (By simplex we always mean closed simplex unless stated otherwise.) The complex $Y^o$ is a spine for Teichmüller space, we will see shortly that it has dimension $4g-3$. Let $A^{oo}$ and $A^{oo}_\infty$ be the second barycentric subdivisions of $A$ and $A_\infty$ respectively. We define $W$ to be the collection of all simplices of $A^{oo}$ which have no face in $A^{oo}_\infty$. $W$ is a regular neighborhood of $Y^o$ and the group $\Gamma_g^1$ acts on the pair $(W,Y^o)$ with both $W/\Gamma_g^1$ and $Y^o/\Gamma_g^1$ compact.

### ii) Description of $\underline{Y}$

First we will study the spine $Y^o$. The key fact is that $Y^o$ is the first barycentric subdivision of the dual complex $Y$ of $A$. This allows us to describe $Y$ directly: $Y$ has a k-cell for each rank $6g-4-k$ arc-system which fills $F$ and the cell corresponding to $\{\alpha_i\}$ is a face of the cell corresponding to $\{\beta_j\}$ if $\{\beta_j\} \subset \{\alpha_i\}$. It is instructive to enumerate the low dimensional cells of $Y$: a 0-cell of $Y$ corresponds to a maximal arc-system in $F$, maximality means that the curves of the arc-system triangulate $F$. A 1-cell of $Y$ corresponds to an arc-system consisting of $6g-4$ curves; these curves cut F into 4g-4 triangles and 1 square. The 1-cell is attached to the two 0-cells which correspond to the two possible completions of the arc-system to a maximal one (figure 3.1). A 2-cell of Y corresponds to an arc-system with 6g-5 curves which cut F into either 4g-5 triangles and 1 pentagon, or 4g-6 triangles and 2 squares. The corresponding 2-cells of Y are illustrated in figure 3.2. This process continues until we reach the

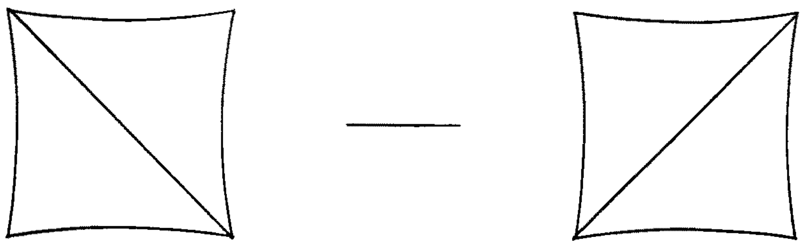


Figure 3.1

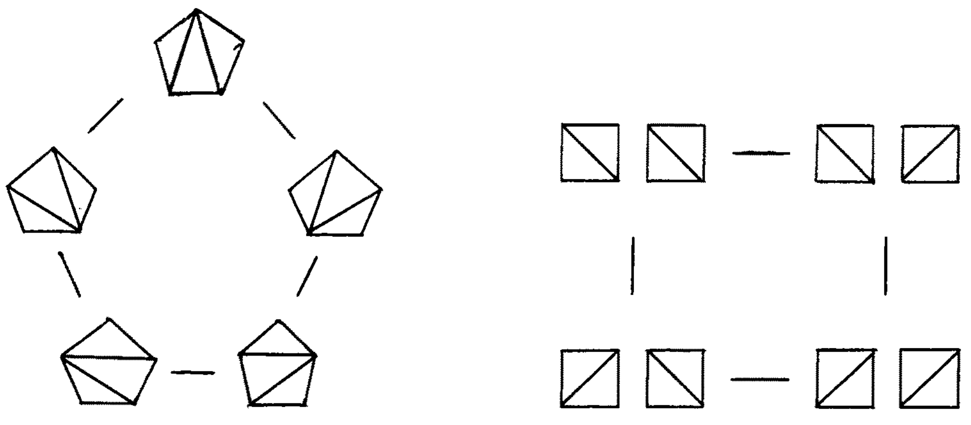


Figure 3.2

case of an arc-system consisting of $2g$ curves, since this is the minimum number of curves mecessary to fill $F$. In this situation the curves cut $F$ into a single $4g$-gon and the possible completions to higher rank arc-systems describe cells which fit together to form a $4g-4$ sphere. $Y$ has a $4g-3$ cell attached along this $4g-4$ sphere.

As a subset of $T_g^1$, $Y$ corresponds to those surfaces $R$ with basepoint $p$ such that the graph $\Omega$ given by the Strebel differential $\phi$ has "enough" edges of maximal $\phi$-length. By this we mean that if $\ell$ is the largest length of any edge of $\Omega$, then we may collapse the edges of $\Omega$ whose length is less than $\ell$ without changing the topological type of $R$. The dimension of the (open) cell of $Y$ which contains $(R,p,[f])$ is $6g-3$ minus the number of maximal length edges of $\Omega$.

The genus 1 case is illustrated in figure 3.3; the vertices of $Y$

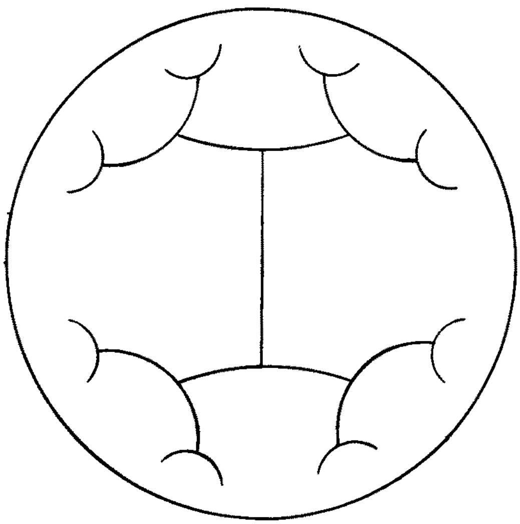


Figure 3.3

correspond to cut-systems with 3 curves while the edges correspond to cut-systems with 2 curves. In this picture we can see why the vertices of $A$ have no dual 2-cells: the link of a vertex is PL-homeomorphic to $\mathbb{R}$, not to $S^1$ as it would have to be. We can also see how to retract $T_1^1$ onto $Y$ as follows. The complement of $Y$ in $T_1^1$ has one component for each point of $A_\infty$ and $T_1^1$ acts transitively on these components. Furthermore, every point in $T_1^1 - Y$ lies on a unique geodesic ray which begins on $Y$ and tends to a point of $A_\infty$.

Flowing in along these geodesics rays collapses $T_1^1$ onto Y; by choosing the flow in one component of $T_1^1 - Y$ and using the action of $\Gamma_1^1$ to extend it to all of $T_1^1$ it may be made $\Gamma_1^1$-equivariant. This means the flow descends to moduli space $M_1^1$; here $M_1^1$ is homeomorphic to $\mathbb{R}^2$, but is collapsed onto $Y/\Gamma_1^1$ which is an interval.

The general case is directly analogous to the case where $g = 1$. The dimension of the complex Y is $4g-3$ because the cells in $A_\infty^o$ have links which are contracible, not spherical, so they have no dual cells. The Teichmüller space $T_g^1$ can be $\Gamma_g^1$-equivariantly retracted onto Y by "flowing" along straight lines (in the simplicial structure provided by A) away from $A_\infty$ onto Y. The construction of these simplicial flow lines is somewhat technical and will be omitted (the construction requires a proof that the entire complex A is contractible; see [H3]).

iii) Description of W

Now we pass to the study of the regular neighborhood W. Recall that the first barycentric subdivision $A^o$ of A has a vertex for each rank k arc-system in F and an r-cell for each chain of r + 1 inclusions of arc-systems. In symbols we write a k-cell of $A^o$ as

$$\beta_o \subset \beta_1 \subset \ldots \subset \beta_k$$

where each $\beta_i$ corresponds to an arc-system $\alpha_o^i, \ldots, \alpha_{n_i}^i$. The second barycentric subdivision $A^{oo}$ has a similar description; its k-cells are written

$$\gamma_o < \gamma_1 < \ldots < \gamma_k$$

where each $\gamma_i$ denotes a chain

$$\beta_o^i \subset \ldots \subset \beta_{m_i}^i$$

and $\gamma_i < \gamma_{i+1}$ means the chain for $\gamma_i$ is obtained from the chain for $\gamma_{i+1}$ by omitting some terms. The cell $\gamma_o < \ldots < \gamma_k$ lies in W if and only if the top term $\beta_{m_i}^i$ fill F for every i and it lies in $\partial W$ when, in addition, the bottom term $\beta_o^i$ does not fill F for every i. Using this description it is not hard now to check that W is a PL manifold with boundary by analyzing the links of the cells in $W-\partial W$ and in $\partial W$. For details of this in the general case we refer to reader to [H3]; here we will only deal with the case $g = 1$ where we can actually draw W.

A picture of $W$ is given in figure 3.4. To keep the illustration form getting too cluttered we have drawn only the outline of $W$ and the spine $Y$. To see the full

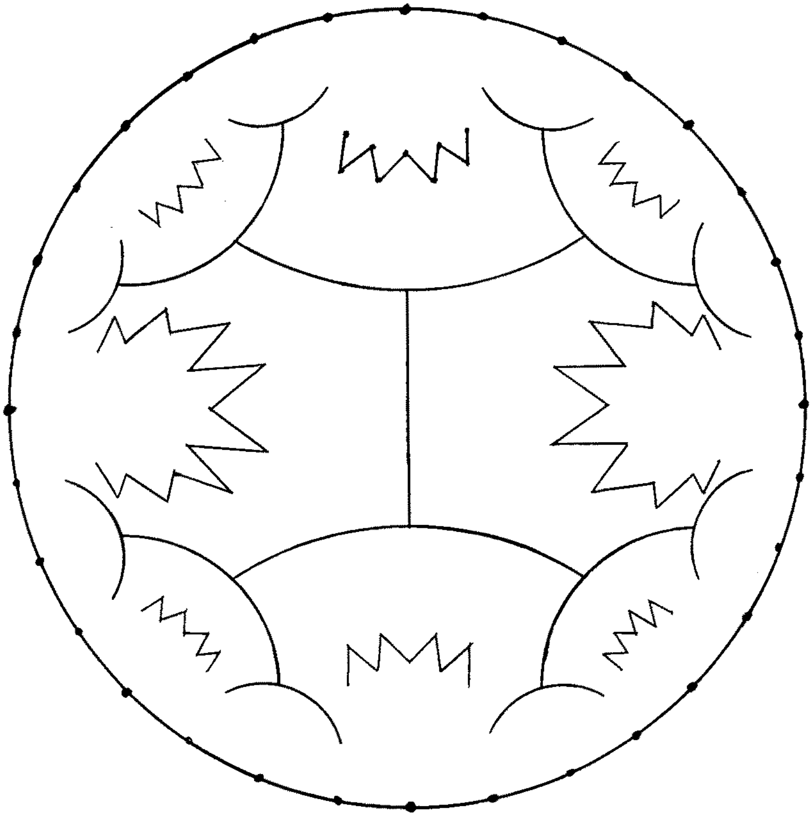


Figure 3.4

picture we have pulled out a simplex of $A$ in figure 3.5. The vertices of the simplex are labeled 0, 1, 2, corresponding to three curves $\alpha_o$, $\alpha_1$, $\alpha_2$ of a maximal arc-system and the vertices of $A^{oo}$ have been labeled using the notation introduced above. The part of $W$ contained in the simplex is shaded.

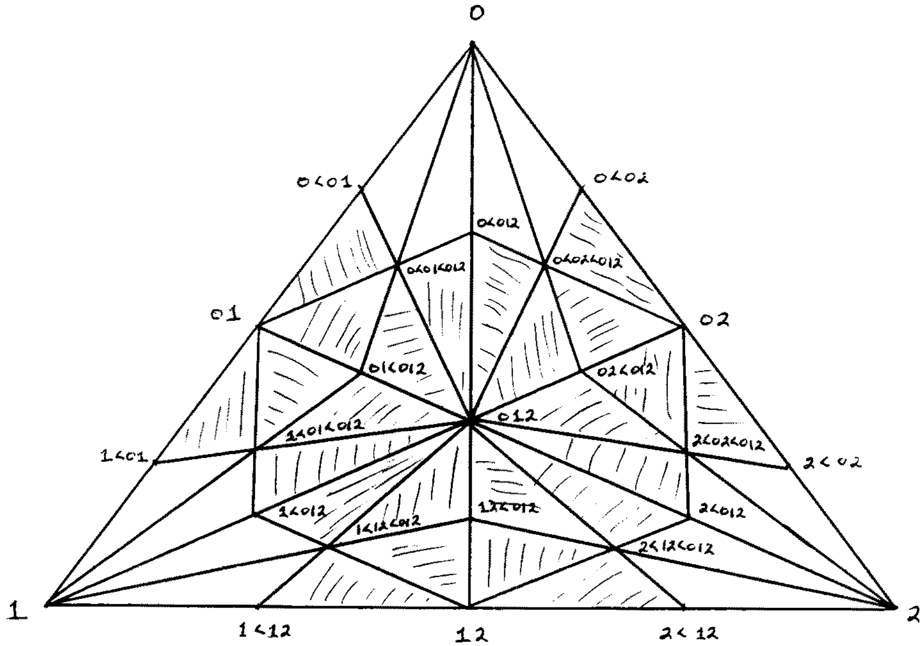



Figure 3.5

Now we come to the question: In what sense is $W$ a bordification of $T_g^1$? After all, $W$ was constructed inside $T_g^1$ and the action of $\Gamma_g^1$ is given by restriction not by extension. The solution to this is to look once again at the flow we mentioned earlier which collapses $T_g^1$ onto $Y$. This flow is $\Gamma_g^1$-equivariant and can be parameterized so that it collapses $T_g^1$ onto $Y$ at time $t = 1$ and provides a homeomorphism of $T_g^1$ into itself at any time $t < 1$. After some rescaling by linear coordinate changes in the simplices of $A^o$ we can arrange that the flow at time $t = 1/2$ gives a homeomorphism of $T_g^1$ onto $W - \partial W$.

If we remove $Y$ and add on $A_\infty$ the inverse of the flow is well-defined and retracts everything onto $A_\infty$. This provides the necessary tool to prove

Lemma 3.3: $\partial W$ is $\Gamma_g^1$-equivariantly homotopy equivalent to $A_\infty$.

We omit the details of the proof (see[H3]). This lemma will be used in chapter 4 to prove that the groups $\Gamma_g^s$ are virtual duality groups.

## §2 The Geometric Construction

In this section we will give a geometric construction of Harvey's bordification of Teichmüller space. This construction has the advantage over the one in §1 that it works for any value $s \geq 0$ but the disadvantage that it does not provide a spine of the correct dimension (compare §3). While these two constructions do not give the exact same object, they are nevertheless equivalent in the sense that for $s > 0$ there is a map from the combinatorial bordification to the geometric bordification which is a $\Gamma_g^s$-equivariant homeomorphism on $T_g^s$ and a homotopy equivalence on the boundary.

### i) Geodesic Length Functions

Let $F$ be a surface of genus $g$ with $s$ punctures and let $(X,f)$ represent a point of $T_g^s$ where $X$ is complete hyperbolic of finite area and $f\colon X \to F$ is a homeomorphism. For any simple closed curve $C \subset F$ there is a unique closed geodesic $\gamma \subset X$ such that $f(\gamma)$ is freely homotopic to $C$; $\gamma$ will be simple. We define the geodesic length function $\ell_C\colon T_g^s \to \mathbb{R}^+$ by associating to $(X,f)$ the length of $\gamma$. It is a standard result that the function $\ell_C$ is $C^\infty$ for every $C$.

Before we define $W$ we state (without proof) an elementary result from hyperbolic geometry.

Lemma 3.4: There exists a number $\varepsilon > 0$ such that if $X$ is any complete hyperbolic surface and $\gamma_1, \gamma_2$ are two simple closed geodesics of length $\leq \varepsilon$ then $\gamma_1$ and $\gamma_2$ are disjoint.

### ii) Definition of W

Fix $\varepsilon > 0$ as in the lemma and set:

$$W = \{(X,[f]) \in T_g^s\colon \ell_C((X,[f])) \geq \varepsilon \text{ for every simple closed curve } C \subset F\}$$

It is clear that the action of $\Gamma_g^s$ on $T_g^s$ restricts to $W$ where it is properly discontinuous. A theorem of Mumford says that $W/\Gamma_g^s$ is compact. We summarize the main facts about $W$.

Proposition 3.5: $W$ is a real analytic manifold-with-corners which admits a properly discontinuous action of $\Gamma_g^s$ such that $W/\Gamma_g^s$ is compact. There is a diffeomorphism $W - \partial W \to T_g^s$ which commutes with the action of $\Gamma_g^s$.

The interior of $W$ is $\overset{\circ}{W} = \{(X,[f])\colon \ell_C((X,[f])) > \varepsilon$ for every $C\}$ and $W - \overset{\circ}{W} = \partial W$ is $\{(X,[f])\colon \ell_C((X,[f])) = \varepsilon$ for at least one $C\}$. It is not hard to see that $\overset{\circ}{W}$ is open in $T_g^s$:

let $(X,[f]) \in \overset{\circ}{W}$ and suppose $\gamma_1,\ldots,\gamma_t \subset X$ have the same length and have shorter length than any other closed geodesics in $X$. Since the length spectrum on $X$ is discrete there is a neighborhood of $(X,[f])$ in $T_g^s$ so that if $(X',[f'])$ lies in this neighborhood and has shortest curves $\gamma_1',\ldots,\gamma_n'$ then each $f'(\gamma_i')$ is freely homotopic to some $f(\gamma_j)$. If we put $C_j = f(\gamma_j)$ then the intersection of this neighborhood with $\cap \ell_{C_i}^{-1}((\varepsilon,\infty))$ lies in $\overset{\circ}{W}$.

iii) <u>The Structure of $\partial W$; Definition of $Z_g^s$</u>

The first step in analyzing $\partial W$ is to define a new simplicial complex $Z_g^s$. The isotopy class of a family $\{C_0,\ldots,C_k\}$ of disjoint simple closed curves in $F$ will be called a <u>rank-k curve-system</u> if the curves satisfy the nontriviality condition that no $C_i$ is freely homotopic to a point and no $C_i$ and $C_j$ are freely homotopic to one another $(i\neq j)$. The complex $Z_g^s$ has a k-simplex $\langle C_0,\ldots,C_k\rangle$ is a face of $\langle C_0',\ldots,C_\ell'\rangle$ exactly when $\{C_i\} \subset \{C_j'\}$.
Since a curve system can contain at most $3g-3+s$ curves, $Z_g^s$ has dimension $3g-4+s$. We will show in Chapter 4 that for $s = 1$ $Z_g^1$ is homotopy equivalent to $A_\infty$ (this is also true for $s > 1$, but we have not given the definition of $A_\infty$ in this case; it can be found in [H3]).

The complex $Z_g^s$ will act as parameter space for $\partial W$: suppose $\mathcal{C} = \{C_0,\ldots,C_k\}$ is a rank-k curve-system in $F$. Define

$$T_{\mathcal{C}} = \{(X,[f]) \in T_g^s\colon \ell_{C_i}((X,[f])) = \varepsilon \text{ for every } i\}.$$

When $k$ is maximal $(=3g-4+s)$, the curves of $\mathcal{C}$ form a partition of $F$ so we may use them to define Fenchel-Nielsen coordinates on $T_g^s$. These coordinates provide an identification of $T_{\mathcal{C}}$ with $\mathbb{R}^{3g-3+s}$ since only the twist parameters can vary. When $k$ is not maximal we use the fact that $\mathcal{C}$ can be included in a maximal curve-system to see that $T_{\mathcal{C}}$ is again homeomorphic to Euclidean space. This time the lengths of the extra curves may vary so $T_{\mathcal{C}}$ has dimension $6g-5+2s-k$ (codimension $k+1$).

When the curve-system $\mathcal{C}$ has rank 0, the subspace $T_{\mathcal{C}}$ has codimension 1; these subspaces act like walls which cut $W$ out of $T_g^s$. Not all of the wall $T_{\mathcal{C}}$ will border $W$; define

$$T_{\mathcal{C}}^+ = \{(X,[f]) \in T_{\mathcal{C}}\colon \ell_{C'} > \varepsilon \quad \text{whenever} \quad C' \notin \mathcal{C}\}$$

and

$$\overline{T}_{\mathcal{C}}^+ = \{(X,[f]) \in T_{\mathcal{C}}\colon \ell_{C'} \geq \varepsilon \text{ for every } C'\}.$$

$$\partial W = \bigcup_{C \text{ rank } 0} \overline{T}_C^{\,+} = \coprod_C T_C^+$$

By lemma 3.4, if $C = \{C\}$ and $C' = \{C'\}$ are two rank-0 curve-systems then $\overline{T}_C^+$ intersects $T_{C'}^+$ if and only if $C$ and $C'$ are isotopic to disjoint curves, and in that case $\overline{T}_C^{\,+} \cap \overline{T}_{C'}^{\,+} = \overline{T}_{C\cup C'}^{\,+}$.

Each of the spaces $T_C$ may be thought of as a kind of Teichmüller space; let $F_C = F - C$ (we treat the new holes as punctures), then $T_C$ may be identified naturally with $\mathbb{R}^{k+1}$ × (Teichmüller space of $F_C$) where the first factor corresponds to the twist-parameters on the k+1 curves of $C$. Using flows like the one we will construct in iv, $T_C^+$ may be identified with $T_C$ and $\overline{T}_C^{\,+}$ with the Borel-Serre bodification of $T_C$. In particular, this tells us that $\overline{T}_C^{\,+}$ is contractible and proves:

<u>Lemma</u> <u>3.6</u>: $\partial W$ is $\Gamma_g^s$-equivariantly homotopy equivalent to the complex $Z_g^s$.

Our analysis of $W$ is now easy to complete. Each point of $T_C^+$ has a neighborhood $U$ in $T_g^s$ so that if $(X,[f]) \in U$, then the curves having the shortest length in $X$ lie in $C$. If we complete $C$ (arbitrarily) to a partition, the resulting Fenchel-Nielen coordinates identify $U \cap W$ with $\mathbb{R}^n \times Q$ where $N = 6g-7+2s-k$ and $Q$ is the upper orthant in $\mathbb{R}^{k+1}$ where $\ell_{C_i} \geq 0$ for every $C_i$ in $C$. These coordinates give $W$ the required structure of manifold-with-corners (see [BS]).

iv) <u>Deformation of</u> $T_g^s$ <u>onto</u> $W - \partial W$

To finish the proof of Theorem 3.6 we must show how to map $T_g^s$ onto $W - \partial W$ equivariantly. This will be done by constructing a flow on $T_g^s$ which moves the surfaces which have short curves in a direction which increases the lengths of those curves, but fixes the surfaces which do not have short curves. This is easy to do one curve-system at a time using Fenchel-Nielsen coordinates; it is considerably harder to find a $\Gamma_g^s$-equivariant flow which simultaneously increases the length of all short curves. The one we describe here is due to Scott Wolpert.

Begin by selecting $\varepsilon > 0$ such that $3\varepsilon$ satisfies the conditions of Lemma 3.4; that is, any two closed geodesics of length $\leq 3\varepsilon$ are disjoint. Choose a $C^\infty$ function $\phi\colon [0,\infty] \to [0,1]$ such that

$$\phi[0,\varepsilon] \equiv 1, \quad \phi[2\varepsilon,\infty) \equiv 0$$

and $\phi$ is decreasing on $[\varepsilon,2\varepsilon]$.

The flow will be constructed using the gradients of the functions $\ell_C$. To regulate the flow we need a lemma:

Lemma 3.7: There exist functions $\kappa_C \in C^\infty(\mathcal{T}_g)$, one for every isotopy class of simple closed curve in $F$, such that at each point $(X,[f])$ of $\mathcal{T}_g^s$, the formula

$$\sum_{C:\ell_C\le 3\varepsilon} \kappa_C(\text{grad } \ell_C)\, \ell_{C'} = \phi(\ell_{C'})$$

holds for every $C'$ such that $\ell_{C'}(X,[f]) \le 3\varepsilon$.

Proof: Let $C_1,\ldots,C_n$ be disjoint simple closed curves in $F$. Twisting on $C_i$ generates a vector field $t_{C_i}$ on $\mathcal{T}_g^s$ (see chapter 5).

The set $\{C_i\}$ can be completed to a partition of $F$ where the $t_{C_i}$ become coordinate vector fields in the resulting Fenchel-Nielsen coordinates; this implies the $t_{C_i}$ are linearly independent.

Define $g(\ ,\ )$ to be the Weil-Peterson metric on $\mathcal{T}_g^s$ and let $\omega$ be the corresponding 2-form $(\omega(v,w) = g(Jv,w)$ where $J$ is the complex structure on $\mathcal{T}_g^s)$. The duality formula (again see chapter 5) of Wolpert says

$$\omega(t_C,\ ) = g(it_C,\ ) = -d\ell_C$$

In particular, this means that grad $\ell_C = it_C$ and tells us that $(\text{grad } \ell_{C_i})d\ell_{C_j} = g(\text{grad } \ell_{C_i}, \text{grad } \ell_{C_j}) = g(it_{C_i}, it_{C_j}) = g(t_{C_i}, t_{C_j})$.

Linearly independence of the $t_{C_i}$ then implies that the matrix $((\text{grad } \ell_{C_i})d\ell_{C_i})$ is positive definite and therefore nonsingular.

From this we see immediately that the functions $\kappa_C$ exist at each point $(X,[f]) \in \mathcal{T}_g^s$ such that $\ell_C(X,[f]) \le 3\varepsilon$. It is not hard to check that $\kappa_C = 0$ if $\ell_C(X,[f]) > 2\varepsilon$ so setting $K_C = 0$ where it is not already defined extends $\kappa_C$ to all of $\mathcal{T}_g^s$. Using the fact that $\ell_C$ and $g(\ ,\ )$ are $C^\infty$ one proves easily that $\kappa_C$ is $C^\infty$. □

Now we can define the flow on $\mathcal{T}_g^s$. Let $V$ be the vector field whose value at $(X,[f])$ is given by

$$V = \sum_{C:\ell_C\le 3\varepsilon} \kappa_C(\text{grad } \ell_C).$$

The flow $\psi_t$ generated by $V$ is clearly $\Gamma_g^s$-invariant; the length functions satisfy $\frac{d\ell_C}{dt} = \phi(\ell_C)$. At time $t = \varepsilon$, $\psi_\varepsilon$ is the required differomorphism of $T_g^s$ onto $W - \partial W$. □

§3. Thurston's Spine for $T_g^0$

The method which we described in §1 for constructing a spine $Y \subset T_g^1$ may be adapted to work for any value of $s \geq 1$. It does not, however, work when $s = 0$ and there is at present no known combinatorial description of a spine for $T_g^0$. Thurston has given a geometric description of what ought to be the spine in this case and shown how to retract $T_g^0$ onto it $\Gamma_g^0$-equivariantly [T2]. Unfortunately we are unable to say whether it is best possible (lowest dimension) as we could in the earlier case.

The subspace $Y \subset T_g^0$ is easy to describe; it consists of all marked hyperbolic surfaces $X$ which have the property that the shortest closed geodesics $\gamma_1, \ldots, \gamma_n$ (length $\gamma_1 = \ldots =$ length $\gamma_n$ and all other closed geodesics are longer) fill the surface $X$. It is easy to see that two shortest closed geodesics can meet in at most one point so the number $n$ is bounded for fixed $g$. Notice that when $(X,[f])$ lies in $Y$ all the curves in $\{\gamma_i\}$ must be nonseparating, otherwise they could not fill $X$. This means that a flow on Teichmüller space which collapeses everything onto $Y$ cannot preserve shortest geodesics; any separating shortest geodesic must end up longer than some non-separating one.

Thurston constructs a flow of $T_g^0$ onto $Y$ as follows. Let $X$ be a hyperbolic surface with marking $f$ and suppose $\{\gamma_1, \ldots, \gamma_n\}$ is a set of simple closed geodesics in $X$ which do not fill $X$. For later use we set $\Lambda = \{f(\gamma_i)\}$. Choose a simple closed geodesic $\gamma \subset X$ such that each $\gamma_i$ is either disjoint from $\gamma$ or equals $\gamma$ and let $X_0$ be obtained by splitting $X$ along $\gamma$ ($X_0$ may be disconnected). The surface $X_0$ is naturally included in a complete hyperbolic surface $X_1$ as a deformation retract ($X_1$ has infinite area and flares out at $\partial X_0$). Any geodesic arc $\alpha$ properly imbedded in $X_0$ extends uniquely to a bi-infinite geodesic in $X_1$ which is embedded; split $X_1$ open along this geodesic line and insert a strip from $\mathbb{H}^2$ bounded by two nearby lines. The surface $X_0$ is replaced by a new surface with the property that any closed geodesic in $X_0$ which meets $\alpha$ is now longer. If we perform this operation on several arcs every geodesic in $X_0$ can be lengthened.

By matching the change in the lengths of the two curves of $\partial X_0$ we may reglue to form a new surface which has the property that any geodesic not meeting $\gamma$ has been lengthened (as has $\gamma$). The infinitesimal version of this construction defines a vector field on all of $T_g^0$ which we denote $V_\Lambda$. For completeness, when $\Lambda$ is a finite set of simple closed curves which fill $F$ we set $V_\Lambda = 0$.

The next step is to patch these vector fields together with a partition of unity. For any $\varepsilon > 0$ and any $\Lambda$ as above define $U_\Lambda$ to be the set of all $(X,[f]) \in T_g^0$ such that

$$\{C: \ell_C((X,[f])) \leq L_X + |\Lambda|\varepsilon\}$$

is exactly $\Lambda$, where $|\Lambda|$ denotes the cardinality of $\Lambda$ and $L_X$ is the length of the shortest geodesic in $X$. For $\varepsilon$ small enough the sets $U_\Lambda$ form a covering of $T_g^0$: choose $B$ such that for every $(X,[f])$ the number of closed geodesics of length less than $L_X + 1$ is not more than $B$ and let $\varepsilon < 1/B$. Let $\gamma_1,\ldots,\gamma_n$ be the geodesics on $X$ of length less than $L + 1$ with $\ell(\gamma_i) \leq \ell(\gamma_{i+1})$ for every $i$. If there is a first $i > 1$ such that $\ell(\gamma_i) > L + \varepsilon i$, set $\Lambda = \{f(\gamma_1),\ldots,f(\gamma_{i-1})\}$; otherwise set $\Lambda = \{f(\gamma_1),\ldots,f(\gamma_n)\}$. In either case $(X,[f])$ lies in $U_\Lambda$.

Now we choose a partition of unity $\{\lambda_\Lambda\}$ subordinate to $\{U_\Lambda\}$ and define a vector field $V_\varepsilon$ on $T_g^0$ by the formula

$$V_\varepsilon = \sum_\Lambda \lambda_\Lambda V_\Lambda.$$

This makes sense because if two sets $U_\Lambda$ and $U_{\Lambda'}$ intersect, then either $\Lambda \subset \Lambda'$ or $\Lambda' \subset \Lambda$. The flow generated by $V_\varepsilon$ is $\Gamma_g^0$-equivariant and deforms $T_g^0$ into

$$Y_\varepsilon = \{(X,[f]): \{C: \ell_C((X,[f])) < L_X + \varepsilon\} \text{ fills } F\}.$$

Letting $\varepsilon$ go to zero gives the desired retraction onto $Y$.

## Chapter 4: How Close is the Mapping Class Group to Being Arithmetic?

Let $G$ be an algebraic subgroup of $GL_n$ defined over $\mathbb{Q}$, $G_{\mathbb{Q}}$ the group of $\mathbb{Q}$-points of $G$ and $G_{\mathbb{Z}} = G_{\mathbb{Q}} \cap GL_n\mathbb{Z}$. A subgroup $\Gamma < G_{\mathbb{Q}}$ is called arithmetic if it is commensurable with $G_{\mathbb{Z}}$. The mapping class groups have many properties in common with the arithmetic groups. In the following list (taken from [Se 1]) $\Gamma$ denotes an arithmetic group or a mapping class group:

(1) $\Gamma$ is finitely presented,

(2) $\Gamma$ has only finitely many conjugacy classes of finite subgroups,

(3) $\Gamma$ is residually finite,

(4) $\Gamma$ is virtually torsion free,

(5) for any torsion free subgroup $\hat{\Gamma}$ of finite index in $\Gamma$ there exists a finite complex which is a $K(\hat{\Gamma},1)$,

(6) the virtual cohomological dimension (vcd) of $\Gamma$ is finite.

References for properties (1)-(6) when $\Gamma$ is arithmetic may be found in [Sl]. For the mapping class group see [Wa] and [HT[ for (1), [Mc] for (3); (4), (5) and (6) may be proven using the complex $Y$ of Chapter 3 (although all three are well-known and follow from more standard results).

In §1 of this chapter we will establish for the mapping class group the next property on Serre's list for arithmetic groups:

(7) $H^q(\Gamma;\mathbb{Z}[\Gamma])$ is zero except for a single value of $q(q = \mathrm{vcd}(\Gamma))$ for which it is a free $\mathbb{Z}$-module $I$; thus $\Gamma$ is a virtual duality group as defined by Bieri and Eckmann [Bi Ec].

This is Corollary 4.2 below. When $\Gamma$ is arithmetic in an algebraic group $G$ which is simple and has $\mathbb{Q}$-rank $r_{\mathbb{Q}} \geq 2$, the following holds:

(8) Every normal subgroup of $\Gamma$ is either finite index, or is finite and central.

This property fails for $\Gamma_g^s$, because the Torelli group $T_g^s = \mathrm{Ker}$ $(\Gamma_g^s \to Sp(2g;\mathbb{Z}))$ is normal and is neither finite nor finite index. In §2 we will use this to show $\Gamma_g^s$ is not arithmetic in $G$ when $r_{\mathbb{Q}}(G) > 1$. Concerning the possibility that $\Gamma_g^s$ might be arithmetic in $G$ with $r_{\mathbb{Q}}(G) = 1$, we will also see in §2 that vcd $\Gamma_g^s$ turns out to be the wrong value for this to be true $(g \geq 3)$. Thus we will have proven the result (announced first by Ivanov) that $\Gamma_g^s$ is not arithmetic for $g \geq 3$. In fact, one extra step will show that $\Gamma_g^s$ cannot be a lattice (discrete, cofinite volume) in any algebraic group $G$. The more general question of whether $\Gamma_g^s$ admits any faithful representation at all in an algebraic group remains open (compare [M^cCa], [L1] and [L2]).

## §1. The Mapping Class Group is a Virtual Duality Group

Let $\Gamma$ be an arithmetic group in a linear algebraic group $G$ and let $X$ be the symmetric space associated to $G$. If $\overline{X}$ denotes the Borel-Serre bordification of $X$, then the action of $\Gamma$ on $X$ extends to a properly discontinuous action on $\overline{X}$ (in fact the action of $G_{\mathbb{Q}}$ extends to $\overline{X}$). The boundary $\partial\overline{X}$ is homotopy equivalent to the Tits building of $G$, which in turn has the homotopy type of a wedge of $d$ spheres where $d = r_{\mathbb{Q}}(G) - 1$. In particular, $H_d(\partial\overline{X}) = I$, the Steinberg module of $G_{\mathbb{Q}}$, is free abelian of infinite rank (unless $\Gamma$ is co-compact, in which case it has rank 1). It follows that when $\Gamma$ is torsion free,

$$H^k(\Gamma;\mathbb{Z}\Gamma) \cong H_c^k(\overline{X}) \cong H_{n-k-1}(\partial\overline{X}) \cong \begin{cases} I, & k = n-d-1 \\ 0, & \text{otherwise.} \end{cases}$$

($n = \dim X$, $H_c^*$ = cohomology with compact supports), so that $\Gamma$ is a duality group in the sense of Bieri and Eckmann. This is equivalent to the statement that if $M$ is any $\mathbb{Z}\Gamma$-module, then

$$H^k(\Gamma;M) \cong H_{v-k}(\Gamma;M\otimes I)$$

where $v = n-d-1$ is the cohomological dimension of $\Gamma$. (The cohomological dimension of $\Gamma$ $(\mathrm{cd}(\Gamma))$ is the smallest integer $v$ such that there exists a $\mathbb{Z}\Gamma$-module $M$ with $H^v(\Gamma;M) \neq 0$.

To follow the outline above for the mapping class groups it remains only to show:

Theorem 4.1: The boundary $\partial W$ is homotopy equivalent to a wedge of $d$-spheres where $d = 2g-2$ when $s = 0$, $g > 0$, $d = 2g-3+s$ when $s > 0$, $g > 0$ and $d = s-4$, $g = 0$.

An equivalent result is that $\Gamma_g^s$ is a virtual duality group:

Corollary 4.2: If $\Gamma$ is any torsion-free subgroup of finite index in $\Gamma_g^s$, then

$$H^k(\Gamma;\mathbb{Z}\Gamma) \cong \begin{cases} I = H_d(\partial W), & k = n-d-1 \\ 0 & , \text{otherwise,} \end{cases}$$

where $n = \dim T_g^s = 6g-6+2s$. Thus

$$H^k(\Gamma;M) \cong H_{v-k}(\Gamma;M\otimes I)$$

with

$$v = n-d-1 = \begin{cases} 4g-5 & s = 0 \text{ and } g > 1, \\ 4g-4+s & s > 0 \text{ and } g \geq 1, \\ 1 & s = 0,\ g = 1, \\ s-3 & s > 2,\ g = 0. \end{cases}$$

The integer $v$ is the virtual cohomological dimension (vcd) of $\Gamma_g^s$; that is, $v$ is the cohomological dimension of any torsion free subgroup of finite index in $\Gamma_g^s$.

i) Reduction to the case where $s = 0$.

Let

$1 \to A \to B \to C \to 1$

be an exact sequence of torsion free groups. When $A$ and $C$ are duality groups, Bieri and Eckmann prove that $B$ is also a duality group, $cd(B) = cd(A) + cd(C)$ and the dualizing module $I_B$ is isomorphic to $I_A \otimes I_C$. We apply this to the exact sequence (derived from (S1) in Chapter 1):

$$1 \to \pi_1(F_g^s) \to \hat{\Gamma}_g^{s+1} \to \hat{\Gamma}_g^s \to 1 \qquad (S_2)$$

where $\hat{\Gamma}_g^s$ is torson free, finite index in $\Gamma_g^s$ and $\hat{\Gamma}_g^{s+1} = \eta^{-1}(\hat{\Gamma}_g^s)$ has the same properties in $\Gamma_g^{s+1}$. This means that for fixed $g$ if we show that $\Gamma_g^0$ is a virtual duality group then the same will hold for $\Gamma_g^s$, $s > 0$. It also means that for fixed $g > 1$ Theorem 4.1 need only be verified for $s = 0$.

We will prove Theorem 4.1 by induction on $g$. To start the induction we note that the cases $g = 0, 1,$ and $2$ follow from more elementary results: For $g = 0$, $\Gamma_0^3$ is trivial (as are $\Gamma_0^0$, $\Gamma_0^1$ and $\Gamma_0^2$) so induction on $s$ using $(S_2)$ establishes Corollary 4.2 and therefore Theorem 4.1 for all $\Gamma_0^s$, $s \geq 3$. For $g = 1$, $\Gamma_1^0 \cong \Gamma_1^1 \cong SL_2\mathbb{Z}$ which is arithmetic; applications of $(S_2)$ with $s \geq 1$ give the general case of $\Gamma_1^s$. For $g = 2$, it is known that $\Gamma_2^0$ is an extension of a finite group by $\Gamma_0^6$. Any subgroup of finite index in $\Gamma_2^0$ which is torsion free will map isomorphically onto a subgroup of $\Gamma_0^6$ having the same properties. This proves 4.2 and therefore 4.1 for $\Gamma_2^0$, thus for all $\Gamma_2^s$.

From here on we will assume $g \geq 2$. In view of the results of Chapter 3, §2 and the comments above, proving Theorem 4.1 is equivalent to showing $Z_g^0 \simeq \vee S^{2g-2}$.

ii) $Z_g^0$ is $2g-3$ connected.

The subcomplex $A_\infty \subset A$ was defined to be the simplices of $A$ corresponding to arc-systems which do not fill the surface $F$. It takes $2g$ curves to fill $F$ so $A_\infty$ contains the $2g-2$ skeleton of $A$. It can be shown that $A \simeq W$ so $A$ is contractible and $A_\infty$ is $2g-3$ connected.

By forgetting the base point we may define a map $\eta : Z^1_g \to Z^0_g$. This map has a right inverse $\omega : Z^0_g \to Z^1_g$ defined as follows. Choose a hyperbolic metric on $F$ so that all simple closed curves in $F$ are represented by geodesics. Select a point $p \in F$ such that no simple closed geodesic passes through $p$; this gives a map from the curve systems in $F$ to the curve systems in $F-p$ and defines $\omega$. Clearly $\eta \circ \omega = 1$, so $\eta_*$ is surjective on each $\pi_k$.

By the remarks above the following will imply $Z^1_g$ and $Z^0_g$ are $2g-3$ connected.

Lemma 4.3: The complexes $A_\infty$ and $Z^1_g$ are (equivalently) homotopy equivalent.

Proof: If $C \subset F_g - \{p\}$ is any simple closed curve, define $A_C = \{\langle\alpha_0,\ldots,\alpha_k\rangle : \alpha_i \cap C'$ is empty for some $C'$ isotopic to $C\}$. Clearly $A_C \subset A_\infty$ and $\bigcup_C A_C = A_\infty$. $A_C$ may be identified with the arc-system complex of the component of $F-C$ which contains $p$; it is therefore contractible. Finally, for any $C_1,\ldots,C_k$ (disjoint or not), $\bigcap_{i=1}^{k} A_{C_i}$ is either empty or contractible. These facts mean that $A_\infty \simeq N$, the nerve of the cover $\{A_C\}$.

Let $C_0,\ldots,C_k$ be a collection of $k+1$ simple closed curves (not necessarily disjoint, but nontrivial and nonisotopic) in $F-p$. Isotope the curves to have minimal geometric intersection and let

$F(C_0,\ldots,C_k) = F -$ (open regular neighborhood of $\cup C_i$). Set $F_0(C_0,\ldots,C_k)$ equal to the component of $F(C_0,\ldots,C_k)$ that contains $p$. The nerve $N$ may be described combinatorially as the complex which has a $k$-cell $[C_0,\ldots,C_k]$ for each isotopy class of simple closed curves $C_0,\ldots,C_k$ such that $F_0(C_0,\ldots,C_k)$ is not simply connected and has the usual face relations given by inclusion. From this description it is clear that $Z^1_g$ may be identified with a subcomplex of $N$.

Let $N^o$ and $Z^o$ denote the first barycentric subdivisions of $N$ and $Z^1_g$ respectively; we define a retraction $r : N^o \to Z^o$ as follows. Given a collection $C_0,\ldots,C_k$, set $\partial' F(C_0,\ldots,C_k)$ equal to the curve-system in $F$ obtained from $\partial F(C_0,\ldots,C_k)$ by omitting null-homotopic curves and redundancies. Let $B(C_0,\ldots,C_k)$ denote the barycenter of the cell $[C_0,\ldots,C_k]$ of $N$ and, when $C_0,\ldots,C_k$

form a curve system, the barycenter of the cell $\langle C_0,\ldots,C_k\rangle$ of $Z_g^1$. Now set $r(B(C_0,\ldots,C_k)) = B(\partial' F(C_0,\ldots,C_k))$ and extend linearly. Clearly $r|Z^o$ is the identity. To see that $r \simeq 1$ triangulate $N^o \times I$ by identifying $N^o \times 0$ and $N^o \times 1$ with $N^o$ and joining each

$$[C_0] < [C_1] < \ldots < [C_k]$$

in $N^o \times 1$ to each chain

$$[C_0'] < \ldots < [C_\ell']$$

in $N^o \times 0$ whenever $C_\ell' \subset C_0$. Since $\partial' F(C_0,\ldots,C_k)$ is disjoint from each $C_i$ it is clear that $1|N^o \times 0 \coprod r|N^o \times 1$ extends linearly to $N^o \times I$, providing the needed homotopy.

iii) $Z_g^0$ <u>has the homotopy type of a</u> $2g-2$ <u>complex</u>

To finish the proof of Theorem 4.1 we must now show that $Z_g^0$, which has dimension $3g-4$, has the homotopy type of a complex of dimension $2g-2$ (recall we are assuming $g \geq 2$). Let $Z^o$ denote the first barycentric subdivision of $Z_g^0$, we build $Z^o$ piece by piece, including the subcomplexes spanned by vertices of descending weight.

Let $X_k$ be the subcomplex of $Z^o$ consisting of simplices whose vertices have weight $\geq k$. Assume Theorem 4.1 for all $Z_h^s$ with $h < g$ and assume that $X_{k+1}$ has been shown to be homotopy equivalent to a complex of dimension $\leq 2g-2$. $X_k$ is obtained from $X_{k+1}$ by adding the simplices which have exactly one vertex of weight $k$ (since no two vertices of a simplex of $Z^o$ have the same weight). Let $v$ be a vertex of $X_k - X_{k+1}$ corresponding to the curves $C_0,\ldots,C_k$. If $F^1,\ldots,F^t$ are the components of the surface obtained by splitting $F$ along $\{C_i\}$ and $Z(F^i)$ denotes the complex of curve-systems on $F^i$, then the link of $v$ in $X_{k+1}$ is easily identified with the join of the $Z(F^i)$. If $F^i$ has genus $g_i$ and $r_i$ boundary components, then $g = \sum g_i + k - t + 2$ with $g_i < g$ for all $i$ and $\sum r_i = 2k+2$. Since each $Z(F^i)$ is by assumption homotopy equivalent to a complex of dimension $\leq 2g_i + r_i - 3$, the link of $v$ is homotopy equivalent to one of dimension $\leq \sum(2g_i+r_i-3) + (t-1) = 2g-3$. This verifies that $X_k$ is homotopically of dimension $\leq 2g-2$ for each $k$; since $X_0 = Z^o$ the proof of Theorem 4.1 is now complete.

## §2. The Mapping Class Group is not Arithmetic

Suppose that the group $\Gamma = \Gamma_g^s$ is an arithmetic subgroup of the linear algebraic group $G$. Then Corollary 4.2 can be combined with the results of Borel and Serre to see that if $n$ is the dimension of the symmetric space $X$ associated to $G$, then $n - r_{\mathbb{Q}}(G) = \mathrm{vcd}(\Gamma)$. The number $r_{\mathbb{Q}}(G)$ can be computed directly from $\Gamma$; it is the maximal rank of an abelian subgroup of $\Gamma$. In [BLM] this is shown to be $3g-3+s$, so $n = 7g-8$ when $g > 1$, $s = 0$ and $n = 7g-7+2s$ for $g > 0$. Thus Teichmüller space cannot be identified with $X$ when $s \geq 0$, $g > 2$ or $s > 0$, $g = 2$.

This still does not mean $\Gamma$ cannot be arithmetic; to prove it cannot be we argue as follows (this argument was shown to us by Bill Goldman). Suppose first that $G$ has rank 1 so that $X$ is hyperbolic space. Two elements of infinite order in the group of isometries of $X$ then commute if and only if they have the same fixed point set on the sphere at $\infty$. This means that commuting is an equivalence relation on the elements of infinite order in $\Gamma$. For the mapping class group this is absurd: simply take curves $C_0$, $C_1$ and $C_2$ with $C_0$ disjoint from $C_1$ and $C_2$ but $C_1$ intersecting $C_2$ (say in one point). If $\tau_i$ denotes the Dehn twist on $C_i$, then $\tau_0$ commutes with $\tau_1$ and $\tau_2$ but $\tau_1$ and $\tau_2$ do not commute.

An alternative argument for the rank 1 case goes as follows. If $\Gamma$ is cocompact, the dualizing module $I$ will be isomorphic to $\mathbf{Z}$. It is not hard to show this is not true; on the contrary, it has infinite rank. If $\Gamma$ is not cocompact it must at least be cofinite volume. The V-manifold $X/\Gamma$ will have cusps, modeled on horoballs/$\Gamma$. The boundary horosphere $\Sigma$ of one of these horoballs has a Euclidean structure and since $\Sigma/\Gamma$ is compact it is covered by an $n-1$ dimensional torus. This implies $\Gamma$ has an abelian subgroup of rank $n-1$ which is impossible by the dimension count above.

Next consider the case where $G$ has rank $\geq 2$. As mentioned earlier, property (8) and the existence of the Torelli group say that $G$ cannot be simple. Suppose then that $G$ is semisimple; a stronger version of (8), also proven by Margulis, says that if $\Gamma$ is an irreducible lattice in $G$, then once again any normal subgroup in $G$ is either finite (and central) or finite index. This means $\Gamma$ must be reducible, so there is a subgroup of finite index in $\Gamma$ which is a direct product of infinite groups. An analysis of the centralizers of the elements of $\Gamma$ (compare [M$^c$Car]) shows this is not possible. Finally, if $G$ is not semisimple we need to look at solvable subgroups of $\Gamma$. A theorem of Birman, Lubotsky and M$^c$Carthy [BLM] says that every solvable subgroup of $\Gamma$ is virtually abelian. Such a subgroup will not be normal in $\Gamma$, so the map $G \to G/\mathrm{rad}(G)$ imbeds $\Gamma$ in the semisimple group $G/\mathrm{rad}(G)$; this is impossible.

# Chapter 5. The Weil-Petersson Geometry of Teichmüller Space

In this chapter we will describe some results of Scott Wolpert on the geometry of Teichmüller space ([Wol 1]-[Wol 5]). The theme we will be following is the counterpart to that of chapter 4, namely: How close is Teichmüller space to being a symmetric space? We will translate this into the question: How much of the formal geometry of a symmetric space does Teichmüller space have? The metric we will study on $T_g^s$ is the Weil-Petersson metric; it is Kähler and we will see that its Hermitian and symplectic geometry arise from the hyperbolic geometry of the surface. The metric is also invariant under the action of the mapping class group $\Gamma_g^s$; on $M_g^s$ is it not complete, rather it admits a continuous extension to the Deligne-Mumford compactification $\overline{M}_g^s$ ([Mas]). The corresponding Kähler form $\omega_{wp}$ extends to $\overline{\omega}_{wp}$ on $\overline{M}_g^s$; in §3 we will show how Wolpert uses $\omega_{wp}$ to give an analytic proof that $\overline{M}_g^s$ is projective.

## §1. The Symplectic Geometry of the Weil-Petersson Form

We begin with the definition of the Weil-Petersson metric on $T_g^s$. Let $R$ be a Riemann surface and let $\lambda$ be the hyperbolic line element on $R$. Teichmüller space is a complex manifold and the holomorphic cotangent space at $R$ may be identified with $Q(R)$, the space of integrable holomorphic quadratic differentials on $R$ (tensors of type $dz \otimes dz$). If $\phi,\psi \in Q(R)$, the Hermitian product

$$\langle\phi,\psi\rangle = \frac{1}{2}\int_R \phi\,\overline{\psi}\,\lambda^{-2}$$

defines the Weil-Petersson metric at $R$. This metric is Kähler; its corresponding Kähler form is denoted $\omega_{wp}$. The first thing we will do is to give Wolpert's formula for $\omega_{wp}$ in terms of Fenchel-Nielsen coordinates.

### i) $\omega_{wp}$ in Fenchel-Nielsen Coordinates.

Let $C = \{C_1,\ldots,C_n\}$ be a maximal curve system in $F$ and let $(\tau_i,\ell_i)$ be the corresponding Fenchel-Nielsen coordinates for $T_g^s$.

**Theorem 5.1** $\omega_{wp} = \sum_i d\ell_i \wedge d\tau_i$.

Several things about this statement are surprising. First of all, the Weil-Petersson metric is Kähler, while Fenchel-Nielsen coordinates are only real analytic; the simplicity of the formula is therefore unexpected. Secondly, the Weil-Petersson metric is invariant under the action of the mapping class group, so $\sum d\ell_i \wedge d\tau_i$ must be also. Actually, Theorem 5.1 says more since it shows that $\sum d\ell_i \wedge d\tau_i$ is independent of the curve-system $C$. By contrast the change of coordinates from one curve-system to another can be quite complicated.

Theorem 5.1 is a consequence of the duality formula which was used already in chapter 3. To state this we must first define the Fenchel-Nielsen twist vector fields $t_C$. Let $(X,[f])$ represent a point of Teichmüller space with $X$ hyperbolic and let $\alpha$ be the closed geodesic on $X$ representing the free homotopy class $f^{-1}(C)$ where $C \subset F$ is a nontrivial simple closed curve. Cut $X$ along $\alpha$, rotate one side of the cut and then reglue the sides. The hyperbolic structure in the complement of the cut extends naturally to a hyperbolic structure on the new surface. Varying the amount of rotation gives a flow on $T_g^s$ and the twist vector field $t_C$ (sometimes denoted $t_\alpha$) is the tangent vector field to this flow. We will always normalize $t_C$ so that the hyperbolic displacement of two points on opposite sides of the geodesic $\alpha$ increases at unit speed (thus for example a full rotation about $\alpha$ occurs at time $t$ = length of $\alpha$).

Let $C \subset F$ be any nontrivial simple closed curve. The Duality Formula now states:

Theorem 5.2: $\omega_{wp}(t_C, \ ) = -d\ell_C$.

ii) Proof of Theorem 5.2

To prove this formula we introduce $H(R)$, the space of harmonic Beltrami differentials on $R$. An element $\mu \in H(R)$ is a tensor of type $\frac{\partial}{\partial z} \otimes d\bar{z}$ and is harmonic with respect to the Laplace-Beltrami operator for the hyperbolic metric on $R$. The holomorphic tangent space at $R$ may be identified with $H(R)$ and the Weil-Petersson metric on $T_g^s$ has the dual expression

$$\langle\mu,\nu\rangle = \int_R \mu\bar{\nu}\,\lambda^2,$$

$\mu, \nu \in H(R)$ and $\lambda$ the hyperbolic line element as before.

The underlying Riemannian structure to $\langle\ ,\ \rangle$ is of course given by the symmetric tensor

$$g(\mu,\nu) = 2\mathrm{Re}\langle\mu,\nu\rangle,$$

and the Weil-Petersson Kähler form is defined by the equation

$$\omega_{wp}(\mu,\nu) = g(J\mu,\nu)$$

where $J$ is the complex structure on $\mathcal{T}_g^s$. This means that $\omega_{wp}(t_C, \ )$ is the Riemannian dual of $Jt_C$.

The next step is to write down formulas for $t_C$ and $d\ell_C$ in terms of Poincaré series. Let $X = \mathbb{H}^2/\Gamma$ where $\Gamma$ is Fuchsian and let $\alpha$ be the simple closed geodesic representing $C$ in $X$. When $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents $\alpha$ in $\Gamma < PSL_2\mathbb{R}$, define $\Omega_A(\zeta) = (tr^2A-4)(c\zeta^2+(d-a)\zeta-b)^{-2}$. If $\langle A \rangle$ denotes the infinite cyclic group generated by $A$, we set

$$\Theta_C = \sum_{B \in \Gamma/\langle A\rangle} \Omega_{B^{-1}AB};$$

this is a relative Poincaré series and it converges uniformly and absolutely on compact sets. A formula of Gardiner [G] expresses $d\ell_C$ in terms of $\Theta_C$: if $\mu \in H(X)$ represents a tangent vector, then

$$\mathrm{Re}(d\ell_C(\mu)) = \frac{2}{\pi}\,\mathrm{Re}\int_X \mu\,\Theta_C.$$

In particular, we may write $-d\ell_C = -\frac{2}{\pi}\Theta_C$. On the other hand, Wolpert uses the Bers embedding of $\mathcal{T}_g^s$ into the vector space of $\Gamma$ invariant holomorphic quadratic differentials to show that

$$t_C = \frac{i}{\pi}(\mathrm{Im} z)^2\,\overline{\Theta}_C$$

[Wol 1]. Therefore $Jt_C = -\frac{1}{\pi}(\mathrm{Im} z)^2\,\overline{\Theta}_C$ so $-d\ell_C$ is also the Riemannian dual to $Jt_C$. Theorem 5.2 follows. □

iii) Proof of Theorem 5.1

Next we derive Theorem 5.1 from 5.2. First note that if $\{C_i\}$ is a partion of $F$ giving Fenchel-Nielsen coordinates $\{\tau_i, \ell_i\}$, then the twist vector fields $t_{C_i}$ are just the coordinate vector fields $\frac{\partial}{\partial\tau_i}$. Furthermore, the duality formula implies that $\omega_{wp}$ is invariant under any twist flow. Combining this with the fact that the coordinate vector fields $\{\frac{\partial}{\partial\tau_i}, \frac{\partial}{\partial\ell_i}\}$ commute, it follows that the coefficients of $\omega_{wp}$ in the basis $\{d\tau_i\wedge d\tau_j, d\ell_i\wedge d\tau_j, d\ell_i\wedge d\ell_j\}$ are independent of $\tau_i$. From this one can show that it suffices to compute $\omega_{wp}$ at those surfaces $X$ which admit an orientation reversing isometry $\rho$ fixing the partition $\{C_i\}$ ($\rho(\alpha_i) = \alpha_i$ for each $\alpha_i$ where $\{\alpha_i\}$ are the geodesics representing $\{C_i\}$ on $X$). The functions $\ell_{C_i}$ are invariant under $\rho$ since the length of a curve does not depend on the orientation of the surface. On the other hand, the twist parameter

$\tau_i$ does depend on this orientation since right and left are reversed by $\rho$. One makes this precise by showing:

$$\rho^* d\ell_{C_i} = d\ell_{C_i} \quad \text{and} \quad \rho^* d\tau_{C_j} = -d\tau_{C_j} + \frac{n_j}{2} d\ell_{C_j}$$

for some integers $n_j$. Since $\rho$ corresponds to an element of the mapping class group which acts antiholomorphically,

$$\rho^*\omega_{wp} = -\omega_{wp}.$$

Now since the coefficients of $d\tau_i \wedge d\tau_j$ and $d\ell_i \wedge d\ell_j$ are even relative to a $\rho$ substitution, while $\omega_{wp}$ is odd, these coefficients are identically zero. Finally, the fact that the coefficient of $d\ell_i \wedge d\tau_j$ is the Kronecker delta $\delta_{ij}$ follows directly from 5.2. □

iv) Consequences of Theorems 5.1 and 5.2

The first consequence of Theorem 5.2 is that the vector fields $t_C$ are Hamiltonian for $\omega_{wp}$; that is, the Lie derivative $L_{t_C}\omega_{wp}$ vanishes. This follows from the general formula $L_X\omega_{wp} = (d\omega_{wp})(X, \ , \ ) + d(\omega_{wp}(X\ , \ ))$, the fact that $\omega_{wp}$ is Kähler (thus $d\omega_{wp} = 0$) and the duality formula. Thus we see that $\omega_{wp}$ and the vector fields $t_C$ define a symplectic geometry on $M_g^s$ and $T_g^s$ (later we will see that $\omega_{wp}$ extends smoothly to $\overline{M}_g^s$ where it remains symplectic). By analogy with symmetric spaces we may use $\omega_{wp}$ to define a Lie algebra: just take the vector space of all vector fields $X$ on $T_g^s$ such that $L_X\omega_{wp} = 0$. It can be shown that this Lie algebra is generated over the $C^\infty$ functions by the $L_{t_C}$; however, it is infinite dimensional.

An idea suggested by the preceeding Theorems is that the hyperbolic geometry on the surface is reflected in the symplectic geometry of Teichmüller space. There are three main formulas that come out; they are the cosine formula, the sine-length formula and the Lie bracket formula. We state them without proof.

Cosine Formula

Let $C_1$ and $C_2$ be two nontrivial simple closed curves in $F$; then at $X \in T_g^s$

$$t_{C_1}\ell_{C_2} = \omega_{wp}\left(t_{C_1}, t_{C_2}\right) = \sum_{p \in \alpha \# \beta} \cos\Theta_p$$

where $\alpha$ and $\beta$ are the geodesics in $X$ representing $C_1$ and $C_2$, $\alpha \# \beta = \alpha \cap \beta$ unless $\alpha = \beta$ in which case it is empty, and $\Theta_p$ denotes the angle between $\alpha$ and $\beta$. Here $t_{C_1}\ell_{C_2}$ means the Lie derivative of $\ell_{C_2}$ by the vector field $t_{C_1}$.

Sine-Length Formula

Let $C_0$, $C_1$, $C_2$ be three nontrivial simple closed curves in F and let $\alpha$, $\beta$, $\gamma$ represent $C_0$, $C_1$, $C_2$ respectively in $X \in T_g^s$. Then

$$t_{C_0} t_{C_1} \ell_{C_2} = \sum_{(p,q)\in\alpha\#\gamma\times\beta\#\gamma} \frac{e^{m_1}+e^{m_2}}{2(e^{\ell_\gamma}-1)} \sin\Theta_p \sin\Theta_q$$

$$- \sum_{(r,s)\in\alpha\#\beta\times\beta\#\gamma} \frac{e^{n_1}+e^{n_2}}{2(e^{\ell_\beta}-1)} \sin\Theta_r \sin\Theta_s$$

where the two possible routes from p to q along $\gamma$ have length $m_1$ and $m_2$ and the two routes from r to s along $\beta$ have length $n_1$ and $n_2$.

Lie Bracket Formula

Renormalize the vector fields $t_C$ by setting $T_C = 4(\sinh \frac{\ell_C}{2}) t_C$. Then, with notation as above,

$$[T_{C_1}, T_{C_2}] = \sum_{p\in\alpha\#\beta} T_{\alpha_p\beta^+} - T_{\alpha_p\beta^-}$$

where $\alpha_p\beta^+$ and $\alpha_p\beta^-$ are the curves in F corresponding to the configurations in Figure 5.1.

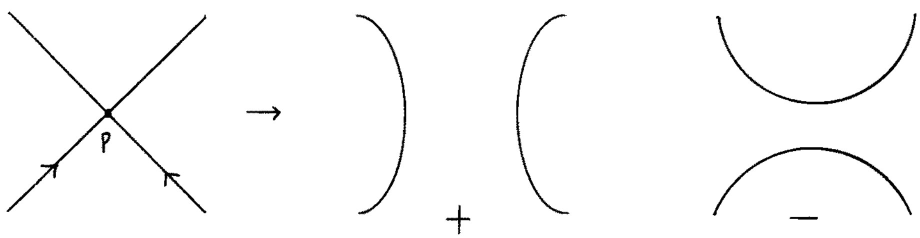



Figure 5.1

Actually, in the third formula we have done something illegal in that the curves $\alpha_p\beta^+$ and $\alpha_p\beta^-$ may not be simple. Nevertheless, the formula makes sense because it is possible to define the functions $\ell_C$ and the vector fields $t_C$ even when C is not simple. (It is clear how to do this for $\ell_C$, and $t_C$ is given by duality). This suggests that a Lie algebra over the integers might be constructed as follows. Let $\hat{\pi}$ denote the conjugacy classes in the fundamental group of the surface $F_g$ and let $\mathbb{Z}\hat{\pi}$ denote the free abelian group on the elements of $\hat{\pi}$. Fix a hyperbolic metric on $F$; each element of $\hat{\pi}$ is represented by a unique closed (but not necessarily simple) geodesic. Goldman [Gold] defines a bracket on $\mathbb{Z}\pi$ by setting

$$[\alpha,\beta] = \sum_{p\in\alpha\#\beta} \alpha_p\beta^+ - \alpha_p\beta^-$$

and extending linearly. The proof that [ , ] satisfies the hypotheses to be a Lie bracket on $\mathbb{Z}\hat{\pi}$ is purely topological.

Unfortunately, all these definitions give infinite dimensional Lie algebras and there is no indication that they contain any interesting finite dimensional subalgebras in any natural way. We will not be discouraged, however, and we will continue to study the Weil-Petersson geometry with an eye towards the formal geometry of a symmetric space. In the next section we dig deeper into the interplay between the hyperbolic geometry of the surface and the symplectic geometry of $T_g^s$.

## §2. The Thurston-Wolpert Random Geodesic Interpretation of the Weil-Petersson Metric

The key step in Kerckhoff's proof of the Nielsen conjecture is his observation that the geodesic length functions are convex along earthquake paths. While it would be out of place here to discuss either the Nielsen conjecture or earthquakes, we can still abstract from this the idea that a geodesic length function $\ell_C$ can be thought of as the square of a distance function (measured from the minimum of $\ell_C$). This means that its second derivative should define a metric tensor at that minimum. Thurston makes this idea precise by introducing the notion of a <u>random geodesic</u> on a hyperbolic surface X and showing that the corresponding "length function" does indeed have its minimum at X. He uses this idea to define a metric on Teichmüller space; Wolpert ([Wol 4]) then proves that this metric is actually the Weil-Petersson metric. This new interpretation of the metric proves to be quite valuable for it gives the following remarkable formula for the complex structure on $T_g$:

$$Jt_C = 3\pi(g-1)\lim_j \frac{[t_C,t_{\beta_j}]}{\ell_{\beta_j}}$$

where $\beta_j$ are random geodesics.

i) Definition of the Random Geodesic

Let $X$ be a hyperbolic surface and $\beta_j \subset X$ a sequence of closed geodesics (not necessarily simple). If $T_1X$ denotes the unit tangents to $X$, then each $\beta_j$ has a unique lift $\tilde{\beta}_j$ to $T_1X$. Define $\{\beta_j\}$ to be uniformly distributed if for all open sets $U \subset T_1X$

$$\lim_j \frac{\ell(\tilde{\beta}_j \cap U)}{\ell(\tilde{\beta}_j)} = \frac{\text{Volume }(U)}{\text{Volume}(T_1X)} .$$

Here we identify $T_1X$ with $T_1\mathbb{H}^2/\Gamma$ where $X = \mathbb{H}^2/\Gamma$ and compute volume via the isomorphism $T_1\mathbb{H}^2 \cong PSL_2\mathbb{R}$. Let $<\ ,\ >$ denote the Weil-Petersson metric.

Theorem 5.3: For any nontrivial simple closed curves $C_1$, $C_2$,

$$<t_{C_1}, t_{C_2}> = 3\pi(g-1) \lim_j \frac{t_{C_1} t_{C_2} \ell_{\beta_j}}{\ell_{\beta_j}},$$

where $\{\beta_j\}$ is uniformly distributed on $X$.

In the formula we are writing $\ell_{\beta_j}$ when we really mean $\ell_{f(\beta_j)}$ where $f\colon X \to F$ is the marking on $X$.

ii) Outline of the proof of Theorem 5.3

First we explain why $<t_{C_1}, t_{C_2}>_T = \lim_j \frac{1}{\ell_{\beta_j}} t_{C_1} t_{C_2} \ell_{\beta_j}$ is symmetric. Since $[t_{C_1}, t_{C_2}]$ is a tangent vector for every $C_1$ and $C_2$, this will follow if we show $\lim_j \frac{1}{\ell_{\beta_j}} V \ell_{\beta_j} = 0$ for any tangent vector $V$. Since twist vector fields span the tangent space to $T_g^s$, it suffices to show $\lim_j \frac{1}{\ell_{\beta_j}} t_C \ell_{\beta_j} = 0$ for every $C$. But the $\beta_j$ are uniformly distributed in $T_1X$, so for any closed geodesic $\alpha \subset X$, in the limit each intersection of $\alpha$ with $\beta_j$ of angle $\theta$ is accompanied by another of angle $\pi - \theta$. Applying the cosine formula completes the argument.

The backbone of the rest of the argument is the fact that $<\ ,\ >_T$ is constructed naturally with respect to the $PSL_2\mathbb{R}$ geometry on the unit tangent bundle $T_1X$. To make use of this, Wolpert observes that there exist tensors $K_1$ and $K_2$ on $X$ such that for any two holomorphic quadratic differentials $\phi$, $\psi$ on $X$

even continuous in complex coordinates on $\overline{M}_g$. In fact $\overline{\omega}_{wp}$ is only a current and thus its positivity and rationality are difficult to check.

The approach consists of three parts; the extension of the Kähler form to $\overline{M}_g$, the rationality of the extension and of course the positivity of the resulting line bundle.

i) <u>The extension of</u> $\omega_{wp}$ <u>to</u> $\overline{M}_g$

Masur [Mas] was the first to consider the extension of $\omega_{wp}$ to $\overline{M}_g$. At a generic point of $D = \overline{M}_g - M_g$ with normal coordinate $z$, he gave the formula

$$\overline{\omega}_{wp} \sim \frac{i/2\, dz \wedge d\overline{z}}{|z|^2 (\log 1/|z|)^3},$$

showing that the extension is not continuous in the complex coordinates. By contrast, since Fenchel-Nielsen coordinates on $\overline{M}_g$ are obtained simply by allowing the $\ell_i$ to be 0, the formula $\omega_{wp} = \sum d\ell_i \wedge d\tau_i$ shows that $\omega_{wp}$ extends smoothly in Fenchel-Nielsen coordinates. This means that the complex structure $\overline{M}_g^{\mathbb{C}}$ and the real-analytic structure $\overline{M}_g^{FN}$ are not subordinate to a common smooth structure on $\overline{M}_g$. Nevertheless, in [Wol 5] it is proved that the identity map $\overline{M}_g^{FN} \to \overline{M}_g^{\mathbb{C}}$ is Lipschitz continuous. From the point of view of cohomology this means we can integrate $\overline{\omega}_{wp}$ over 2-cycles in Fenchel-Nielsen coordinates and use the answer to study $\overline{M}_g^{\mathbb{C}}$.

Masur's formula shows that the form $\overline{\omega}_{wp}$ has singularities along D. Wolpert deals with this by first showing that $\overline{\omega}_{wp}$ is a closed, positive (1,1) current and then that $\overline{\omega_{wp}}$ is the limit of smooth, closed, positive (1,1) forms.

ii) <u>Rationality of</u> $\frac{1}{\pi^2}\overline{\omega}_{wp}$

The second part of the proof is to show that $\frac{1}{\pi^2}\overline{\omega}_{wp}$ is rational. By Theorem 7.2 below ([H1]), $H_2(M_g;\mathbb{Q})$ is rank 1 for $g \geq 3$. An application of Mayer-Victoris then shows $H_2(\overline{M}_g;\mathbb{Q}) \cong H^{6g-8}(\overline{M}_g;\mathbb{Q})$ has rank 2 + [g/2]. We will describe dual bases for $H_2$ and $H_{6g-8}$.

The divisor $D = \overline{M}_g - M_g$ is the sum of 1 + [g/2] components $D_0,\ldots,D_{[g/2]}$ where $D_i$ generically consists of the surfaces with a node which are obtained by collapsing a simple closed curve to a point. For $D_0$ the curve is nonseparating while for $D_i$, $i > 0$, the curve separates the surface into pieces of genus $i$ and $g-i$. The Poincaré dual of $\overline{\omega}_{wp}$ and $D_0,\ldots,D_{[g/2]}$ give $2+[g/2]$ classes in $H_{6g-8}(\overline{M}_g)$.

$$\langle\phi,\psi\rangle_T = \int_X \phi\bar{\psi}K_1 + \phi\psi K_2.$$

$K_1$ must be a $(-1,-1)$ tensor and $K_2$ a $(-3,1)$ tensor and by the naturality of $\langle\ ,\ \rangle_T$ both must be $PSL_2\mathbb{R}$ invariant. The only possibility is that $K_1$ is a multiple of the hyperbolic area element and $K_2$ is zero. Thus $\langle\ ,\ \rangle_T$ is a multiple of $\langle\ ,\ \rangle_{wp}$.

iii) <u>Description of the Complex Structure on</u> $T_g^s$

Let $C$ be any nontrivial simple closed curve and $\{\beta_j\}$ a sequence of uniformly distributed geodesics in $X$. Then Wolpert uses Theorem 5.3 to show:

<u>Theorem 5.4</u>: $Jt_C = 3\pi(g-1)\lim_j \frac{1}{\ell_{\beta_j}}[t_C, t_{\beta_j}]$ where $J$ is the complex structure on $T_g^s$.

The proof of this uses not only 5.3 but also the Lie Bracket formula, the duality formula and the formula

$$t_{\alpha_1}t_{\alpha_2}\ell_\beta + t_{\alpha_2}t_\beta\ell_{\alpha_1} + t_\beta t_{\alpha_1}\ell_{\alpha_2} = 0$$

from [Wol 3]. By skew symmetry $t_{\alpha_1}\ell_{\alpha_2} = -t_{\alpha_2}\ell_{\alpha_1}$, so $t_{\alpha_1}t_{\alpha_2}\ell_\beta + [t_{\alpha_2}, t_\beta]\ell_{\alpha_1} = 0$. Dividing by $\ell_\beta$ and taking $\beta$ in a sequence which is uniformly distributed gives

$$\langle t_{\alpha_1}, t_{\alpha_2}\rangle + 3\pi(g-1)\omega_{wp}(\lim_j \frac{1}{\ell_{\beta_j}}[t_{\alpha_2}, t_{\beta_j}], t_{\alpha_1}) = 0.$$

Since $\alpha_1$ is arbitrary $t_\alpha + 3\pi(g-1)\, J \lim_j \frac{1}{\ell_{\beta_j}}[t_\alpha, t_{\beta_j}] = 0$.

Theorem 5.4 now follows. □

## §3. The Projective Embedding of $\overline{M}_g$

As a final illustration of the importance of the Weil-Petersson geometry we will give a sketch of Wolpert's beautiful proof that $\overline{M}_g$ is projective. The outline is very simple: first he shows that $\omega_{wp}$ extends to $\overline{\omega}_{wp}$ on $\overline{M}_g$ . Next he proves that $\frac{1}{\pi^2}\overline{\omega}_{wp}$ is rational and therefore $\frac{n}{\pi^2}\overline{\omega}_{wp}$ is $c_1$ of a line bundle $L$ for some positive integer $n$. Since $\omega_{wp}$ is Kähler, $L$ is postive; the Kodaira theorem now provides the imbedding $\overline{M}_g \to \mathbb{C}P^n$.

Of course things are not really as simple as all that. The extension $\overline{\omega}_{wp}$ is smooth in Fenchel-Nielsen coordinates but is not

For $H_2(\overline{M}_g)$ take one of the $2+[g/2]$ configurations of Figure 5.2 and consider the subset of $D$ described by allowing the structure

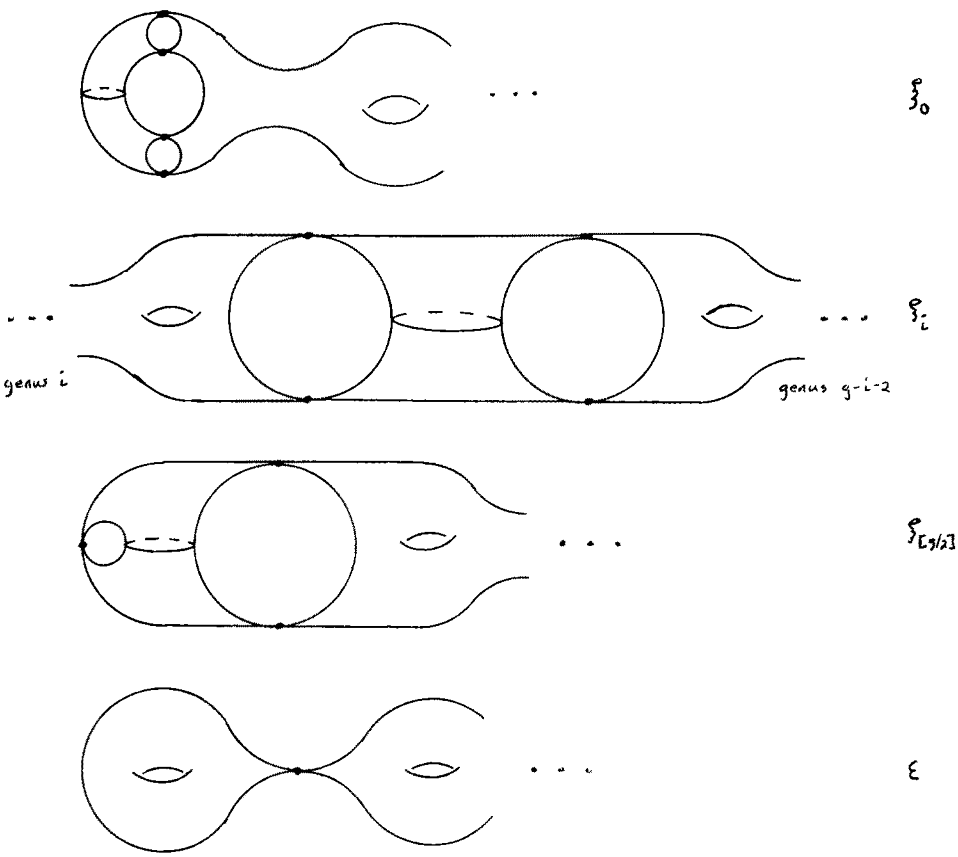



Figure 5.2

on the component labeled $A$ to vary while that on the rest of the surface remains fixed. Each of these describes an analytic 2-cycle in $D$; $\varepsilon$ is isomorphic to $\overline{M}\,{}_1^1$ while $\xi_0,\ldots,\xi_{[g/2]}$ are isomorphic to $\overline{M}_0{}^4$. The next thing to do is to compute the intersection matrix between these two collections; it will be nonsingular, showing that both sets are actually bases. This is straightforward except for evaluating the integrals $\int_\varepsilon \overline{\omega}_{wp}$ and $\int_{\xi_i} \overline{\omega}_{wp}$. Now Wolpert computes directly that $\int_{\overline{M}{}_1^1} \overline{\omega}_{wp} = \pi^2/6$. Passing to a 4-fold cover gives

$\int_{\overline{M}_1{}^4} \overline{\omega}_{wp} = 2\pi^2/3$ and since $F_1^4$ double covers $F_0^4$, $\int_{\overline{M}_0{}^4} \overline{\omega}_{wp} = \pi^2/3$ .

Using Theorem 5.1 it is easy to see that $\overline{\omega}_{wp}$ restricts, i.e. $\overline{\omega}_{wp}|\varepsilon = \overline{\omega}_{wp}$ on $\overline{M}_1{}^1$ and $\overline{\omega}_{wp}|\xi_i = \overline{\omega}_{wp}$ on $\overline{M}_0{}^4$. This shows not only that the collections are bases, but simultaneously evaluates $\overline{\omega}_{wp}$ and shows $\frac{1}{\pi^2}\overline{\omega}_{wp}$ is rational.

iii) <u>The Positive Line Bundle</u>

Now $\frac{1}{\pi^2}\overline{\omega}_{wp}$ is a rational, closed positive (1,1) current on $\overline{M}_g$. We would like to say that $\frac{n}{\pi^2}\overline{\omega}_{wp}$ is $c_1$ of a positive line bundle $L$ on $\overline{M}_g$. Recall that for any $L$, $c_1(L) = \frac{1}{2\pi i}\partial\overline{\partial}\log\|s\|^2$ where $s$ is any section of $L$ and $\|\ \|$ is a metric for $L$. Heuristically, if $\|s\|^2 = e^{1/\log(1/|z|^2)}, |z| < 1$, then we find

$$\partial\overline{\partial}\log\|s\|^2 = \partial\overline{\partial}\ \frac{1}{\log(1/|z|^2)} = \frac{2}{|z|^2(\log(1/|z|^2))^3}\ ;$$

the principal term of Masur's formula. Now by our calculation $e^{1/\log(1/|z|^2)}$ is a $C^0$ metric of positive curvature in the sense of currents. Similarly Wolpert finds that $\overline{\omega}_{wp}$ is the positive curvature form of a $C^0$ metric for a line bundle over $\overline{M}_g$. An argument of Richberg is then used to replace the metric in the line bundle by a smooth metric with positive corvature form. This completes the argument.

## Chapter 6 Stability of the Homology of $\Gamma_g^s$

We return now to a direct study of $H_*(\Gamma_g^s)$. A natural question to ask about $\Gamma_g^s$ is whether its homology stabilizes as $g \to \infty$. This kind of result is known (rationally, at least) for many classes of arithmetic groups, e.g. $SL_n$ and $Sp_{2g}$ ([B]), and the techniques to prove it are readily available ([C],[Q],[V],[W]). We will now combine these techniques with the results of Chapter 2 and 4 to show that in fact $H_k(\Gamma_g^s)$ is independent of $g$ when $g >> k$.

Let $F_{g,r}^s$ denote an oriented surface of genus $g$ with $r$ boundary components and $s$ punctures. The mapping class group $\Gamma_{g,r}^s$ is defined to be the group of all isotopy classes of homeomorphisms of $F$ which are the identity on $\partial F$ and fix the punctures individually. We emphasize that isotopies must fix $\partial F$ pointwise, otherwise there would be no distinction between $\Gamma_{g,r}^s$ and $\Gamma_{g,0}^{r+s}$. Define three maps

$$\phi: F_{g,r}^s \to F_{g,r+1}^s \quad , r \geq 1,$$

$$\psi: F_{g,r}^s \to F_{g+1,r-1}^s \quad , r \geq 2,$$

$$\eta: F_{g,r}^s \to F_{g+1,r-2}^s \quad , r \geq 2$$

as follows. For $\psi$ sew a pair of pants ( a copy of the surface $F_{0,3}^0$) to $F_{g,r}^s$ along two components of its boundary. For $\phi$ sew a pair of pants to $F_{g,r}^s$ along one component of its boundary. Finally, for $\eta$ sew two components of $\partial F_{g,r}^s$ together. The maps $\phi$, $\psi$, $\eta$ induce maps of mapping class groups (extend by the identity on $F_{0,3}^0$ for $\phi$ and $\psi$; $\eta$ is obvious); let $\phi_*$, $\psi_*$, $\eta_*$ be (respectively) the maps they induce on homology. We may now state the main result of $[H_2]$:

<u>Theorem 6.1</u>: $\phi_*: H_k(\Gamma_{g,r}^s) \to H_k(\Gamma_{g,r+1}^s)$ is an isomorphism for $g \geq 3k-2$,

$\psi_*: H_k(\Gamma_{g,r}^s) \to H_k(\Gamma_{g+1,r-1}^s)$ is an isomorphism for $g \geq 3k-1$,

$\eta_*: H_k(\Gamma_{g,r}^s) \to H_k(\Gamma_{g+1,r-2}^s)$ is an isomorphism for $g \geq 3k$.

By combining these maps in various ways we can see that Theorem 6.1 implies $H_k(\Gamma_{g,r}^s)$ is independent of $g$ and $r$ as long as $g \geq 3k+1$.

For the moduli spaces this says that $H_k(M_g^s; \mathbb{Q})$ does not depend on $g$ when $g \geq 3k+1$.

Before giving the proof of 6.1 we make some observations due to Ed Miller [$M_i$]. Let $\Lambda_{g,1} = \mathrm{Diff}^+(F_{g,1})$ be the group of diffeomorphisms fixing the boundary component of $F$ pointwise. Taking boundary connected sum defines a natural homomorphism

$$\Lambda_{g,1} \times \Lambda_{h,1} \to \Lambda_{g+h,1}$$

which induces a product on classifying spaces

$$\lambda: B\Lambda_{g,1} \times B\Lambda_{h,1} \to B\Lambda_{g+h,1}.$$

Now set $A = \lim\limits_{\to} H_*(B\Lambda_{g,1}; \mathbb{Q})$ where the limit is defined using the natural inclusions $\to \Lambda_{g,1} \to \Lambda_{g+1,1} \to \cdots$ . Theorem 6.1 implies $A$ is finite type; therefore $A$, under the product $\lambda_*$ induced from $\lambda$, is a commutative, cocommutative Hopf algebra of finite type. A theorem of Milnor and Moore [MM] then implies:

Corollary 6.2: $A$ is the tensor product of a polynomial algebra on even dimensional generators with an exterior algebra on odd dimensional generators.

We move now to the proof of Theorem 6.1.

§1) The Cell Complexes

The first half of the proof is an excursion through a maze of different cell complexes which are constructed from configurations of arcs and circles in a surface. There are six of these; their names are $X$, $Z$, $AX(\Delta)$, $AZ(\Delta)$, $BX(\Delta,\Delta')$ and $BZ(\Delta,\Delta')$. The complex $Z$ is the one we defined in chapter 3 using curve-systems, while $AZ(\Delta)$ is a generalization of the arc-system complex $A$ we used in Chapter 2 to triangulate $T_g^1$. All the others are derived from these; in fact we have $X \subset Z$, $AX(\Delta) \subset AZ(\Delta)$ and $BX(\Delta,\Delta') \subset BZ(\Delta,\Delta') \subset AZ(\Delta)$.

i) The Complexes $AZ(\Delta)$ and $BZ(\Delta,\Delta')$

Let us be more specific. Fix $F = F_{g,r}^s$ and let $\Delta$ be a collection of points in $\partial F$. Also let $\Delta'$ be any proper subset of $\Delta$. A $\Delta$-arc will be the isotopy class of any $C^\infty$ imbedded path in $F$ from one point of $\Delta$ to another, or a $C^\infty$ imbedded loop in $F$ based at a point of $\Delta$. The $\Delta$-arc $\alpha$ is nontrivial if it is not null-homotopic and is not homotopic (rel $\partial\alpha$) into $\partial F - \Delta \cup \partial\alpha$.

A $\Delta$-arc system of rank $k$ is then any family $\alpha_0,\ldots,\alpha_k$ of non-trivial $\Delta$-arcs which are disjoint, except that they may intersect at their endpoints, such that no two distinct $\alpha_i$ are homotopic (rel endpoints). Also define a $(\Delta,\Delta')$-arc to be any $\Delta$-arc with one end in $\Delta'$ and the other in $\Delta - \Delta'$ and a $(\Delta,\Delta')$-arc-system of rank $k$ to be any $\Delta$-arc-system of rank k consisting only of $(\Delta,\Delta')$-arcs.

Now following the same pattern as for A let $AZ(\Delta)$ be the simplicial complex which has a k-cell $\langle\alpha_0,\ldots,\alpha_k\rangle$ for each rank k $\Delta$-arc-system in F, with $\langle\alpha_0,\ldots,\alpha_k\rangle$ identified as a face of $\langle\beta_0,\ldots,\beta_\ell\rangle$ if and only if $\{\alpha_i\} \subset \{\beta_j\}$. Also define $BZ(\Delta,\Delta')$ to be the subcomplex of $AZ(\Delta)$ consisting of simplices $\langle\alpha_0,\ldots,\alpha_k\rangle$ where $\{\alpha_i\}$ is a $(\Delta,\Delta')$-arc-system.

We say that a simplicial complex $\sum$ of dimension n is spherical if $\sum \simeq \vee S^n$. The first step in the proof of Theorem 6.1 is:

Theorem 6.3: (a) The complex $AZ(\Delta)$ is contractible, except in the special cases where F is a 2-disk, a punctured 2-disk or an annulus with $\Delta$ contained in one component of $\partial F$, in which case $AZ(\Delta)$ is homeomorphic to a sphere.

(b) The complex $BZ(\Delta,\Delta')$ is spherical. If no component of $\partial F$ contains points of both $\Delta'$ and $\Delta - \Delta'$, it is contractible.

We refer the reader to [H2] for the proof.

Along with these two complexes we may define $Z = Z^s_{g,r}$ just as we did $Z^s_g$; in fact $Z^s_{g,r} \cong Z^{r+s}_g$ because isotopy classes of simple closed curves in the interior of a surface aren't effected if we remove the boundary curves to create punctures. The mapping class group $\Gamma = \Gamma^s_{g,r}$ acts on each of the complexes $Z$, $AZ(\Delta)$ and $BZ(\Delta)$ in the obvious way.

ii) The Complexes $X$, $AX(\Delta)$, $BX(\Delta,\Delta')$

The problem with the complexes $Z$, $AZ(\Delta)$ and $BZ(\Delta,\Delta')$ is that their quotients by $\Gamma$, while finite complexes, are difficult to work with because they have too many cells. To rectify this problem we define in each case a natural subcomplex which is invariant under the action of $\Gamma$ and has a simpler quotient.

The definition is the same in all 3 cases:

the subcomplexes $X$, $AX(\Delta)$ and $BX(\Delta,\Delta')$ are obtained by restricting in each case to curve or arcsystems which do not separate the surface (the entire system must be nonseparating, not just the individual curves or arcs). The complex $X$ is $g - 1$ dimensional since $g$ curves whose union does not separate $F$ will cut it into a planar surface where all further curves are separating. Similar reasoning shows that $AX(\Delta)$ and $BX(\Delta,\Delta')$ are $2g-2+r'$ dimensional where $r'$ is the number of boundary components of $F$ which contain points of $\Delta$.

Theorem 6.4: The complexes $X$, $AX(\Delta)$ and $BX(\Delta,\Delta')$ are spherical.

This theorem is proved by a combinatorial analysis of the inclusion into the corresponding larger complex. For example, any map $f:S^n \to X$ with $n<g-1$ extends to $\bar{f}: D^{n+1} \to Z$ by Theorem 4.1. We then use the fact that $n$ is small, so that simplices of $im(\bar{f})$ do not involve too many curves in $F$, to show how to homotope $\bar{f}$ (rel $f$) into $X$. Similar arguments work for $AX(\Delta)$ and $BX(\Delta,\Delta')$ using an induction on $g,r$, $\#\Delta$ and $\#\Delta'$. This is why we had to allow $\Delta$ and $\Delta'$ to be arbitrary. From here on, however, we will assume that for $AX(\Delta)$, $\Delta$ consists of a single point and for $BX(\Delta,\Delta')$ $\Delta$ consists of 2 points on distinct boundary components of $F$ with $\Delta'$ equal to one of the points. For brevity we then write $AX = AX(\Delta)$ and $BX = BX(\Delta,\Delta')$.

## iii) Description of the Quotient Complexes

Let $G$ be a group acting simplicially on the complex $\Sigma$. The quotient $\Sigma/G$ inherits a natural cell structure from $\Sigma$ only if the action has the property that whenever $g \in G$ fixes a simplex of $\Sigma$ setwise, if does so pointwise. This property can always be arranged, if necessary, by passing to the first barycentric subdivision $\Sigma^o$ of $\Sigma$. In our case this will only be necessary for $X$; the property above is already true for $AX$ and $BX$.

The quotient of $X^o$ by $\Gamma$ is a simplex of dimension $g-1$. This is because any two rank-k curve systems which do not separate $F$ are identified by $\Gamma$, so any top dimensional simplex of $X^o$ maps homeomorphically to $X^o/\Gamma$.

The quotients of $AX$ and $BX$ are more complicated. Orient $\partial F$ and consider any k-cell $\langle\alpha_0,\dots,\alpha_k\rangle$ of $AX$. In a neighborhood of the component of $\partial F$ which contains the point $q$ of $\Delta$, the picture is as in Figure 6.1. Number the edges emanating from $q$ in the order determined by the orientation; then $\langle\alpha_o,\dots,\alpha_k\rangle$ determines a pairing of the integers $\{1,\ldots,2k+2\}$.

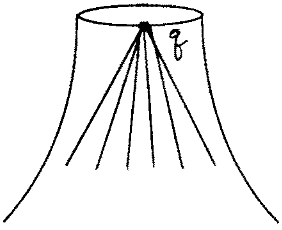


Figure 6.1

Two cells of AX are identified by $\Gamma$ if and only if they give the same pairing. Furthermore, if $k < g$ any pairing may occur.

A similar analysis holds for BX. Number the edges emanating from $q_1$ as in figure 6.1; the order that they come into $q_2$ gives a permutation of $k+1$ elements. Any two cells of BX are identified by $\Gamma$ if and only if they determine the same permutation and for $k \le g$ any permutation can occur.

In the next section we will use these descriptions to show how stability is proven.

§ 2 Spectral Sequence Arguments

To prove Theorem 6.1 we use Theorem 6.4 and the spectral sequence techniques of [C], [Q], [W], [V]. Begin with the

Inductive assumptions;

$(1_{k-1})$ $\quad \phi_*: H_n(\Gamma_{g,r}) \to H_n(\Gamma_{g,r+1})$ is an isomorphism,

$g \ge 3n-2,\ 1 < n < k$ and $r \ge 1$.

$\phi_*: H_1(\Gamma_{g,r}) \to H_1(\Gamma_{g,r+1})$ is an isomorphism,

$g \ge 2,\ r \ge 1$.

$(2_{k-1})$ $\quad \psi_*: H_n(\Gamma_{g,r}) \to H_n(\Gamma_{g+1,r-1})$ is surjective,

$g \ge 3n-2,\ 1 \le n < k,\ r \ge 2$, and an isomorphism,

$g \ge 3n-1,\ 1 < n < k,\ r \ge 2$.

$\psi_*: H_1(\Gamma_{g,r}) \to H_1(\Gamma_{g+1,r-1})$ is an isomorphism,

$g \ge 3,\ r \ge 2$.

$(3_{k-1})$ $\eta_*: H_n(\Gamma_{g,r};\mathbb{Q}) \to H_n(\Gamma_{g+1,r-2};\mathbb{Q})$ is an isomorphism, $g \geq 3n-1$, $n < k$, $r \geq 2$.

$\eta_*: H_n(\Gamma_{g,r}) \to H_n(\Gamma_{g+1,r-2})$ is an isomorphism, $g \geq 3n$, $n < k$, $r \geq 2$.

Both maps are surjective at one smaller value of $g$.

It is easy to check $(1_k)$, $(2_k)$ and $(3_k)$ directly for $k \leq 2$ using the results of [H1] (see chapter 7). Assume $k \geq 3$ and that $(1_j)$, $(2_j)$ and $(3_j)$ have been verified for all $j < k$. There are then three spectral sequence arguments to perform to prove $(1_k)$, $(2_k)$ and $(3_k)$. We will show how $(1_k)$ goes (it is the easiest); the reader is refered to [H2] for the other two.

First notice that $\phi_*$ is injective for every $g$, $k$ when $r \geq 1$. This is because we may define $\theta: \Gamma_{g,r+1} \to \Gamma_{g,r}$ by plugging one of the extra boundary components of the pair of pants sewed on in defining $\phi$ and we will have $\theta \circ \phi = 1$.

Now look at the action of $\Gamma = \Gamma_{g,r+1}$ on the complex BX defined on $F_{g,r+1}$. Let $K_* \to \mathbb{Z}$ be the augmented chain complex of BX. Choose $E_* \to \mathbb{Z}$, a free $\mathbb{Z}\Gamma$ resolution of $\mathbb{Z}$. The double complex $E_* \otimes_{\mathbb{Z}\Gamma} K_*$ gives rise to a spectral sequence converging to zero for $p+q \leq 2g-1$, with

$$E^1_{p,q} = H_q(\Gamma;K_p), \quad p \geq -1,$$

(see [C], [Q], [W], [V]). By Shapiro's lemma there is a decomposition

$$H_q(\Gamma;K_p) \cong \bigoplus_{i=1}^{n_p} H_q(\Gamma^i_p)$$

where $K_p = K^1_p \oplus \ldots \oplus K^{n_p}_p$ with $\Gamma(K^i_p) \subset K^i_p$, $\Gamma$ acts transtively on the generators of each $K^i_p$ and $\Gamma^i_p$ denotes the stabilizer of a chosen p-cell $\sigma^i_p$, a generator from $K^i_p$.

<u>Lemma 6.5</u>: $H_p(BX/\Gamma) = 0$, $1 \leq p \leq g-1$.

<u>Proof</u>: Recall each p-cell $\sigma^i_p$ may be defined by the permutation it gives of $0,1,\ldots,p$. Label this permutation $(i_0,\ldots,i_p)$ where $i_j$ is the image of $j$ under the permutation. The boundary map of $BX/\Gamma$ is now

$$\partial_p(i_0,\ldots,i_p) = \sum_{j=0}^{p} (-1)^j(\tau_j(i_0),\ldots,\hat{i}_j,\ldots,\tau_j(i_p))$$

where $\tau_j$ is the shift map

$$\tau_j(n) = \begin{cases} n, & n < i_j \\ n-1, & n > i_j. \end{cases}$$

Consider the map

$$C_p(BX/\Gamma) \xrightarrow{D_p} C_{p+1}(BX/\Gamma)$$

$$(i_0,\ldots,i_p) \to (p+1,i_0,\ldots,i_p).$$

The formula $D\partial + \partial D = 1$ is easy to verify, so the identity and the zero map on $C_*(BX/\Gamma)$ are chain homotopic in the range $p < g$. This proves 6.5. $\square$

Next we look at the $E^1$ term of the spectral sequence. If necessary, rechoose the $\sigma_p^i = \langle\beta_0^i,\ldots,\beta_p^i\rangle$ so that for each $p$ there is an embedding

$$\omega_p : F_{g-p,1} \to F - \{\beta_t^i; 0 \le t \le p, 1 \le i \le n_p\},$$

with $\omega_p = \omega_{p-1} \cdot \psi \cdot \phi$ for each $p$. This is possible because the genus of $F - \{\beta_j^i; 0 \le j \le p\}$ is at least $g - p$. If $(\sigma_p^j)_i$ is the ith face of $\sigma_p^j$, choose $f_i$ in $\Gamma$ identifying $(\sigma_p^j)_i$ with the appropriate $\sigma_{p-1}^{k_i}$, making sure that $f_i|\omega_p(F_{g-p,1}) = 1$. The map

$$d^1 : E_{p,q}^1 \to E_{p-1,q}^1$$

has components

$$(-1)^i (f_i)_* : H_q(\Gamma_p^j) \to H_q(\Gamma_{p-1}^{k_i}).$$

By $(1_{k-1})$ and $(2_{k-1})$, the maps $(\omega_{p-1})_*$, $(\omega_p)_*$, $\psi_*$ and $\phi_*$ are isomorphisms as long as $g - p \ge 3q$, $q < k$, and are surjective when $g-p = 3q-1$, $q < k$. Since

$$(\omega_{p-1})_* \cdot \psi_* \cdot \phi_* = (f_i)_*(\omega_p)_*,$$

$(f_i)_*$ is an isomorphism or is surjective for the same values of $p$, $q$ and $g$.

Consider the diagram

$$\begin{array}{ccc} H_q(\Gamma_p^i) & \to & H_q(\Gamma) \\ {\scriptstyle k_i}\downarrow{\scriptstyle (f_i)_*} & & \downarrow{\scriptstyle (f_i)_*} \\ H_q(\Gamma_{p-1}^i) & \to & H_q(\Gamma) \end{array} \quad .$$

The horizontal maps are induced by the inclusions of the stabilizers in $\Gamma$. Since $F_i$ is an element of $\Gamma$, it induces the identity on homology, so that the right hand vertical map is the identity. The horizontal maps are compositions of the maps $\phi_*$ and $\psi_*$; so for $p \geq 1$ and $q < k$, they are isomorphism. This means that $E^1_{*,q}$ is isomorphic as a complex to $C_*(BX/\Gamma) \otimes H_q(\Gamma)$, where the boundary map is the tensor product of the boundary map of $BX/\Gamma$ with the identity on $H_q(\Gamma)$. In particular,

$$E^2_{p,k-p} \cong H_p(BX/\Gamma; H_{k-p}(\Gamma)) = 0,$$

as long as $g - (p+1) \geq 3(k-p) - 1$. Since $p \geq 1$, this holds for $g \geq 3k-2$.

Now we use the fact that $E^\infty_{p,q} = 0$, $p+q<2g$, to conclude that

$$d^1_{0,k} : E^1_{0,k} \to E^1_{-1,k}$$

is surjective, $g \geq 3k-2$ ($k>1$ and $g \geq 3k-2$ imply $g>k$). As $d^1_{0,k}$ is the map $\phi_*$, $(1_k)$ is verified.

The arguments for $(2_k)$ and $(3_k)$ are more difficult and will be omitted.

## Chapter 7. Computations of the Cohomology of $\Gamma_g^s$

Finally the time has come to construct some cohomology classes for $\Gamma_g^s$. In this chapter we will discuss which Betti numbers of $\Gamma_g^s$ and $M_g^s$ are known explicitly. Then we will give Miller's construction of a polynomial algebra on even generators in the stable cohomology of $\Gamma_g^s$ [Mi], and describe unstable relations for these classes due to Mumford [M1], Morita [Mor] and Harris [Ha].

### §1. The first two Betti numbers of $M_g^s$

In Theorem 3.2 we observed that $\Gamma_g^s$ has no rational cohomology above dimension 4g-4+s (4g-5 when $g > 1$, $s = 0$), while $M_g^s$ has no integral homology above dimension 4g-4+s, $s > 0$. Only two other Betti numbers, $\beta_1$ and $\beta_2$, are known. Mumford [M] proved that $H_1(\Gamma_g)$ is torsion of order dividing 10 $(g > 1)$ and Powell [P] proved that $H_1(\Gamma_g) = 0$ when $g > 2$. The general statement is:

Theorem 7.1:

$$H_1(\Gamma_g^s) \cong \begin{cases} 0 & \text{for } g > 2, \\ \mathbb{Z}/10\mathbb{Z} & \text{for } g = 2 \quad \text{and} \\ \mathbb{Z}/12\mathbb{Z} & \text{for } g = 1. \end{cases}$$

Another thing Mumford did in [M] was to define $\mathrm{Pic}(M_g^s)$ and prove $\mathrm{Pic}(M_g^s) \cong H^2(\Gamma_g^s)$, $g > 1$. In [H1] this latter group was computed; the result is:

Theorem 7.2: $H_2(\Gamma_g^s) \cong \mathbb{Z}^{s+1}$ for $g > 4$.*

This result also holds rationally for $\Gamma_3^s$ and $\Gamma_4^s$ while $H_2(\Gamma_2^s;\mathbb{Q})$ and $H_2(\Gamma_1^s;\mathbb{Q})$ have rank s. We do not know the torsion in $H_2(\Gamma_g^s)$ when $2 \le g \le 4$.

The proofs of Theorem 7.1 and 7.2 will not be presented here as that would lead us too far afield. Instead we content ourselves with a description of generators for $H_2(\Gamma_g^s)$. Geometrically, we saw in Chapter 5 that $\frac{1}{\pi^2}\omega_{WP}$ generates $H^2(M_g;\mathbb{Q})$. We now give a purely topological description of generators. Let $\Lambda = \mathrm{Diff}^+(F_g^s)$ be the group of orientation preserving diffeomorphism of $F_g$ which fix s points, equipped with the $C^\infty$-topology. The group $\Lambda$ has one

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* Actually, the statement in $[H_1]$ was that $H_2(\Gamma_g^0)$ has torsion, but this was in error, the correction appears in the same journal somewhat later.

component for each element of $\Gamma_g^s$ and, when $2-2g-s<0$, Earle and Eells [EE] have proved that each of these components is contractible. This means that the classifying space $B\Lambda$ is a $K(\Gamma_g^s,1)$.

Let $\Omega_2(B\Lambda)$ denote the second bordism group of $B\Lambda$. An element of $\Omega_2(B\Lambda)$ is represented by a map $\phi : X \to B\Lambda$ where $X$ is a closed oriented surface and two maps $\phi_1 : X_1 \to B\Lambda$ and $\phi_2 : X_2 \to B\Lambda$ represent the same element if and only if there is a map $\psi : M^3 \to B\Lambda$ where $M^3$ is an oriented 3-manifold with $\partial M^3 = X_1 \coprod -X_2$ and $\psi | X_i = \phi_i$. Bordism groups and homology groups agree in low dimensions; in particular we have:

$$H_2(\Gamma_g^s) \cong H_2(B\Lambda) \cong \Omega_2(B\Lambda).$$

Any map $f : X \to B\Lambda$ induces a bundle over $X$ with fiber $F$: use $f$ to pull back the bundle $E\Lambda \times_\Gamma F \to B\Lambda$ where $E\Lambda$ is the universal cover of $B\Lambda$. Bordant maps pull back bordant bundles, i.e. bundles which cobound a bundle over a 3-manifold.

Now we may define $H_2(\Gamma_g^s) \to \mathbb{Z}^{s+1}$. Each $X \in H_2(\Gamma_g^s)$ gives rise as above to a fiber bundle $F_g \to W^4 \to X$ with structure group in $\Lambda$. This bundle then has $s$ canonical sections $\sigma_1,\ldots,\sigma_s$; the invariants we associate to $X$ are signature$(W)/4$ and the self-intersection numbers $[\sigma_i(X)]^2$ (the signature of $W^4$ is always divisible by 4 so these invariants are all integers). Theorem 7.2 states that this map is an isomorphism.

§2. Miller's Polynomial Algebra

Let $\Lambda = \mathrm{Diff}^+(F_g^0)$ and consider the universal $F_g$ bundle $p : E \to B\Lambda$. Define $\tau$ to be the bundle of tangents to the fibers of $p$ and set $\omega = -c_1(\tau)$. Now define

$$\kappa_i = p_* \omega^{i+1};$$

$\kappa_i \in H^{2i}(B\Lambda) \cong H^{2i}(\Gamma_g^0)$. When we have an analytic family $\pi : X \to B$ which coincides with the pull back of $E \to B\Lambda$:

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B\Lambda, \end{array}$$

then $\hat{f}^*(\omega)$ is $c_1$ of the relative dualizing sheaf. Mumford [M1] shows that $f^*(\kappa_i)$ may be regarded as a Chow cohomology class in $A^i(B) \otimes \mathbb{Q}$. Miller proves:

Theorem 7.3: Let $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$ be the polynomial algebra on the $\kappa_i$. Then the induced map

$$\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to H^*(\Gamma_g^0;\mathbb{Q})$$

is injective in dimensions less than $g/3$.

To prove this theorem, Miller first looks at the Hopf algebra $A = \varinjlim H_*(\Gamma_{g,1};\mathbb{Q})$ defined in Chapter 6 and proves that each $\kappa_i$ vanishes on the $\lambda_*$ decomposibles. It therefore suffices to construct explicit homology classes $X_n \in H_{2n}(\Lambda;\mathbb{Q})$ such that $[\kappa_n,X_n] \neq 0$. Theorem 7.3 then follows from the stability Theorem 6.1.

i) Construction of the Classes $X_n$

The classes $X_n$ are provided by:

Lemma 7.4: There exists for each $n > 0$ a smooth fibration of projective algebraic varieties $\pi : X^{n+1} \to B^n$ with fiber $F_g$ such that $[\omega^{n+1},X] \neq 0$, where $\omega$ is $c_1$ of the tangents to the fiber of $\pi$.

The construction of the fibration $\pi : X^{n+1} \to B^n$ is modeled on one of Atiyah [A] . First we review Atiyah's construction which gives the example for $n = 1$. Let $C$ be a connected curve of genus $g > 2$ and let $\tau : C \to C$ be a free involution. The base $B^1$ is the $2^{2g}$-fold covering of $C$ determined by the map $\pi_1(C) \to H_1(C;\mathbb{Z}/2\mathbb{Z})$; if $f : B^1 \to C$ is the covering map, then $f^*$ is $0$ on $H^1$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Now look at $B^1 \times C$ and consider the graphs $G_f$ and $G_{\tau f}$. Atiyah's choice of $f$ guarantees that the homology class of $G_f + G_{\tau f}$ in $H_2(B^1 \times C;\mathbb{Z})$ is even, therefore it is possible to form the 2-fold cover $X^2 \to B^1 \times C$ ramified along the divisor $G_f + G_{\tau f}$. Composing with the projection to $B^1$ gives $\pi : X^2 \to B^1$ with fiber of genus $2g$. Atiyah now shows that

$$\omega^2/3 \;=\; \text{signature } (X^2) \;=\; (g-1)2^{2g-1}$$

where $\omega$ is $c_1$ of the tangents to the fibers.

To generalize the construction, fix an epimorphism $H_1(C;\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}$; the composition $\pi_1(C) \to H_1(C;\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z} \oplus \mathbb{Z}/2^n\mathbb{Z}$ has kernel $K_n$ and determines a $4^n$-fold covering $\dot{C}_n \to C$. Miller uses the tower $\to C_n \to \ldots \to C_1 \to C_0 = C$ to inductively define $X^{n+1} \to B^n$ with connected fiber $Y_n$ such that:

(1) $X^{n+1}$ maps onto $X^n$ in such a way that the composition

$$X^{n+1} \to X^n \to \ldots \to X^2 \to B^1 \times C \to C$$

maps $\pi_1(Y^n)$ and $\pi_1(X^{n+1})$ onto $K_{n-1}$, and

(2) $[\omega^{n+1}, X^{n+1}] \neq 0$, $\omega = c_1$ (tangents to the fiber) as above.

Assume inductively that $X^{i+1} \to B^i$ has been constructed for all $i < n$. By (1), $X^n \to C$ lifts to $X^n \to C_{n-2}$ and sends both $\pi_1(Y_{n-1})$ and $\pi_1(X^n)$ onto $K_{n-2}$. Define $X'$ to be the 4-fold cover $X' \to X^n$ induced by the composition $\pi_1(X^n) \to K_{n-2} \to K_{n-2}/K_{n-1} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with a similar definition for $Y' \to Y_{n-1}$. Clearly both $\pi_1(X') \to \pi_1(C)$ and $\pi_1(Y') \to \pi_1(C)$ have image $K_{n-1}$. Form the fiber product

$$\begin{array}{ccc} X' \times_{B^{n-1}} X' & \xrightarrow{p_2} & X' \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ X' & \longrightarrow & B^{n-1} \end{array};$$

the fiber of $p_1$ is $Y'$. We extend this diagram to the left as follows. Let $T^n \to X'$ be the finite covering determined by the kernel of the representation $\pi_1(X') \to \mathrm{Aut}(H_1(Y';\mathbb{Z}/2\mathbb{Z}))$ associated to $p_1$. Also let $B^n \to T^n$ be the finite covering determined by $\pi_1(T^n) \to H_1(T^n; \mathbb{Z}/2\mathbb{Z})$. Then we have the pull back diagram

$$\begin{array}{ccccc} W^{n+1} & \xrightarrow{h} & X' \times_{B^{n-1}} X' & & \\ \downarrow & & \downarrow{\scriptstyle p_1} & & \\ B^n \longrightarrow & T^n \longrightarrow & X' & & \end{array}.$$

Now $X' \times_{B^{n-1}} X'$ admits the free involution $\sigma(x_1, x_2) = (x_1, \tau x_2)$ where $\tau$ is a nontrivial covering translation of $X' \to X$. If $\Delta$ denotes the diagonal section $\Delta(x) = (x,x)$ of $p_1$, then we may look at the divisor $\Delta + \sigma\Delta$ in $X' \times_{B^{n-1}} X'$. Miller proves that the homology class of $h^{-1}(\Delta+\sigma\Delta)$ is even. Therefore it makes sense to define $X^{n+1}$ to be the 2-fold cover of $W^{n+1}$ ramified along $h^{-1}(\Delta+\sigma\Delta)$. The composition $X^{n+1} \to W^{n+1} \to B^n$ is the desired fibration.

The entire construction is summarized in the diagram

$$\begin{tikzcd}
X^{n+1} \arrow[d] & & & & \\
W^{n+1} \arrow[rr, "h"] \arrow[d] & & X' \times X' \arrow[r, "p_2"] \arrow[d, "p_1"'] & X' \arrow[r] \arrow[d] & X^n \arrow[d] \\
B^n \arrow[r] & T^n \arrow[r] & X' \arrow[r] & B^{n-1} \arrow[r, phantom, "="] & B^{n-1} .
\end{tikzcd}$$

The map $p : X^{n+1} \to X^n$ promised in (1) above is the obvious one in this diagram; it is easy to check it has the right properties.

To calculate $[\omega^{n+1}, X^{n+1}]$ choose a holomorphic differential on $C$ and use the maps of (1) to pull it back to $\omega_{n+1}$ on $X^{n+1}$ and $\omega_n$ on $X^n$. One computes that

$$\omega_{n+1} = p^*(\omega_n) - (h^{-1}(\Delta+\sigma\Delta))$$

and then uses this to work back through the diagrams to see

$$[\omega_{n+1}^{n+1}, X^{n+1}] = 16N((1/2)^{n+1}-1)[\omega_n^n, X^n] \neq 0$$

where $N$ is the degree of the covering $B^n \to X'$ (see [Mi] for details).

ii) <u>Comparison with</u> $H^*(Sp(2g;\mathbb{Z});\mathbb{Q})$

In [B] Borel found the stable cohomology of $Sp(2g;\mathbb{Z})$ with $\mathbb{Q}$-coefficients; it is a polynomial algebra on generators in dimensions $4k+2$, $k \geq 0$. Miller considers the problem of computing the map induced on cohomology by $\mu : \Gamma_g \to Sp(2g;\mathbb{Z})$.

The map $\mu$ induces a map $BDiff^+(F_g) \to BSp(2g;\mathbb{Z})$. Using the inclusion $Sp(2g;\mathbb{Z}) \subset Sp(2g;\mathbb{R})$ and the fact that $U(g)$ is the maximal compact subgroup of $Sp(2g;\mathbb{R})$ we obtain a map

$$\bar{\mu} : BDiff^+(F_g) \to BU(g).$$

The map $\bar{\mu}$ sends the $\lambda$ product in $A$ to the product in $BU$ induced by Whitney sum. If $\nu$ denotes the universal bundle over $BU$, the characteristic classes $s_n(\nu)$ (the polynomials in the Chern classes of $\nu$ corresponding to $\sum t_j^n$) vanish on decomposibles. Miller [Mi], Mumford [M1] and Morita [Mor] prove independently that

$$\bar{\mu}^*(s_n(\nu)) = \begin{cases} 0 & n \text{ even}, \\ (-1)^{\frac{n+1}{2}} \dfrac{B_{\frac{n+1}{2}}}{n+1} \cdot \kappa_n & n \text{ odd}, \end{cases}$$

where $B_n$ is the $n^{\underline{th}}$ Bernoulli number. In particular, this implies $\mu^*$ is injective.

iii) Relations among the $\kappa_i$

In [M1], Mumford gave an algebraic construction of the classes $\kappa_i$ and proved relations between them. These and other relations (along with Miller's polynomial algebra) were rediscovered by Morita [Mor] who described them topologically. Harris ([Ha]), while considering the problem of determining when certain linear combinations of divisor classes are ample and/or effective, also find relations among the classes in $\mathrm{Pic}(M_g^1) \otimes \mathbb{Q}$. We will briefly describe these relations.

Let $\eta$ be the $g$-dimensional complex vector bundle over $B\Lambda$ ($\Lambda = \mathrm{Diff}^+F_g$ as before) induced by $\bar{\mu}$, and let $s_i(\eta)$ be the pull backs of the characteristic classes $s_i(\nu)$ by $\bar{\mu}$. As a real bundle, $\eta$ is determined by the map $\Gamma_g \to Sp(2g;\mathbb{R})$, so it is flat. This means all its Pontrjagin classes vanish, or equivalently

$$s_{2i}(\eta) = 0 \qquad \text{for all } i. \tag{R1}$$

This is a relation in $H^{4i}(\Gamma_g^0;\mathbb{Q})$.

Let $\xi = \pi: E \to B\Lambda$ be the universal bundle over $B\Lambda$ and consider the conjugate of the pull back $\overline{\pi^*(\eta)}$ over $E$. The fibers of the bundle $\eta$ over the point $x$ may be identified naturally with $H^1(E_x;\mathbb{R})$ which in turn may be identified with the space of harmonic 1-forms on $E_x$. Using this description, define a map $\phi: \overline{\pi^*(\eta)} \to \xi^*$ by setting

$$\phi(\omega)(v) = \omega(v) + i\omega(v).$$

This gives a sequence

$$0 \to \mathrm{Ker}(\phi) \to \overline{\pi^*(\eta)} \to \xi^* \to 0;$$

Mumford and Morita observe $c_k(\mathrm{Ker}\ \phi) = 0$ for $k \geq g$. This gives

$$\sum_{j=0}^{g} \kappa^{k-j} c_j = 0, \quad k \geq g, \tag{R2}$$

where $\kappa = \pi^*(\kappa_1) \in H^2(E)$ and $c_j$ denotes the pull back to $H^{2j}(E)$ of the $j^{\underline{th}}$ Chern class of $\eta$. Since $E$ is a $K(\Gamma_g^1,1)$, this is a relation in $H^*(\Gamma_g^1;\mathbb{Q})$.

Applying $\pi_*$ to (R2) gives

$$\sum_{j=0}^{g} \kappa_{k-1-j}\, \pi_*(c_j) = 0, \quad k \geq g, \tag{R3}$$

a relation in $H^*(\Gamma_g;\mathbb{Q})$. Mumford shows that these relations imply that $\kappa_k$ is a polynomial in $\kappa_1,\ldots,\kappa_{g-2}$ for all $k \geq g-1$.

In [Ha] Harris considers the following question. Let $\omega$ be $c_1$ of the relative dualizing sheaf as before and let $\kappa = \pi^*\pi_*\omega^2 \in H^2(\mathcal{M}_g;\mathbb{Q}) \cong \mathrm{Pic}(\mathcal{M}_g) \otimes \mathbb{Q}$. Consider complete, nondegenerate families of curves $X \to B$ with associated classifying map $\phi : B \to \mathcal{M}_g$ (non-degenerate means $\phi$ has finite fibers). Then for what $a$, $b$ is $a\omega + b\kappa$ ample on every $X \to B$? He proves this is true for $a > 0$ and $a + 4g(g-1)b > 0$. He then shows that if $B$ is 2-dimensional,

$$\kappa_1^2 > (2g-2)\kappa_2 \tag{R4}$$

and finally, the most amazing relation of all is

$$(4g(g-1)\omega - \kappa)^{g+1} = 0. \tag{R5}$$

We refer the reader to [Ha] for more information and proofs.

## §3. Homology of $\overline{\mathcal{M}}_g^s$

Using Mayer-Vietoris one can easily relate $H_*(\mathcal{M}_g^s)$ to $H_*(\overline{\mathcal{M}}_g^s)$. For example, $\beta_1(\mathcal{M}_g^s) = 0$ implies $\beta_1(\overline{\mathcal{M}}_g^s) = 0$ and we saw in Chapter 5 that $\beta_2(\mathcal{M}_g) = 1$ implies $\beta_2(\overline{\mathcal{M}}_g) = 2 + [g/2]$. The fact that $\kappa_1$ is symplectic on $\overline{\mathcal{M}}_g$ shows that $\kappa_1^{3g-3} \neq 0$. What is more, Wolpert uses intersections of the $D_i$ to show that $\beta_{2k}(\overline{\mathcal{M}}_g) \geq \frac{1}{2}\binom{g-1}{k}$.

Let $\mathfrak{h}_g$ denote the Siegel upper half space of degree $g$,

$$\mathfrak{h}_g = \{Z \in M_g(\mathbb{C}) : Z = Z^t,\ \mathrm{Im}Z > 0\}$$

where $M_g(\mathbb{C})$ is $g \times g$ matrices with complex entries and $t$ denotes transpose. The group $Sp(2g,\mathbb{Z})$ acts on $\mathfrak{h}_g$ by the formula

$$Z \cdot M = (ZC+D)^{-1} \cdot (ZA+B)$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The quotient $\mathfrak{h}_g/Sp(2g;\mathbb{Z})$ is called the <u>Siegel modular space</u> and may be identified with $A_g$, the space of <u>principally polarized abelian varieties</u>. The action of $Sp(2g;\mathbb{Z})$ is properly discontinuous and $\mathfrak{h}_g \simeq *$ so $H^*(A_g;\mathbb{Q}) \cong H^*(Sp(2g;\mathbb{Z});\mathbb{Q})$.

One may associate to each Reimann surface its Jacobian; this defines a mapping

$$J : \mathcal{M}_g \to A_g$$

called the <u>period mapping</u>. The classical Torelli theorem says that $J$ is an embedding. Rationally, the maps $J^*$ and $\mu^*$ are the same on cohomology.

Suppose next that $\overline{A}_g$ denotes the Satake compactification of $A_g$;

it is obtained in a manner similar to $\overline{M}_g$ by adding on copies of $A_k$, $k < n$, at $\infty$. The map $J$ extends to $\overline{J} : \overline{M}_g \to \overline{A}_g$ by normalizing and then taking period matrices as before. Charney and Lee ([CL1]) prove three things in this situation which we summarize as:

Theorem 7.5: (1) For $g \geq i+1$, $H^i(\overline{A}_g;\mathbb{Q}) \cong H^i(\varinjlim_g \overline{A}_g;\mathbb{Q})$.

(2) $H^*(\varinjlim \overline{A}_g;\mathbb{Q}) \cong \mathbb{Q}[x_2,x_6,\ldots] \otimes \mathbb{Q}[y_6,y_{10},\ldots]$, where the $x_i$ live in $H^*(A_g;\mathbb{Q})$.

(3) $\mathrm{Ker}(\overline{J}^*) = \langle y_6,y_{10},\ldots\rangle$; that is, image $(\overline{J}^*)$ = image $(J^*)$ = image$(\mu^*)$.

This result is proven by decomposing $\overline{A}_g$ as a union of simplicial $K(\pi,1)$'s, relating the pieces to K-theory using $\mathbb{Q}$ categories and then applying results from K-theory to compute the cohomology.

## §4. Torsion in $H^*(\Gamma_g;\mathbb{Z})$

For completeness we will state here what is known about torsion in $H^*(\Gamma_g^0;\mathbb{Z})$. By stability, $H^*(\Gamma_g^0;\mathbb{Z}) \cong H^*(\Gamma_{g,1}^0;\mathbb{Z})$ and since $\Gamma_{g,1}^0$ is torsion free, $H^*(\Gamma_{g,1}^0;\mathbb{Z}) \cong H^*(M_{g,1}^0;\mathbb{Z})$ where $M_{g,1}^0$ is the moduli space of triples $(R,p,v)$ with $R$ a Riemann surface, $p$ a point of $R$ and $v$ a unit tangent vector to $R$ at $p$. However, we only have $H^*(\Gamma_g;\mathbb{Q}) \cong H^*(M_g;\mathbb{Q})$; it is not known if the torsion classes we shall describe lie in $M_g$.

Outside of low dimensional phenomena, the first construction of torsion in $H^*(\Gamma_g)$ seems to be due to Charney and Lee [CL]. As a special case of their work on classifying spaces of Hodge structures, they prove that for every odd prime $p$ there exists a $p$-torsion class in $H^{2p-2}(\Gamma^0_{(p-1)/2};\mathbb{Z})$. This result was strengthened by Glover and Mislin [GM] who showed:

Theorem 7.6: The stable cohomology group $H^{4k}(\Gamma_g;\mathbb{Z})$ $(g >> k)$ contains an element of order $E_{2k}$ = denominator of $B_{2k}/2k$, where $B_{2k}$ is the $2k^{\underline{th}}$ Bernoulli number.

For $p$ prime, the number $E_{2k}$ is divisible by $p^\alpha$ if and only if $p^{\alpha-1}(p-1)$ divides $2k$. Together with Theorem 7.6, this can be used to show

Corollary 7.7: Let $p$ be prime and odd and let $\alpha \geq 1$. Then , for every $j \geq 1$, $H^{2jp^\alpha(p-1)}(\Gamma_g^0;\mathbb{Z})$ contains an element of order $p^\alpha$.

## Chapter 8: The Euler Characteristic of Moduli Space

The results in this chapter are all taken from [HZ].

Let $\hat{\Gamma}$ be a torsion free subgroup of finite index in $\Gamma_g^s$. The subgroup $\hat{\Gamma}$ acts properly discontinuously and freely on $T_g^s$, the quotient is therefore a manifold. We define the <u>orbifold Euler characteristic</u> of $\Gamma_g^s$ to be

$$\chi(\Gamma_g^s) = [\Gamma_g^s : \hat{\Gamma}]^{-1} \cdot e(T_g^s/\hat{\Gamma})$$

where $e(T_g^s/\hat{\Gamma})$ is the ordinary Euler characteristic of $T_g^s/\hat{\Gamma}$. The rational number $\chi(\Gamma_g^s)$ does not depend on the choice of the subgroup $\hat{\Gamma}$.

We will use the contractible cell complex $Y$ of Chapter 3 to compute $\chi(\Gamma_g^s)$. Using the exact sequence $(S_1)$ of Chapter 1 together with the fact that orbifold Euler characteristics are multiplicative in short exact sequences, we see that it suffices once again to consider the case where $s = 1$. The main theorem of [HZ] is:

<u>Theorem 8.1</u>: $\chi(\Gamma_g^1) = \zeta(1-2g)$.

Here $\zeta$ denotes the Riemann zeta function; its value at $1-2g$ is given by $\zeta(1-2g) = -B_{2g}/2g$, where $B_{2g}$ is the $2g^{\underline{th}}$ Bernoulli number.

An equivalent formulation of the theorem is given by

$$\chi(\Gamma_g) = \frac{B_{2g}}{4g(g-1)} \qquad (g > 1),$$

and the general expression is

$$\chi(\Gamma_0^s) = \begin{cases} 1 & s \leq 3 \\ (-1)^{s-3}(s-3)! & s \geq 3 \end{cases}$$

$$\chi(\Gamma_1^s) = \begin{cases} -\frac{1}{12} & s \leq 1 \\ \frac{(-1)^s (s-1)!}{12} & s \geq 1 \end{cases}$$

$$\chi(\Gamma_g^s) = (-1)^s \frac{(2g+s-3)!}{2g(2g-2)!} B_{2g} \qquad g \geq 2, \quad s \geq 0.$$

By a result of Ken Brown ([Br]), the ordinary Euler characteristic

$$e(\Gamma_g^s) = \sum_{i\geq 0} (-1)^i \text{ rank } (H_i(\Gamma_g^s;\mathbb{Q}))$$

can be computed from the orbifold Euler characteristics of the centralizers of the elements of finite order in $\Gamma_g^s$. The exact formula is

$$e(\Gamma_g^s) = \sum_{<\sigma>} \chi(Z_\sigma)$$

where the sum is over all conjugacy classes $<\sigma>$ of elements $\sigma$ of finite order in $\Gamma_g^s$ and $Z_\sigma$ denotes the centralizer of $\sigma$. These centralizers are extensions of finite groups by mapping class groups, so their Euler characteristics can be computed from the formulas above. Working this out for $s = 0$ gives:

Theorem 8.2: The numbers $e(\Gamma_g)$ are given by the generating function

$$\sum_{g\geq 1} e(\Gamma_g)t^{2g-2} = \sum_{\substack{k\geq 1\\ m,d|k}} \sum_{\substack{h,s\geq 0\\ s+2h\geq 3}} \frac{\chi(\Gamma_h^s)}{s!} \frac{\mu(m)}{m^2} \phi(d) \left(\frac{k}{m}t^k\right)^{2h-2} \beta_{\frac{k}{m},\frac{d}{(d,m)}}(t^m)^s,$$

where $\mu$ denotes the Möbius function, $\phi$ the Euler totient function and $\beta$ is given by

$$\beta_{k,d}(t) = \sum_{\substack{\ell|k\\ \ell\neq k}} \mu\left(\frac{d}{(d,\ell)}\right) \frac{\phi(k/\ell)}{\phi(d/(d,\ell))} t^{k-\ell}.$$

This theorem can be used to easily derive values of $e(\Gamma_g)$. For example, we have the following table when $g \leq 15$:

| $g$ | $e(\Gamma_g)$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 3 |
| 4 | 2 |
| 5 | 3 |
| 6 | 4 |
| 7 | 1 |
| 8 | -6 |
| 9 | 45 |
| 10 | -86 |
| 11 | 173 |
| 12 | -100 |
| 13 | 2641 |
| 14 | -48311 |
| 15 | 717766 |

An important consequence of Theorem 8.2 is that it can be used to show $e(\Gamma_g) \sim \chi(\Gamma_g)$; therefore the Betti numbers of $\Gamma_g$ grow more that exponentially in $g$ and $\Gamma_g$ has a lot of homology in dimensions congruent to $g-1$ modulo 2. We have seen how to construct even dimensional classes in Chapter 7 (but not enough to account for the size of $e(\Gamma_g)$ even when $g$ is odd); we know of no constructions of odd-dimensional classes.

## §1. Basics of the Proof

There are three parts to the proof of Theorem 8.1; we outline them now. Let $P_n$ be a polygon with $2n$ sides; a pairing of the sides of $P_n$ gives a unique oriented surface of genus $g \leq n/2$. Using these pairings we define three double sequences of numbers: $\varepsilon_g(n)$ is the number of pairings of the edges of $P_n$ which give a surface of genus $g$, $\mu_g(n)$ is the number of parings of the edges of $P_n$ which give a surface of genus $g$, but with no edge paired to its neighbor (figure 8.1 configuration A) and $\lambda_g(n)$ is the number of pairings giving a surface of genus $g$, but without an occurence of either configuration A or B of figure 8.1.

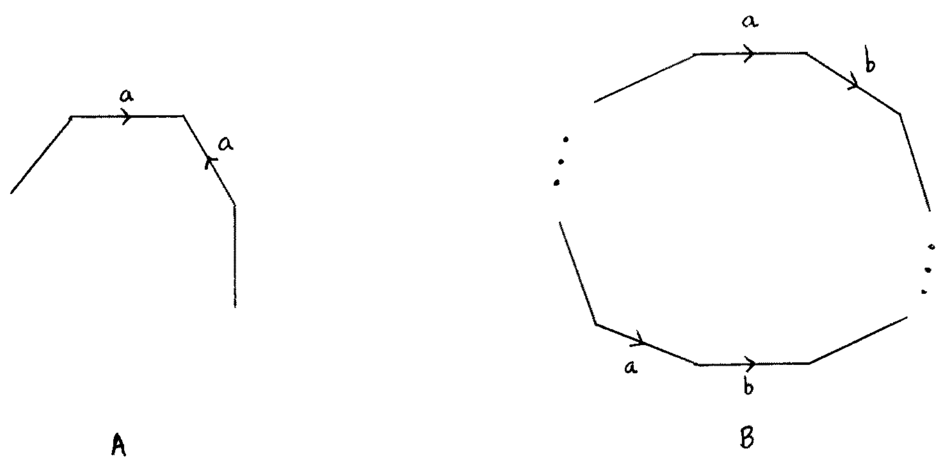


Figure 8.1

After the identification of the sides of $P_n$ has been made, $\partial P_n$ becomes a finite graph $\Omega \subset F$ (compare Chapter 2). The conditions above say that $\varepsilon_g(n)$ counts everything, but $\mu_g(n)$ counts only the pairings for which every vertex of $\Omega$ has valence $\geq 2$ and $\lambda_g(n)$ counts pairings $\Omega$ with vertices of valence $\geq 3$.

By analyzing the action of $\Gamma_g^1$ on $Y$ we will prove in §2:

<u>Theorem 8.3</u>: $\chi(\Gamma_g^1) = \sum_{n=2g}^{6g-3} (-1)^{n-1} \lambda_g(n)/2n.$

It is not difficult to relate the numbers $\varepsilon_g(n)$, $\mu_g(n)$ and $\lambda_g(n)$. The relation is provided by the following lemma whose proof will be omitted

<u>Lemma 8.4</u>: (a) $\varepsilon_g(n) = \sum_{i\geq 0} \binom{2n}{i} \mu_g(n-i)$

(b) $\mu_g(n) = \sum_{i\geq 0} \binom{n}{i} \lambda_g(n-i).$

In §3 we will give the main combinatorial result:

<u>Theorem 8.5</u>: The numbers $\varepsilon_g(n)$ are determined by the formula

$$\varepsilon_g(n) = \frac{(2n)!}{(n+1)!(n-2g)!} \cdot \text{coefficient of } x^{2g} \text{ in } \left(\frac{x/2}{\tanh x/2}\right)^{n+1}.$$

These three statements can be combined to give Theorem 8.1 as follows. Define a polynomial of degree $d = 3g-1$ in $n$ by

$$F(n) = \frac{(n-1)!}{(n-2g)!} \cdot \text{ coefficient of } x^{2g} \text{ in } \left(\frac{x/2}{\tanh x/2}\right)^{n+1}.$$

Clearly $F(-1) = 0$, so $F(n)/(n+1)$ is still a polynomial in $n$ and can be written

$$F(n) = (n+1) \cdot \sum_{r=1}^{d} \frac{r!}{(2r)!} \kappa_g(r) \cdot (n-1)(n-2)\cdot\ldots\cdot(n-r+1),$$

for some numbers $\kappa_g(r)$. The definition of $F$ and Theorem 8.5 give:

$$\text{(c)} \quad \varepsilon_g(n) = \frac{(2n)!}{n!} \sum_{r=1}^{d} \frac{r!}{(2r)!} \frac{\kappa(r)}{(n-r)!}.$$

the relationship between $\kappa$, $\varepsilon$, $\mu$ and $\lambda$ can be understood best by using the generating functions

$$K(x) = \sum_{n\geq 0} \kappa_g(n)x^n, \quad E(x) = \sum_{n\geq 0} \varepsilon_g(n)x^n, \quad M(x) = \sum_{n\geq 0} \mu_g(n)x^n,$$

$$L(x) = \sum_{n\geq 0} \lambda_g(n)x^n.$$

Formulas (a) and (b) of Lemma 8.4 and formula (c) above may then be used to show

$$L(x) = \frac{1}{1+x} M\left(\frac{x}{1+x}\right) = \frac{1}{(1+x)(1+2x)} E\left(\frac{x(1+x)}{(1+2x)^2}\right) = \frac{1}{1+x} K(x(1+x)).$$

Expanding gives

$$\lambda_g(n) = \sum_{r=1}^{d} \kappa_g(r)\binom{r-1}{n-r},$$

and Theorem 8.3 now yields

$$\chi(\Gamma_g^1) = \sum_{n\geq 1} (-1)^{n-1}\lambda_g(n)/2n = \sum_{r=1}^{d} \kappa_g(r) \sum_{n=r}^{2r-1} \frac{(-1)^{n-1}}{2n} \binom{r-1}{n-r}$$

$$= \sum_{r=1}^{d} \kappa_g(r) \frac{(-1)^{r-1}}{2} \frac{(r-1)!^2}{(2r-1)!} = F(0) = -B_{2g}/2g$$

as required.

§2. Counting the Cells of $Y$

The action of $\Gamma_g^1$ on $Y$ is cellular and may be used to compute $\chi(\Gamma_g^1)$. Suppose $\{\sigma_p^i\}$ is a set of representatives for the orbits of the p-cells of $Y$; thus every p-cell is identified to one of the representatives and no two are identified to one another. Then we have the following formula of Quillen ([S], prop. 11):

$$\chi(\Gamma_g^1) = \sum_p (-1)^p \sum_i \chi(G_p^i)$$

where $G_p^i$ is the stabilizer of the cell $\sigma_p^i$.

A p-cell of $Y$ is determined by a rank $n = 6g-3-p$ arc-system which fills $F$; the dual to this arc system is the graph $\Omega \subset F$ whose complement is a disk centered at $*$. Splitting $F$ along $\Omega$ gives a 2n-gon $P_n$ with its center at $*$ and an identification of $F$ with $P_n/\sim$ where $\sim$ is an edge pairing on $P_n$. The pairing $\sim$ satisfies the conditions to be counted for $\lambda_g(n)$ since $\Omega$ has valence $\geq 3$ at each vertex. The stabilizer of this p-cell is the group of rotational symmetries of $\sim$; it is finite of order $\frac{2n}{m}$ and its Euler characteristic is $(2n/m)^{-1}$.

The pairings of the edges of $P_n$ occuring in the count for $\lambda_g(n)$ may be partitioned into equivalence classes, two pairings being equivalent if they differ by a rotation of $P_n$. Choose a representative for each equivalence class, pair the sides of $P_n$ and identify the result with $F$ so that the center of $P_n$ is matched with $*$. This picks out a $p = 6g-3-n$ cell $\sigma_p^i$ for each class and $\{\sigma_p^i\}$ is a set of representatives for the action of $\Gamma_g^1$ on $Y$. If there are $m$ elements in the equivalence class, the identification will have a cyclic symmetry of order $\frac{2n}{m}$ and the corresponding $\sigma^i$ will have isotropy group which is cyclic of order $\frac{2n}{m}$. Counting $(\frac{2n}{m})^{-1}$ for each $\sigma^i$ gives the same answer as counting each of the $m$ elements in each equivalence class with weight $1/2n$. Thus

$$\sum_i \chi(G_p^i) = \lambda_g(n)/2n.$$

Theorem 8.3 follows.

§3. Combinatorics

Now we give the proof of Theorem 8.5.

i) Colorings

A <u>k-coloring</u> of the vertices of $P_n$ is a map $\phi$ from the vertices of $P_n$ to a fixed set of cardinality $k$, called the set of colors. Define $C(n,k)$ to be the number of pairs $(\phi,\tau)$ where $\phi$ is a k-coloring of the vertices of $P_n$ and $\tau$ is an identification of the edges of $P_n$ which is compatible with $\phi$ (thus two edges may be identified by $\tau$ only if the left end of each has the same color as the right end of the other). If we first identify by $\tau$, the number of inequivalent vertices is $n+1-2g$ where $g$ is the genus of the resulting surface. These vertices can be colored in $k^{n+1-2g}$ ways, so we have shown

$$\text{(d)} \quad C(n,k) = \sum_{g=0}^{n/2} \varepsilon_g(n) k^{n+1-2g}.$$

It is easy to see that the numbers $C(n,k)$ determine the numbers $\varepsilon_g(n)$. In fact, Theorem 8.5 is implied by:

<u>Theorem 8.6</u>: $C(n,k) = (2n-1)!!D(n,k)$

where $D(n,k)$ is defined by the generating functions

$$\text{(e)} \quad 1 + 2\sum_{n=0}^{\infty} D(n,k)x^{n+1} = \left(\frac{1+x}{1-x}\right)^k$$

and $(2n-1)!! = (2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1$.

<u>Proof of 8.5</u>: To see how 8.6 implies 8.5, differentiate (e) to obtain:

$$(n+1)D(n,k) = k\cdot \operatorname{Res}_{x=0}\left[\frac{1}{x^{n+1}}\left(\frac{1+x}{1-x}\right)^k \frac{dx}{1-x^2}\right].$$

Making the substitution $x = \tanh\frac{t}{2}$ gives

$$(n+1)D(n,k) = \frac{1}{2}\, k \cdot \mathrm{Res}_{t=0}\left[\left(\frac{1}{\tanh\, t/2}\right)^{n+1} e^{kt}\, dt\right]$$

$$= 2^n k \cdot \text{Coefficient of } t^n \text{ in } e^{kt}\left(\frac{t/2}{\tanh\, t/2}\right)^{n+1}$$

$$= 2^n k \cdot \sum_{r=0}^{n} \frac{k^r}{r!} \text{ Coefficient of } t^{n-r} \text{ in } \left(\frac{t/2}{\tanh\, t/2}\right)^{n+1}.$$

Since $\frac{t/2}{\tanh\, t/2}$ is an even power series, only the coefficients where $n-r$ is even, say $n-r = 2g$, are nonzero. Thus the last equality (multiplied by $\frac{(2n-1)!!}{n+1} = \frac{(2n)!}{2^n(n+1)!}$) may be written

$$(2n-1)!!D(n,k) = \frac{(2n)!}{(n+1)!} \sum_{g=0}^{n/2} \frac{k^{n+1-2g}}{(n-2g)!} \times$$

$$\text{Coefficient of } t^{2g} \text{ in } \left(\frac{t/2}{\tanh\, t/2}\right)^{n+1}. \qquad \square$$

This we are reduced to proving Theorem 8.6.

ii) Full Colorings

A full k-coloring of the vertices of $P_n$ is a k-coloring $\phi$ in which every color is used at least once ($\phi$ is surjective). Let $C_0(n,k)$ be the number of pairs $(\phi,\tau)$ where $\phi$ is a full k-coloring of the vertices of $P_n$ and $\tau$ is a compatible edge pairing. Since the number of inequivalent vertices under any $\tau$ is $n+1-2g$ we have

(f) $C_0(n,k) = 0$ if $k > n+1$.

It is easy to relate $C(n,k)$ and $C_0(n,k)$, the formula is

(g) $$C(n,k) = \sum_{\ell=0}^{k} \binom{k}{\ell} C_0(n,\ell).$$

We will use formulas (f) and (g) together with the following lemma to prove Theorem 8.6.

Lemma 8.7. The function $D(n,k)$ is a polynomial of degree $k-1$ in $n$.

Proof of 8.6. Formula (g) may be inverted to give

$$C_0(n,k) = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} C(n,\ell).$$

Therefore $C_0(n,k) = (2n-1)!!D_0(n,k)$ where $D_0(n,k) = \sum(-1)^{k-\ell}\binom{k}{\ell}D(n,k)$

is again a polynomial of degree $k-1$ in $n$. Formula (f) says that $D_0(n,k)$ vanishes for $n = 0,1,\dots,k-2$, so $D_0(n,k) = \alpha_k\binom{n}{k-1}$ where $\alpha_k$ is independent of $n$.

When $n = k-1$, $C_0(k-1,k)$ may be computed directly: the only possible identifications of the sides of a $2(k-1)$ gon which admits a full k-coloring are those giving a surface of genus 0, hence $C_0(k-1,k) = k!\,\varepsilon_0(k-1)$. The recursion

$$\varepsilon_0(n) = \sum_{a+b=n-1} \varepsilon_0(a)\varepsilon_0(b)$$

with initial value $\varepsilon_0(1) = 1$ is easy to see geometrically; it can be solved to give $\varepsilon_0(n) = \binom{2n}{n}/(n+1)$ (the $n^{\underline{th}}$ Catalin number). Working back we find $\alpha_k = 2^{k-1}$, $C_0(n,k) = (2n-1)!!\,2^{k-1}\binom{n}{k-1}$ and so

$$C(n,k) = (2n-1)!! \sum_{\ell\geq 1} 2^{\ell-1}\binom{k}{\ell}\binom{n}{\ell-1}.$$

Expanding $\left(\frac{1+x}{1-x}\right)^k = \left(1+\frac{2x}{1-x}\right)^k$ by the binomial theorem we see that this implies Theorem 8.6. □

iii) <u>An Integral Formula For</u> $C(n,k)$

Finally we come to the heart of the combinatorial argument as we carry out the proof of Lemma 8.7. We begin the proof by reversing our earlier counting procedure; first we color the vertices of $P_n$ and then we make the identifications.

Let $\phi$ be a coloring of the vertices of $P_n$ and let $n_{ij}$ be the number of edges of $P_n$ which are colored $i-j$ (where $i$ and $j$ are in the color set). Any edge identification $\tau$ must identify an edge colored $i-j$ with one colored $j-i$ but the order the edges occur is unimportant. If $n_{ij} \neq n_{ji}$ for some $i \neq j$ or if for some $i$ $n_{ii}$ is odd, then there are no compatible identifications. Thus we want to count the cases where the matrix $N = (n_{ij})$ is even and symmetric, and when it is the number of compatible edge identifications is $\prod_{i<j} n_{ij}!\,\prod_i (n_{ii}-1)!!$. Therefore

$$C(n,k) = \sum_N C(N)\,\varepsilon(N),$$

with $N=(n_{ij})$ where the $n_{ij}$ are non-negative integers such that $\sum n_{ij} = 2n$, $C(N)$ is the number of k-colorings of $P_n$ having $n_{ij}$ edges colored $i-j$ and

$$\varepsilon(N) = \begin{cases} \prod_{1\le i<j\le k} n_{ij}! \times \prod_{i=1}^{k} (n_{ii}-1)!!, & N \text{ even and symmetric,} \\ 0 & , \text{ otherwise.} \end{cases}$$

To understand the number $c(N)$ associate to each coloring of $P_n$ the product $z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{2n} i_1}$ where $i_j$ is the color of the $j^{\underline{th}}$ vertex. Allowing the variables to commute we see that this equals $\prod_{i,j} z_{ij}^{n_{ij}}$ which we denote $Z^N$ with $Z = (z_{ij})$, $1 \le i,j \le k$. Working out $\mathrm{tr}(Z^{2n})$ explicitly demonstrates the formula

$$\mathrm{tr}(Z^{2n}) = \sum_N C(N) Z^N.$$

The next step is to introduce an integral representation for the function $\varepsilon(N)$. One uses the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^n (x-iy)^m e^{-x^2-y^2}\, dxdy = \delta_{nm} n! \quad \text{and}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases} \quad \text{to see}$$

$$\varepsilon(N) = 2^{-k/2} \pi^{-k^2/2} \int_{H_k} Z^N e^{-\frac{1}{2}\mathrm{tr}(Z^2)}\, d\mu_H$$

where $H_k$ is the space of all $k \times k$ Hermitian matrices $Z = (z_{ij})$ (so $H_k \cong \mathbb{R}^{k^2}$ with variables $x_{ij} (i \le j)$ and $y_{ij} (i < j)$, and

$$z_{ij} = \begin{cases} x_{ii} & i = j, \\ x_{ij} + \sqrt{-1}y_{ij} & i < j, \\ x_{ij} - \sqrt{-1}y_{ij} & i > j,) \end{cases}$$

and $d\mu_H = \prod_{i\le j} dx_{ij} \prod_{i\le j} dy_{ij}$ is Euclidean volume. Thus we have the integral representation

$$C(n,k) = 2^{-k/2} \pi^{-k^2/2} \int_{H_k} \mathrm{tr}(Z^{2n}) e^{-\frac{1}{2}\mathrm{tr}(Z^2)}\, d\mu_H. \qquad (*)$$

iv) Evaluating the Integral

Let $T_k \subset H_k$ be the diagonal matrices ($k \times k$ with real entries), $U_k$ be the unitary group, $\Delta_k$ be the diagonal elements of $U_k$ and $W$ the group of $k \times k$ permutation matrices. Any Hermitian matrix $Z$ is conjugate under $U_k$ to an element of $T_k$ and for all $t \in T_k$ with distinct non-zero entries (therefore for almost all $t$) the choice of $u \in U_k$ with $Z = u^{-1}tu$ is unique up to left multiplication by an element of $\Delta_k \cdot W$. Define

$$T_k \times \Delta_k \backslash U_k \longrightarrow H_k, \qquad \text{by}$$

$$(t,u) \longmapsto u^{-1}tu;$$

this map is generically a covering of degree $k!$

We use the map to make a change of variables in (*) above. It is not difficult to see that

$$d\mu_H = \prod_{i<j} (t_i - t_j)^2 \cdot d\mu_{\Delta \backslash u} \cdot d\mu_T$$

where $d\mu_T$ is Euclidean volume on $T_k \cong \mathbb{R}^k$ and $d\mu_{\Delta \backslash u}$ is the measure induced on $\Delta \backslash u$ by Haar measure. Combining the above with the fact that the function $F(Z) = \operatorname{tr}(Z^{2n}) e^{-\frac{1}{2}\operatorname{tr}(Z^2)}$ is invariant under the action of $U_k$ on $H_k$ by conjugation, we have

$$\int_{H_k} F(Z)\, d\mu_H = \frac{1}{k!} \int_{T_k} \int_{\Delta_k \backslash U_k} F(u^{-1}tu) \prod_{i<j} (t_i - t_j)^2 d\mu_{\Delta \backslash U}\, d\mu_T$$

$$= C_k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_k \end{pmatrix} \prod_{1 \le i < j \le k} (t_i - t_j)^2 dt_1 \ldots dt_k$$

for some $C_k$ which does not depend on $n$. This shows

$$C(n,k) = C_k' \int_{\mathbb{R}^k} (t_1^{2n} + \ldots + t_k^{2n}) e^{-\frac{1}{2}(t_1^2 + \ldots t_k^2)} \prod_{1 \le i < j \le k} (t_i - t_j)^2 dt_1 \ldots dt_k$$

where $C_k'$ is also independent of $n$.

By symmetry of the integrand, we may replace $t_1^{2n} + \ldots + t_k^{2n}$ by $kt_1^{2n}$ without changing the value of the integral. Write

$$\prod_{i<j} (t_i - t_j)^2 = \sum_{r=0}^{2k-2} a_r\, t_1^r$$

where each $a_r$ is a function of $t_2, \ldots, t_k$, and perform the integration over $t_1$ (using the integral representation for $(n-1)!!$ introduced earlier) to obtain

$$C(n,k) = \sum_{s=0}^{k-1} \alpha_{k,s} (2n+2s-1)!!$$

where $\alpha_{k,r}$ is independent of $n$. Since $(2n+2s-1)!! = (2n-1)!!$ times a polynomial of degree $s$ in $n$, we have proven Lemma 8.7 and therefore the main result. □

# REFERENCES

[A] M. F. Atiyah, The signature of fibre bundles, Global Analysis, Papers, in Honor of K. Kodaira, Princeton University Press, (1969).

[Ba] W. L. Bailey, On the imbedding of V-manifolds in projective space, Amer. J. Math., 79 (1957), 403-430.

[BiEc] R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincare duality, Inv. Math., 20 (1973), 103-124.

[BLM] J. Birman, A. Lubotsky, J. McCarthy, Abelian and solvable subgroups of the mapping class group, Duke Math. J. 50 (1983), 1107-1120.

[B] A. Borel, Cohomologie de sous-groupes discrets et representations de groupes demisimple, Asterisque, 32-33 (1976), 73-112.

[BS] A. Borel and J. P. Serre, Corners and arithmetic groups, Comm. Math. Helv., 48 (1973), 436-491.

[BE] B. Bowditch and D. Epstein, On Natural Triangulations Associated to a Punctured Surface, preprint (1984).

[Br] K. Brown, Euler characteristics of discrete groups and G-spaces, Inv. Math., 27 (1974), 229-64.

[C] R. Charney, Homology stability for $Gl_n$ of a Dedekind domain, Inv. Math., 56 (1980), 1-17.

[CL] R. Charney and R. Lee, Characteristic classes for the classifying space of Hodge structures, preprint (1984).

[CL1] R. Charney and R. Lee, Cohomology of the Satake compactification, Topology 22 (1983), 389-423.

[D] M. Dehn, Die gruppe der abbildungsklassen, Acta Math., 69 (1971), 81-115.

[EE] C. Earle and J. Eells, The diffeomorphism group of a compact Riemann surface, Bull. A.M.S., 73 (1967), 557-559.

[G] F. P. Gardiner, Schiffer's interior variation and quasiconformal mapping, Duke Math. J., 42 (1975), 371-380.

[GM] H. Glover and G. Mislin, On the stable cohomology of the mapping class group, preprint (1985).

[Gold] W. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. in Math. 54 (1984), 200-225.

[Gr] E. Grossman, On the residual finiteness of certain mapping class groups, preprint.

[Hard] G. Harder, A Gauss-Bonnet formula for discrete arithmetically defined groups, Annales Sci. E.N.S.

[H1] J. Harer, The second homology group of the mapping class group of an orientable surface, Inv. Math., 72 (1982), 221-239.

[H2] J. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. of Math., 121 (1985), 215-249.

[H3] J. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, Invent. Math. 84 (1986), 157-176.

[HZ] J. Harer and D. Zagier, The Euler characteristic of the moduli space of curves, Invent. Math. 85 (1986), 457-485.

[Ha] J. Harris, Families of smooth curves, preprint, (1984).

[HM] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, Inv. Math., 67 (1982), 23-86.

[Har] W. Harvey, Boundary structure for the modular group, Riemann Surfaces and Related Topics: Proc. 1978 Stony Brook Conf., Princeton University Press, (1978), 245-251.

[HT] A. Hatcher and W. Thurston, A presentation of the mapping class group of a closed, orientable surface, Topology, 19 (1980), 221-237.

[K] S. Kerckhoff, The Nielson realization problem, Ann. of Math. (2) 117 (1983), 235-265.

[L1] D. Long, A note on the normal subgroups of mapping class groups, preprint, (1985).

[L2] D. Long, Maximal subgroups in mapping class groups, preprint, (1985).

[Mas] H. Masur, The extension of the Weil-Petersson metric to the boundary of Teichmüller space, Duke Math. J., 43 (1976), 623-635.

[McCa] J. McCarthy, Normalizers and centralizers of pseudo-Anosov mapping classes, preprint, (1984).

[Mc] J. McCool, Some finitely presented subgroups of the automorphism group of a free group, J. Alg, 35 (1975), 205-213.

[Mi] E. Miller, The homology of the mapping class group, J. Diff. Geom. 24 (1986), 1-14.

[MM] J. Milnor and J. Moore, On the structure of Hopf algebras, Ann. of Math., 81 (1965), 211-264.

[Mor] S. Morita, Characteristic classes of surface bundles I, preprint, (1984).

[Mos] L. Mosher, Pseudo-Anosovs on punctured surfaces, thesis, Princeton University, (1983).

[M] D. Mumford, Abelian quotients of the Teichmüller modular group, J. d'Anal. Math., 18 (1967), 227-244.

[M1] D. Mumford, Towards an enumerative geometry of the moduli space of curves, Arithmetic and Geometry, Vol. II, Birkhäuser (1983), 271-328.

[P] J. Powell, Two theorems on the mapping class group of a surface, Proc. AMS, 68 (1978), 347-350.

[Q] D. Quillen, M.I.T. lectures, (1974-5).

[R] H. Royden, Automorphisms and isometries in Teichmüller space, in Ann. of Math. Studies 66, (1971).

[Se] J. P. Serre, Cohomologie des groupes discrets, Prospects in Mathematics, Ann. of Math. Studies no. 70, Princeton University Press (1971), 77-169.

[Se1] J. P. Serre, Arithmetic groups, Homological Group Theory, edited by C.T.C. Wall, Camb. Univ. Press, (1979).

[S] K. Strebel, On quadratic differentials with closed trajectories and second order poles, J. d'Anal. Math., 19 (1967), 373-382.

[S1] K. Strebel, Quadratic Differentials, Erg. der Math. und ihrer Grenz. 5, Springer-Verlag (1984).

[T1] W. Thurston, Geometry and topology of 3-manifolds, Princeton lectures, (1978).

[T2] W. Thurston, A spine for Teichmüller space, preprint, (1984).

[T3] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, preprint.

[V] K. Vogtmann, Spherical posets and homology stability for $O_{n,n}$, Topology, 20 (1981), 119-132.

[W] J. Wagoner, Stability for the homology of the general linear group of a local ring, Topology, 15 (1976), 417-423.

[Wa] B. Wajnryb, A simple presentation for the mapping class group of an orientable surface, Israel J. Math. 45 (1983), 157-174.

[Wol1] S. Wolpert, The Fenchel-Neilsen deformation, Ann. of Math., 115 (1982), 501-528.

[Wol2] S. Wolpert, A projective embedding of the moduli space of stable curves, preprint (1983).

[Wol3] S. Wolpert, On the symplectic geometry of deformations of a hyperbolic surface, Ann. of Math., 117 (1983), 207-234.

[Wol4] S. Wolpert, Thurston's Riemannian metric for Teichmüller space, J. Diff. Geom. 23 (1986), 143-174.

[Wol5] S. Wolpert, On the Weil-Petersson geometry of the moduli space of curves, Amer. J. Math., 107 (1985), 969-998.