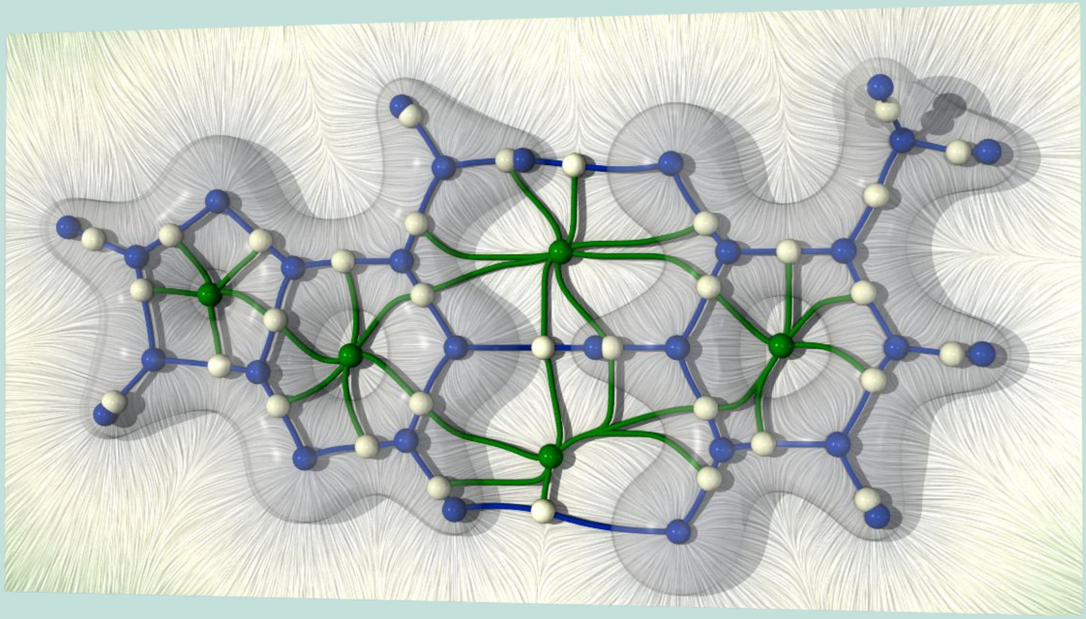
Discrete Morse Theory



Julien Tierny

cnrs

SORBONNE UNIVERSITÉ
CRÉATEURS DE FUTURS DEPUIS 1257

LIP6

Piecewise linear setting

- Input PL scalar data
  - $f : \mathcal{M} \rightarrow \mathbb{R}$

Piecewise linear setting

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Piecewise linear setting

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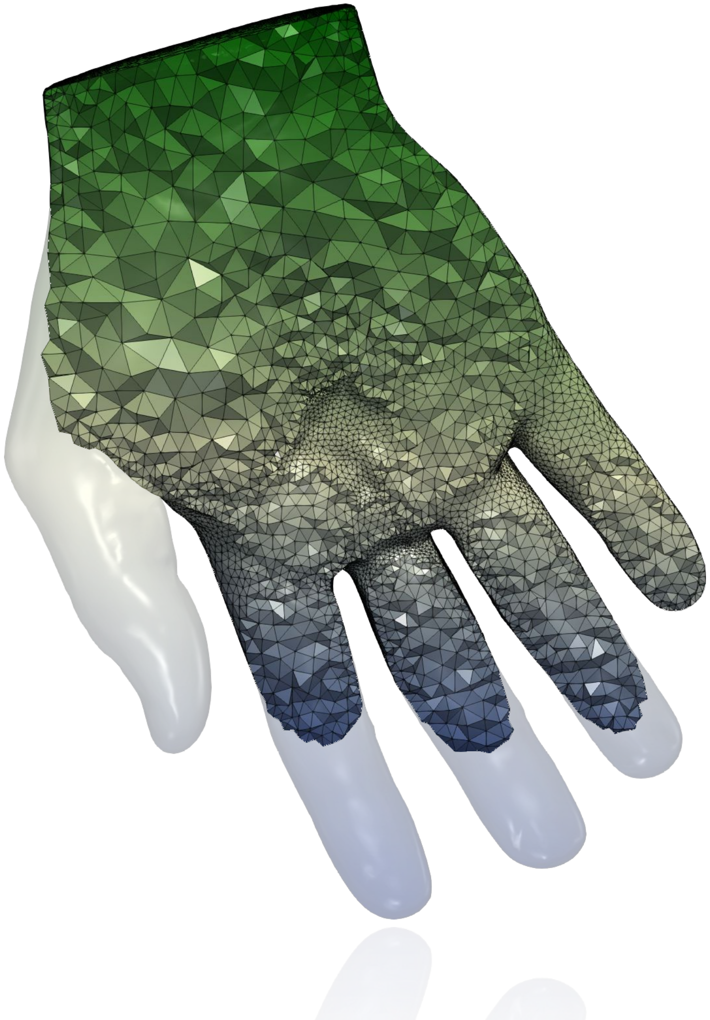
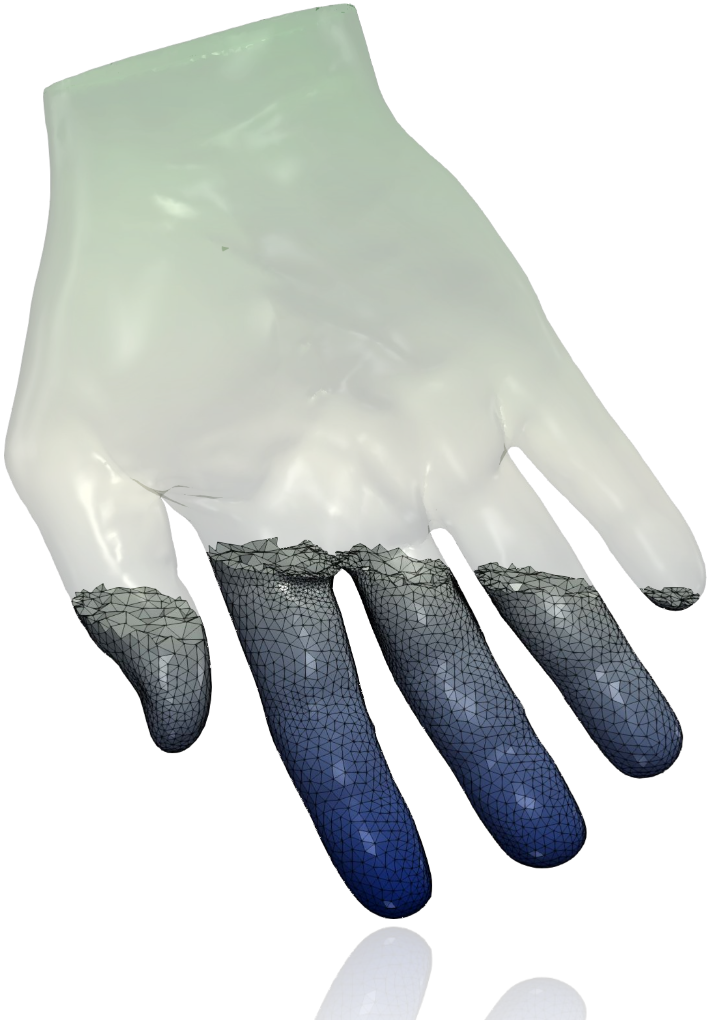


Piecewise linear setting

- Input PL scalar data
  - $f : \mathcal{M} \rightarrow \mathbb{R}$

- Topological abstractions



Piecewise linear setting

- **Input PL scalar data**
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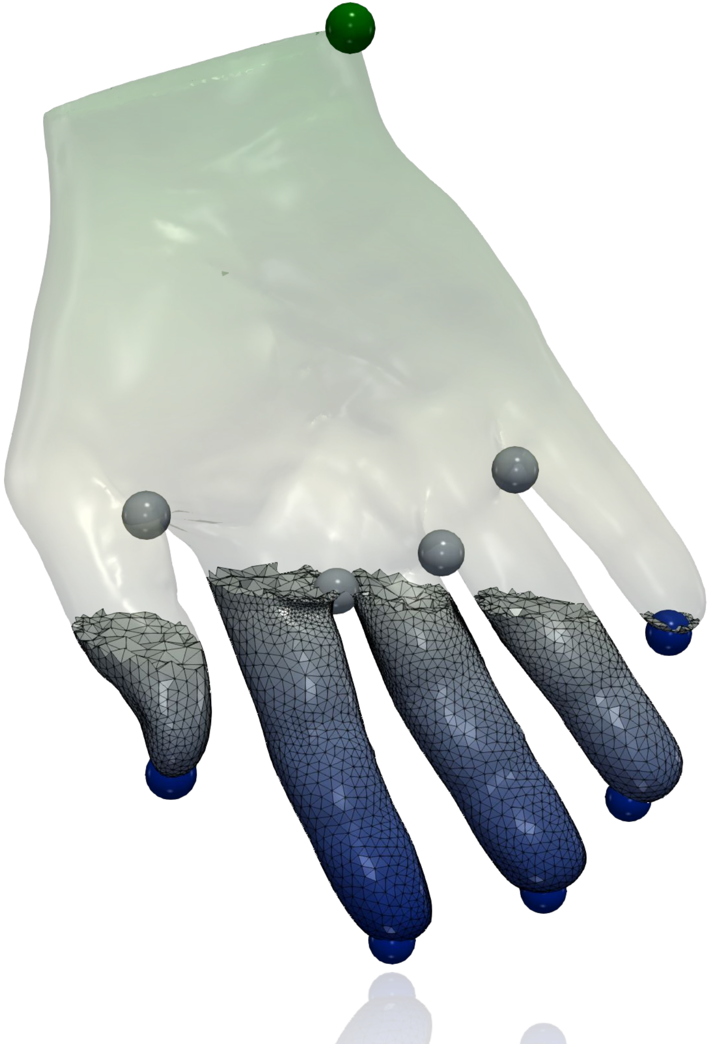
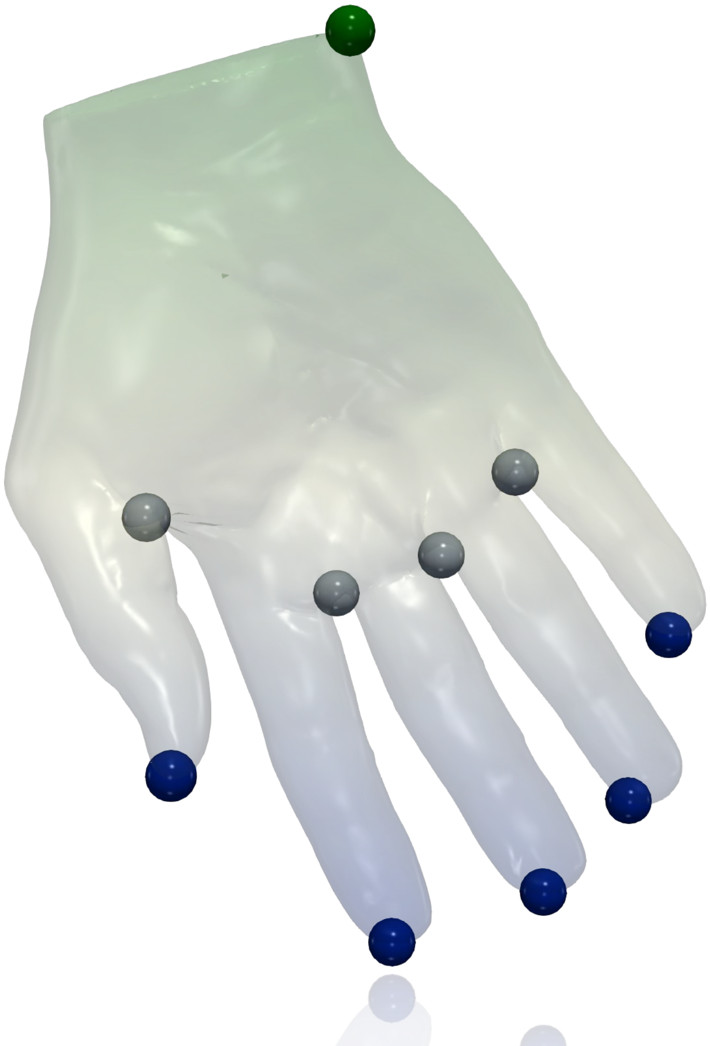


- **Topological abstractions**
  - Critical points

Piecewise linear setting

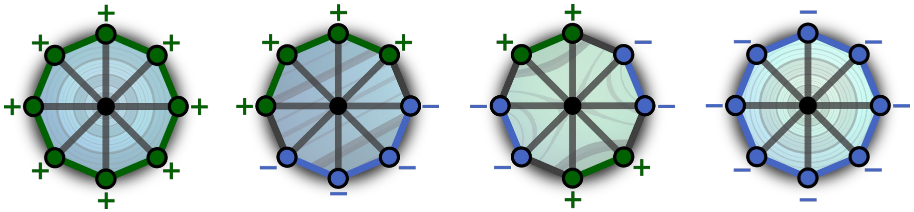
- **Input PL scalar data**
  - $f : \mathcal{M} \rightarrow \mathbb{R}$

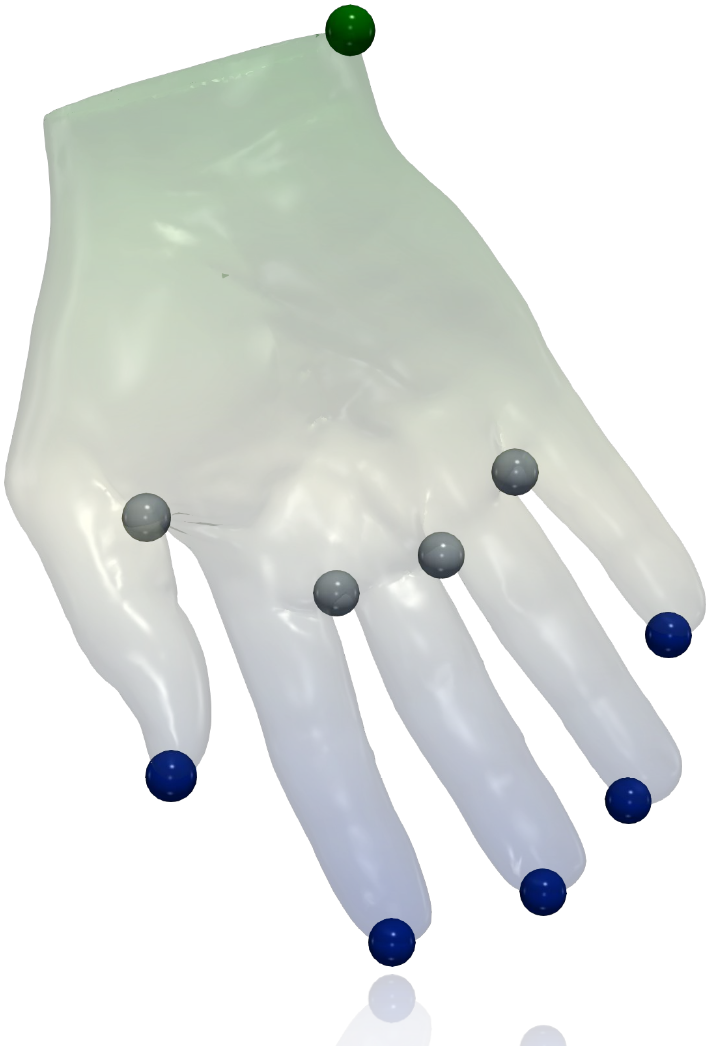


- **Topological abstractions**
  - Critical points

Critical point extraction

- **Local link inspection**
  - Banchoff 1970





Piecewise linear setting

- **Input PL scalar data**
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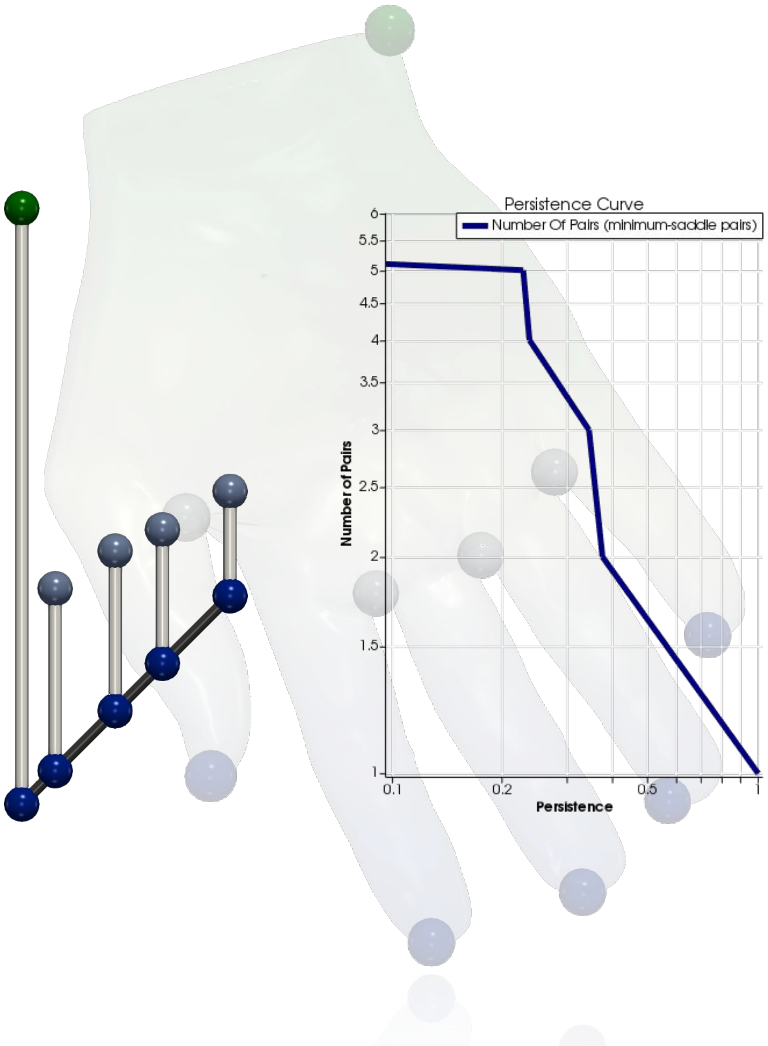
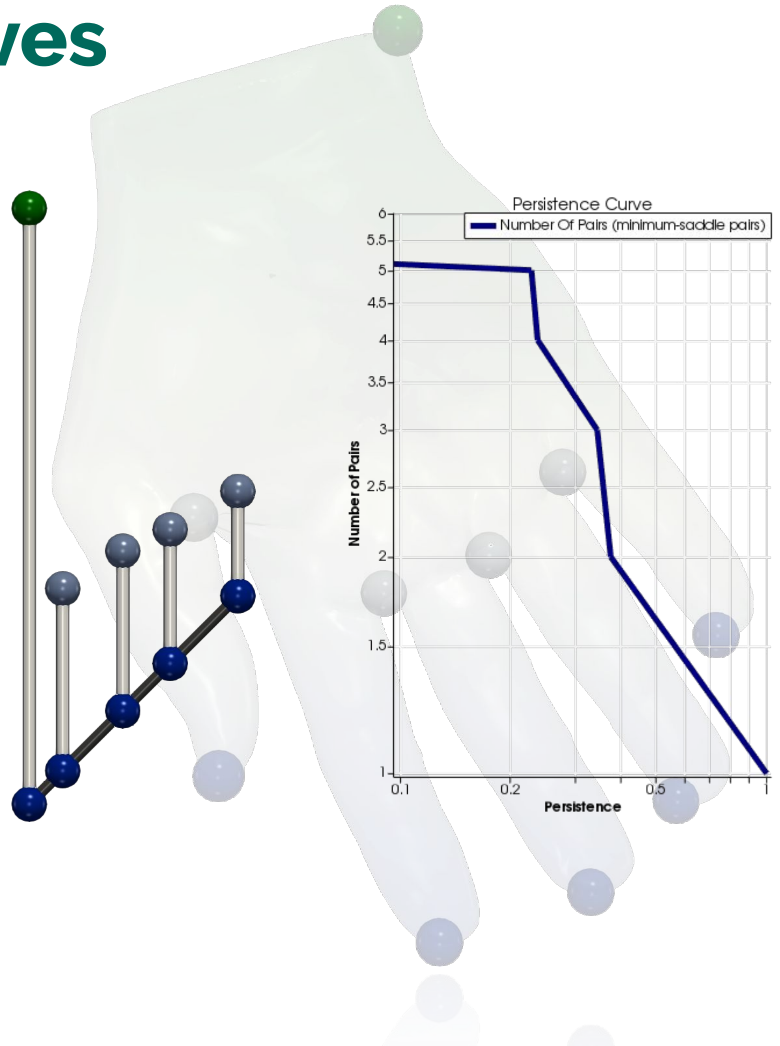


- **Topological abstractions**
  - Critical points
  - Persistence diagrams
  - Persistence curve

Persistence diagrams/curves

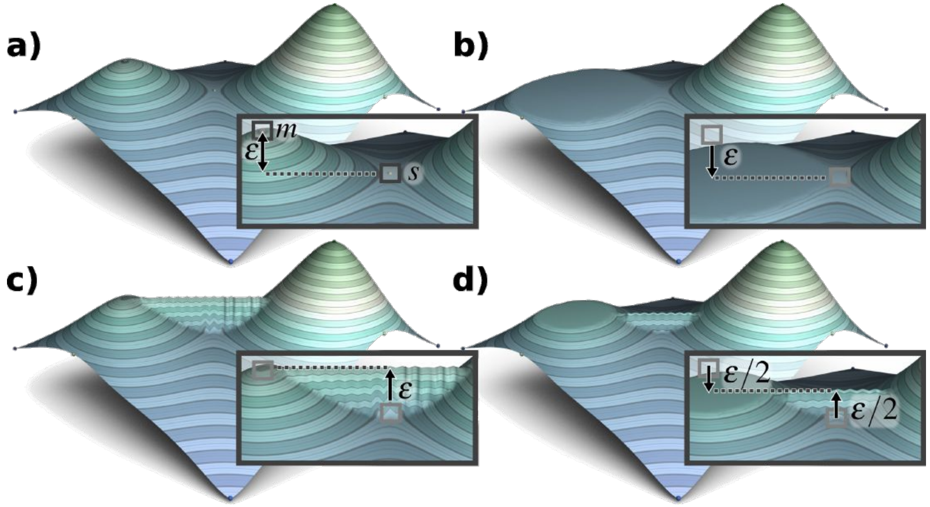
- **Arbitrary dimension**
  - Boundary matrix reduction
  - Edelsbrunner et al. 2002

- **Low-dimensions**
  - Union-Find data structures
    - Min-saddle pairs
    - Saddle-max pairs
    - *Gueunet et al. 2017*
  - Saddle connectors
    - Saddle-saddle pairs



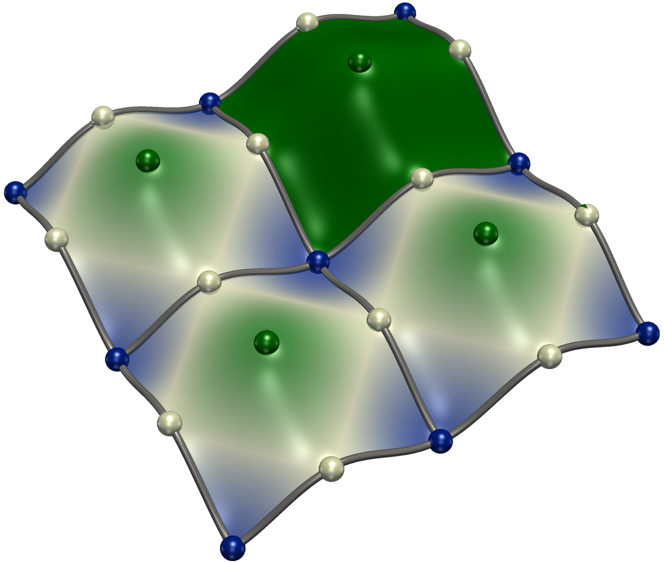
Persistence simplification

- **Simplify the data**
  - Retain only persistent features
- **Algorithms**
  - Edelsbrunner et al. 2006, Attali et al. 2009, *Tierny and Pascucci 2012*, Bauer et al. 2012



Morse complex

- **Descending manifold**
  - Given a critical point p, points of integral lines *terminating* in p
- **Morse complex**
  - All descending manifolds



Morse complex

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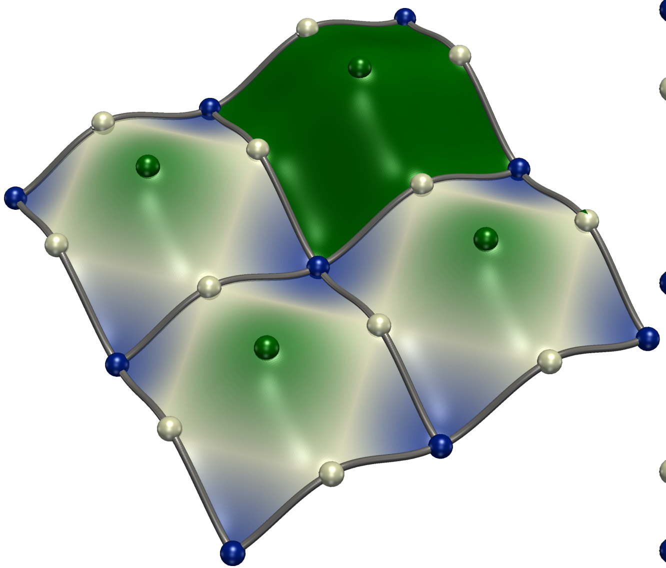

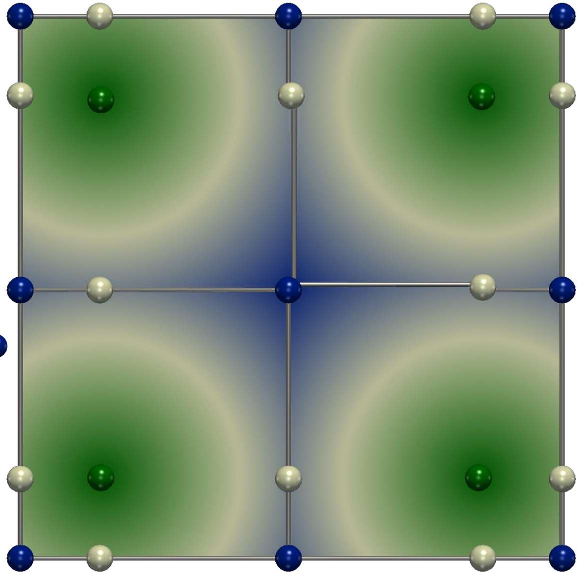

Morse complex

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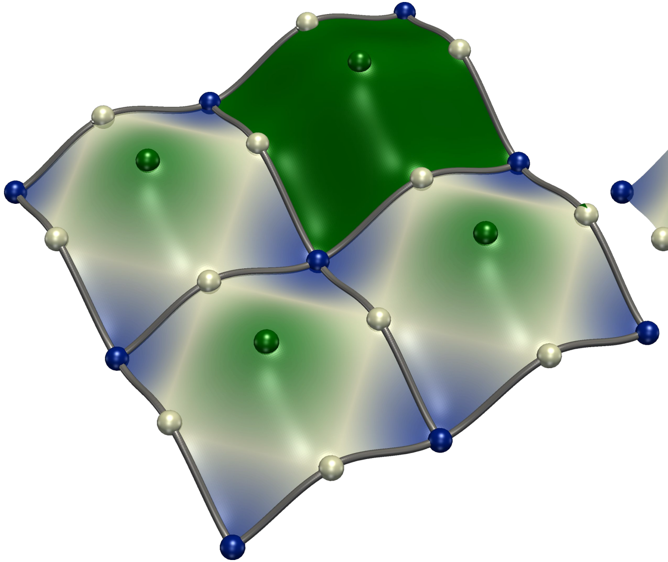
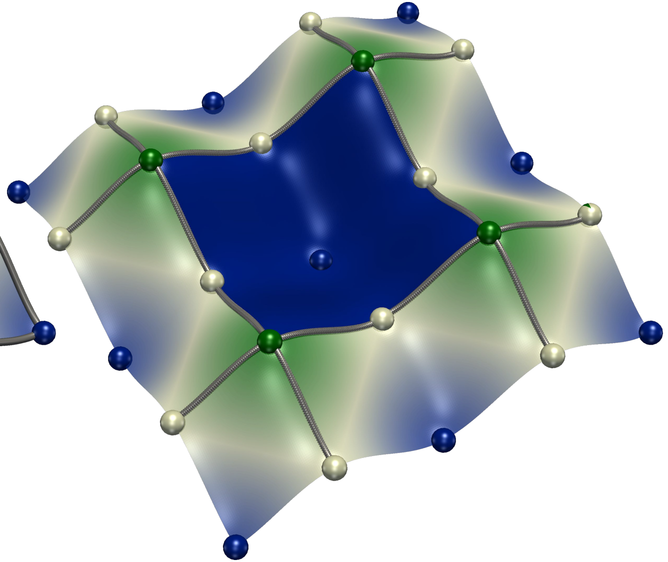
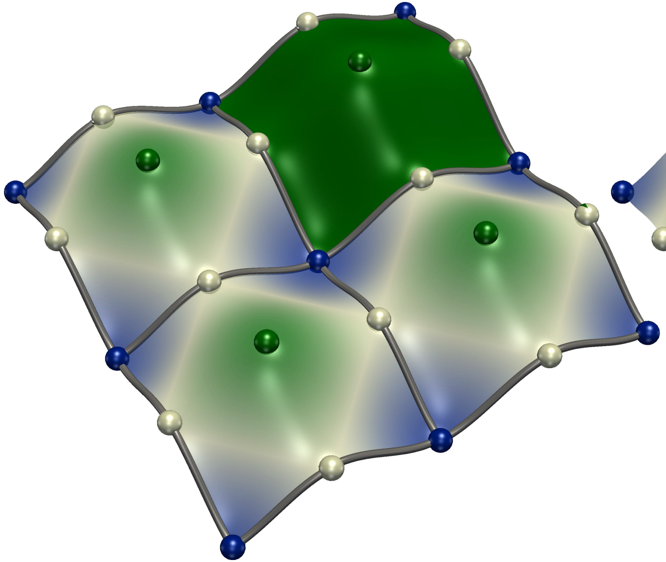
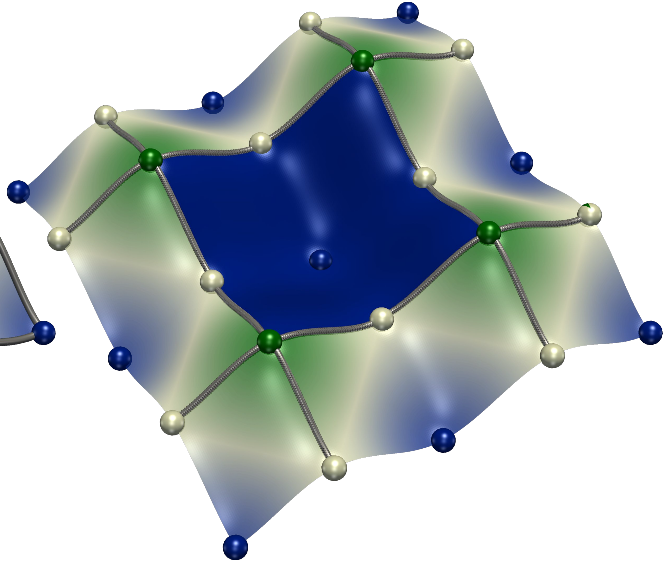



Morse-Smale complex

- **Intersection**
  - Morse complex of f
  - Morse complex of -f





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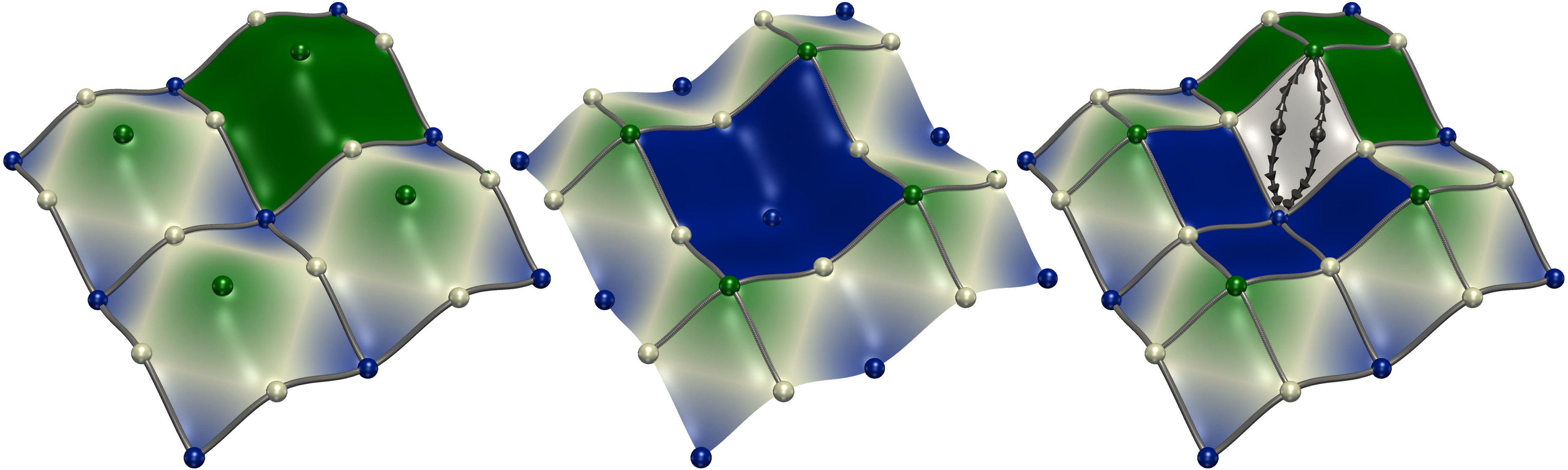
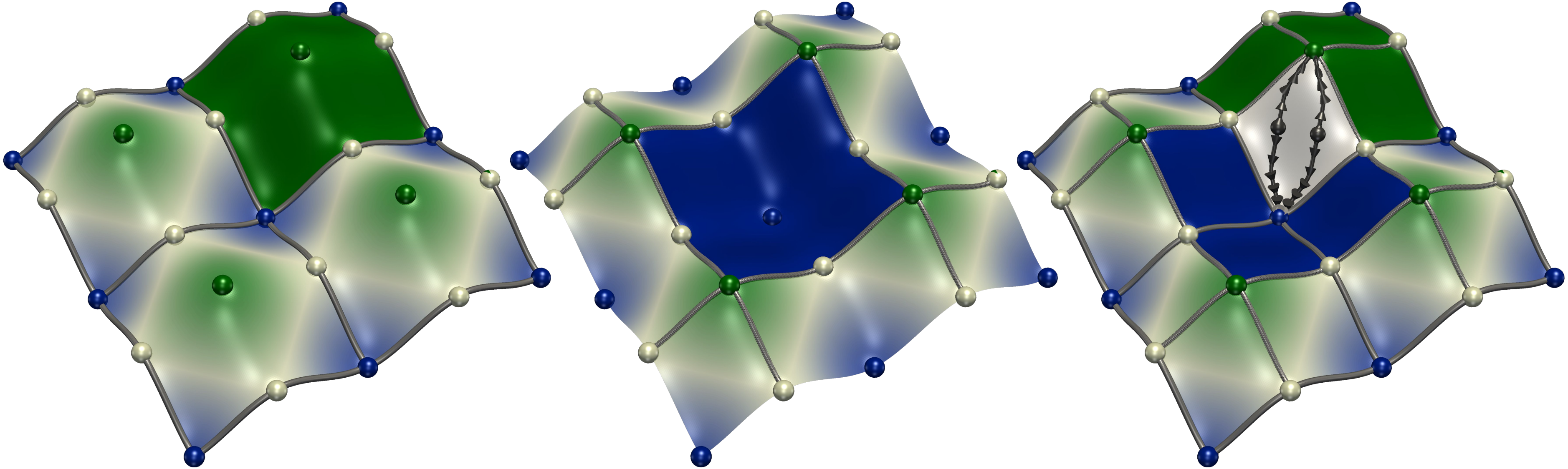
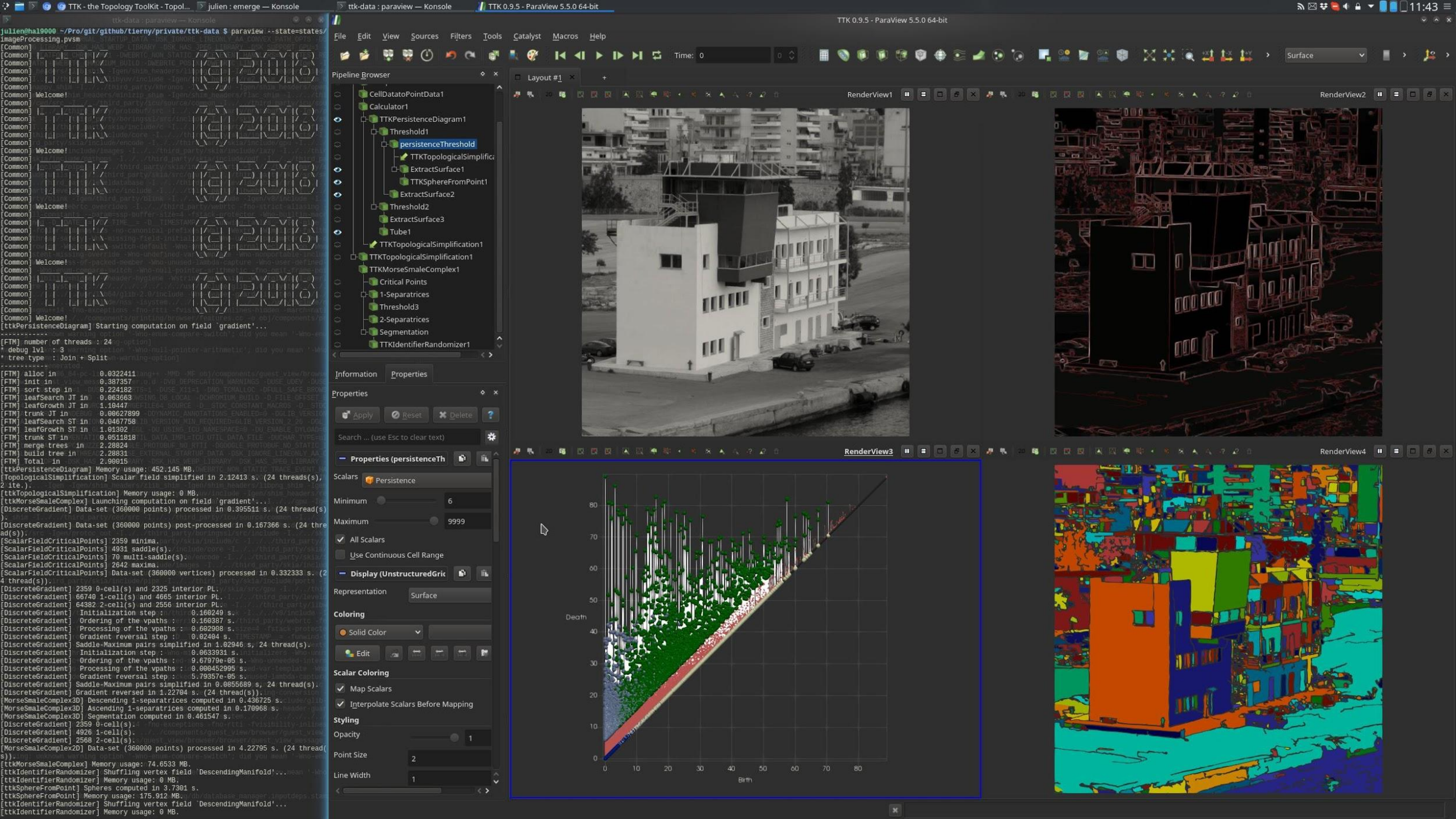


Morse-Smale complex

- **Intersection**
  - Morse complex of f
  - Morse complex of -f
  - Persistence simpliciation
    - Hierarchical complexes





TTK - the Topology ToolKit - Topol...
julien : emerge — Konsole
ttk-data : paraview — Konsole
TTK 0.9.5 - ParaView 5.5.0 64-bit
11:43
julien@hal9000 ~/Pro/git/github/tierny/private/ttk-data $ paraview --state=states/imageProcessing.pvsm
[Common] Welcome!
[ttkPersistenceDiagram] Starting computation on field `gradient'...
[FTM] number of threads : 24
* debug lvl : 3
* tree type : Join + Split
[FTM] alloc in 0.0322411
[FTM] init in 0.387357
[FTM] sort step in 0.224182
[FTM] leafSearch JT in 0.063663
[FTM] leafGrowth JT in 1.10447
[FTM] trunk JT in 0.00627899
[FTM] leafSearch ST in 0.0467758
[FTM] leafGrowth ST in 1.01302
[FTM] trunk ST in 0.0511818
[FTM] merge trees in 2.28824
[FTM] build tree in 2.28831
[FTM] Total in 2.90015
[ttkPersistenceDiagram] Memory usage: 452.145 MB.
[TopologicalSimplification] Scalar field simplified in 2.12413 s. (24 threads(s), 2 ite.).
[ttkTopologicalSimplification] Memory usage: 0 MB.
[ttkMorseSmaleComplex] Launching computation on field `gradient'...
[DiscreteGradient] Data-set (360000 points) processed in 0.395511 s. (24 thread(s)).
[DiscreteGradient] Data-set (360000 points) post-processed in 0.167366 s. (24 thread(s)).
[ScalarFieldCriticalPoints] 2359 minima.
[ScalarFieldCriticalPoints] 4931 saddle(s).
[ScalarFieldCriticalPoints] 70 multi-saddle(s).
[ScalarFieldCriticalPoints] 2642 maxima.
[ScalarFieldCriticalPoints] Data-set (360000 vertices) processed in 0.332333 s. (24 thread(s)).
[DiscreteGradient] 2359 0-cell(s) and 2325 interior PL.
[DiscreteGradient] 66740 1-cell(s) and 4665 interior PL.
[DiscreteGradient] 64382 2-cell(s) and 2556 interior PL.
[DiscreteGradient] Initialization step : 0.160249 s.
[DiscreteGradient] Ordering of the vpaths : 0.160387 s.
[DiscreteGradient] Processing of the vpaths : 0.602908 s.
[DiscreteGradient] Gradient reversal step : 0.02404 s.
[DiscreteGradient] Saddle-Maximum pairs simplified in 1.02946 s, 24 thread(s).
[DiscreteGradient] Initialization step : 0.0633931 s.
[DiscreteGradient] Ordering of the vpaths : 9.67979e-05 s.
[DiscreteGradient] Processing of the vpaths : 0.000452995 s.
[DiscreteGradient] Gradient reversal step : 5.79357e-05 s.
[DiscreteGradient] Saddle-Maximum pairs simplified in 0.0855689 s, 24 thread(s).
[DiscreteGradient] Gradient reversed in 1.22704 s. (24 thread(s)).
[MorseSmaleComplex3D] Descending 1-separatrices computed in 0.436725 s.
[MorseSmaleComplex3D] Ascending 1-separatrices computed in 0.170968 s.
[MorseSmaleComplex3D] Segmentation computed in 0.461547 s.
[DiscreteGradient] 2359 0-cell(s).
[DiscreteGradient] 4926 1-cell(s).
[DiscreteGradient] 2568 2-cell(s).
[MorseSmaleComplex2D] Data-set (360000 points) processed in 4.22795 s. (24 thread(s)).
[ttkMorseSmaleComplex] Memory usage: 74.6533 MB.
[ttkIdentifierRandomizer] Shuffling vertex field `DescendingManifold'...
[ttkIdentifierRandomizer] Memory usage: 0 MB.
[ttkSphereFromPoint] Spheres computed in 3.7301 s.
[ttkSphereFromPoint] Memory usage: 175.912 MB.
[ttkIdentifierRandomizer] Shuffling vertex field `DescendingManifold'...
[ttkIdentifierRandomizer] Memory usage: 0 MB.
File Edit View Sources Filters Tools Catalyst Macros Help
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Surface
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Calculator1
TTKPersistenceDiagram1
Threshold1
persistenceThreshold
TTKTopologicalSimplifica
ExtractSurface1
TTKSphereFromPoint1
ExtractSurface2
Threshold2
ExtractSurface3
Tube1
TTKTopologicalSimplification1
TTKTopologicalSimplification1
TTKMorseSmaleComplex1
Critical Points
1-Separatrices
Threshold3
2-Separatrices
Segmentation
TTKIdentifierRandomizer1
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Solid Color
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Interpolate Scalars Before Mapping
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Line Width 1
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RenderView2
RenderView3
RenderView4
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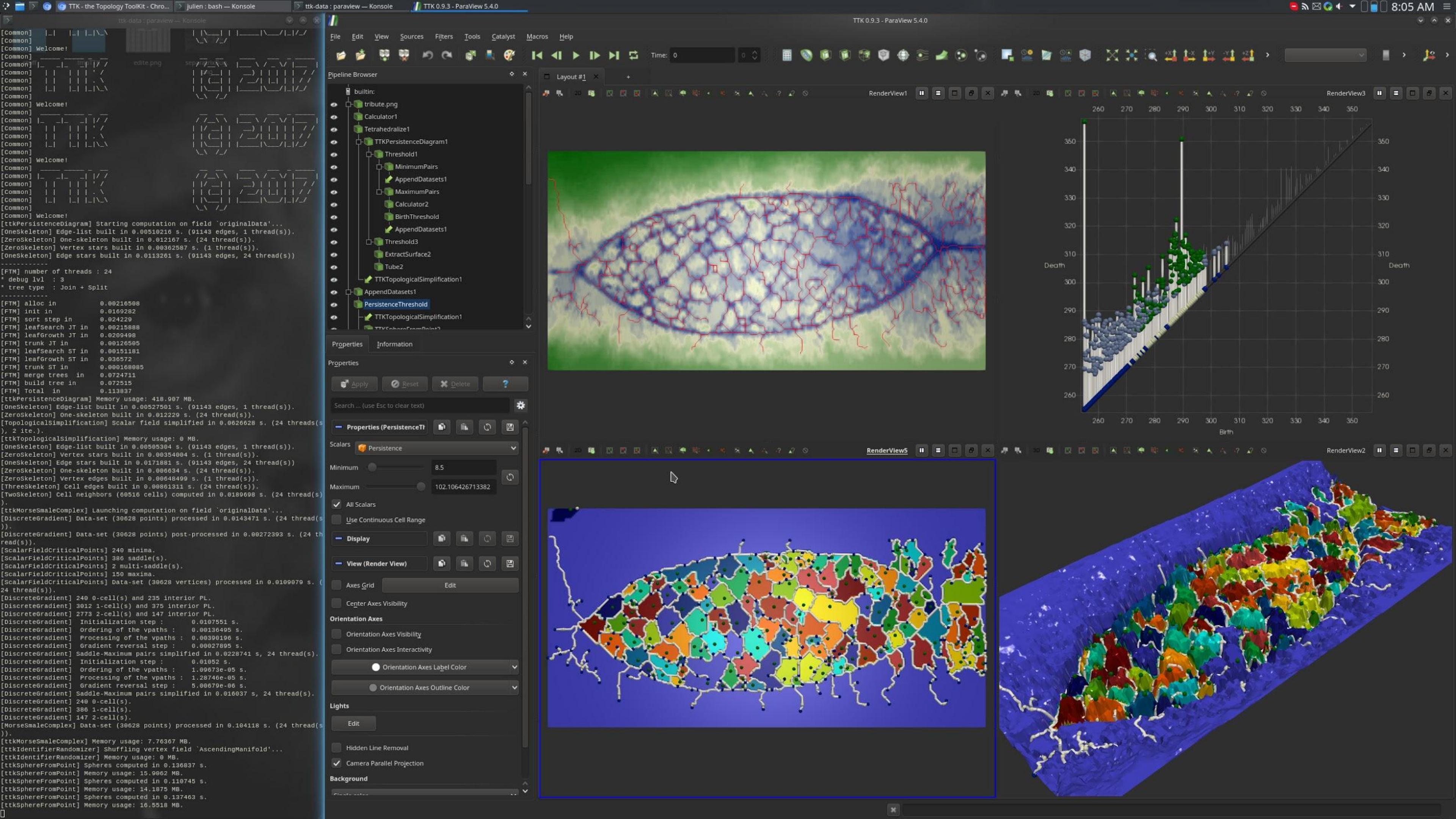
Birth
0 10 20 30 40 50 60 70 80



TTK - the Topology ToolKit - Chro...
julien : bash — Konsole
ttk-data : paraview — Konsole
TTK 0.9.3 - ParaView 5.4.0
8:05 AM
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Threshold1
MinimumPairs
AppendDatasets1
MaximumPairs
Calculator2
BirthThreshold
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ExtractSurface2
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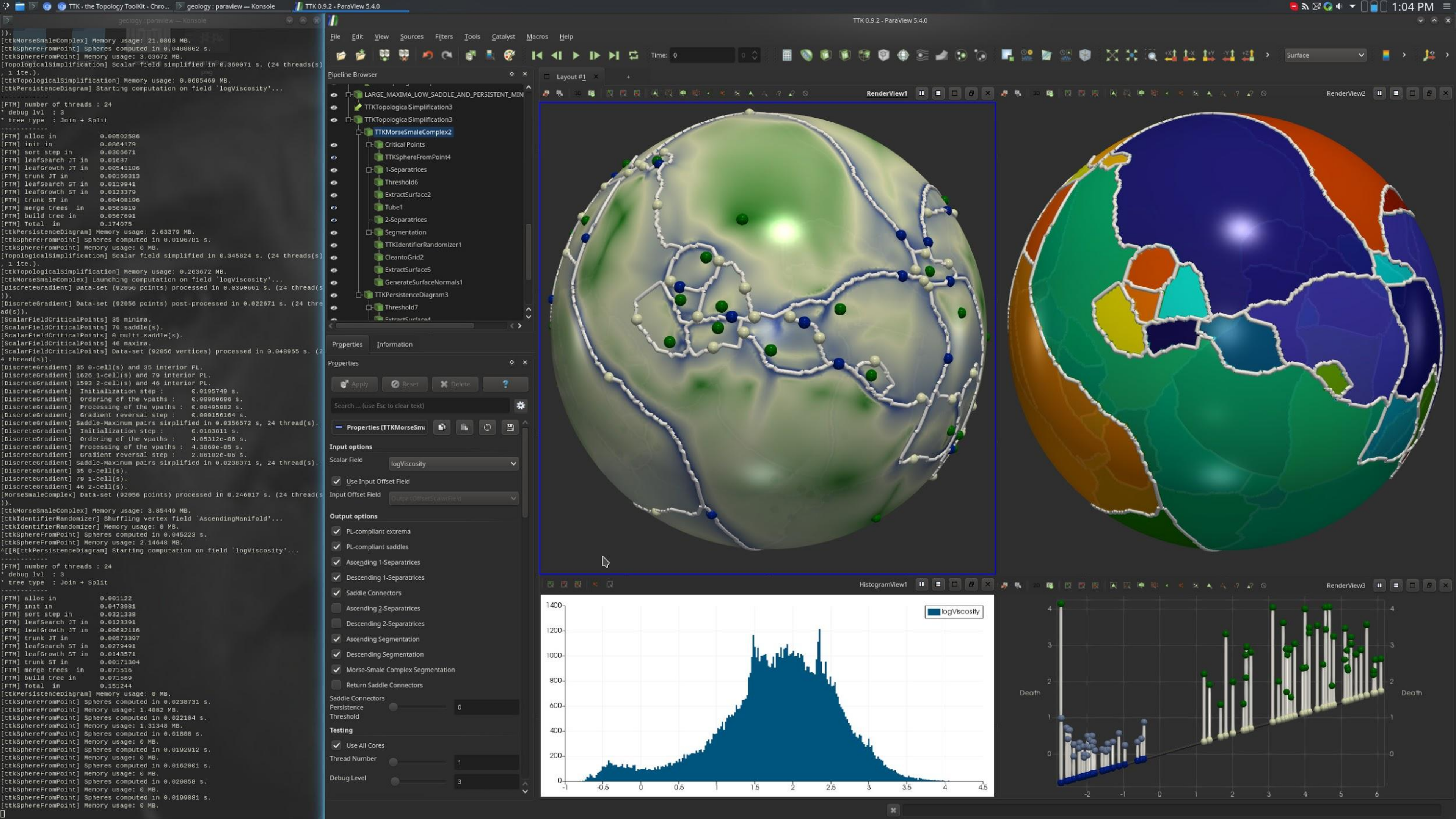
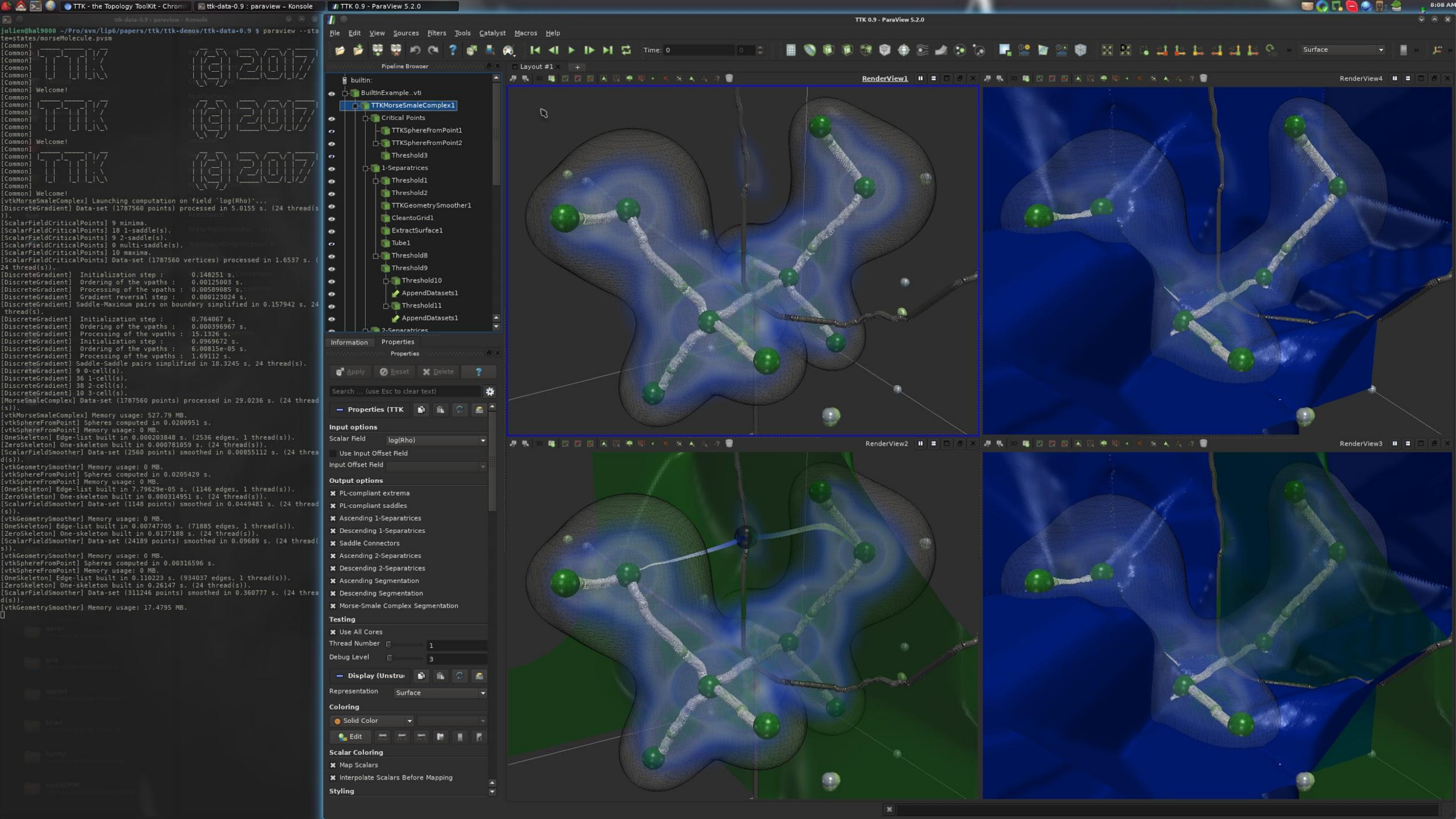
Orientation Axes Label Color
Orientation Axes Outline Color
Lights
Edit
Hidden Line Removal
Camera Parallel Projection
Background
Layout #1
RenderView1
RenderView3
RenderView5
RenderView2
Death
Birth
[ttkPersistenceDiagram] Starting computation on field `originalData'...
[FTM] number of threads : 24
* debug lvl : 3
* tree type : Join + Split
[ttkPersistenceDiagram] Memory usage: 418.907 MB.
[ttkMorseSmaleComplex] Launching computation on field `originalData'...
[ScalarFieldCriticalPoints] 240 minima.
[ScalarFieldCriticalPoints] 386 saddle(s).
[ScalarFieldCriticalPoints] 2 multi-saddle(s).
[ScalarFieldCriticalPoints] 150 maxima.
[ttkMorseSmaleComplex] Memory usage: 7.76367 MB.
[ttkIdentifierRandomizer] Shuffling vertex field `AscendingManifold'...
[ttkSphereFromPoint] Memory usage: 16.5518 MB.


TTK 0.9.2 - ParaView 5.4.0
File Edit View Sources Filters Tools Catalyst Macros Help
Pipeline Browser
LARGE_MAXIMA_LOW_SADDLE_AND_PERSISTENT_MIN
TTKTopologicalSimplification3
TTKMorseSmaleComplex2
Critical Points
TTKSphereFromPoint4
1-Separatrices
Threshold6
ExtractSurface2
Tube1
2-Separatrices
Segmentation
TTKIdentifierRandomizer1
CleantoGrid2
ExtractSurface5
GenerateSurfaceNormals1
TTKPersistenceDiagram3
Threshold7
Properties
Information
Apply
Reset
Delete
Input options
Scalar Field
logViscosity
Use Input Offset Field
Input Offset Field
Output options
PL-compliant extrema
PL-compliant saddles
Ascending 1-Separatrices
Descending 1-Separatrices
Saddle Connectors
Ascending 2-Separatrices
Descending 2-Separatrices
Ascending Segmentation
Descending Segmentation
Morse-Smale Complex Segmentation
Return Saddle Connectors
Saddle Connectors Persistence Threshold
Testing
Use All Cores
Thread Number
Debug Level
Layout #1
RenderView1
RenderView2
HistogramView1
RenderView3
logViscosity
Death
[ScalarFieldCriticalPoints] 35 minima.
[ScalarFieldCriticalPoints] 79 saddle(s).
[ScalarFieldCriticalPoints] 0 multi-saddle(s).
[ScalarFieldCriticalPoints] 46 maxima.
[FTM] number of threads : 24
* debug lvl : 3
* tree type : Join + Split



TTK - the Topology ToolKit - Chromium
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TTK 0.9 - ParaView 5.2.0
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Macros
Help
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Critical Points
TTKSphereFromPoint1
TTKSphereFromPoint2
Threshold3
1-Separatrices
Threshold1
Threshold2
TTKGeometrySmoother1
CleantoGrid1
ExtractSurface1
Tube1
Threshold8
Threshold9
Threshold10
AppendDatasets1
Threshold11
AppendDatasets1
Information
Properties
Apply
Reset
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Search ... (use Esc to clear text)
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Scalar Field
log(Rho)
Use Input Offset Field
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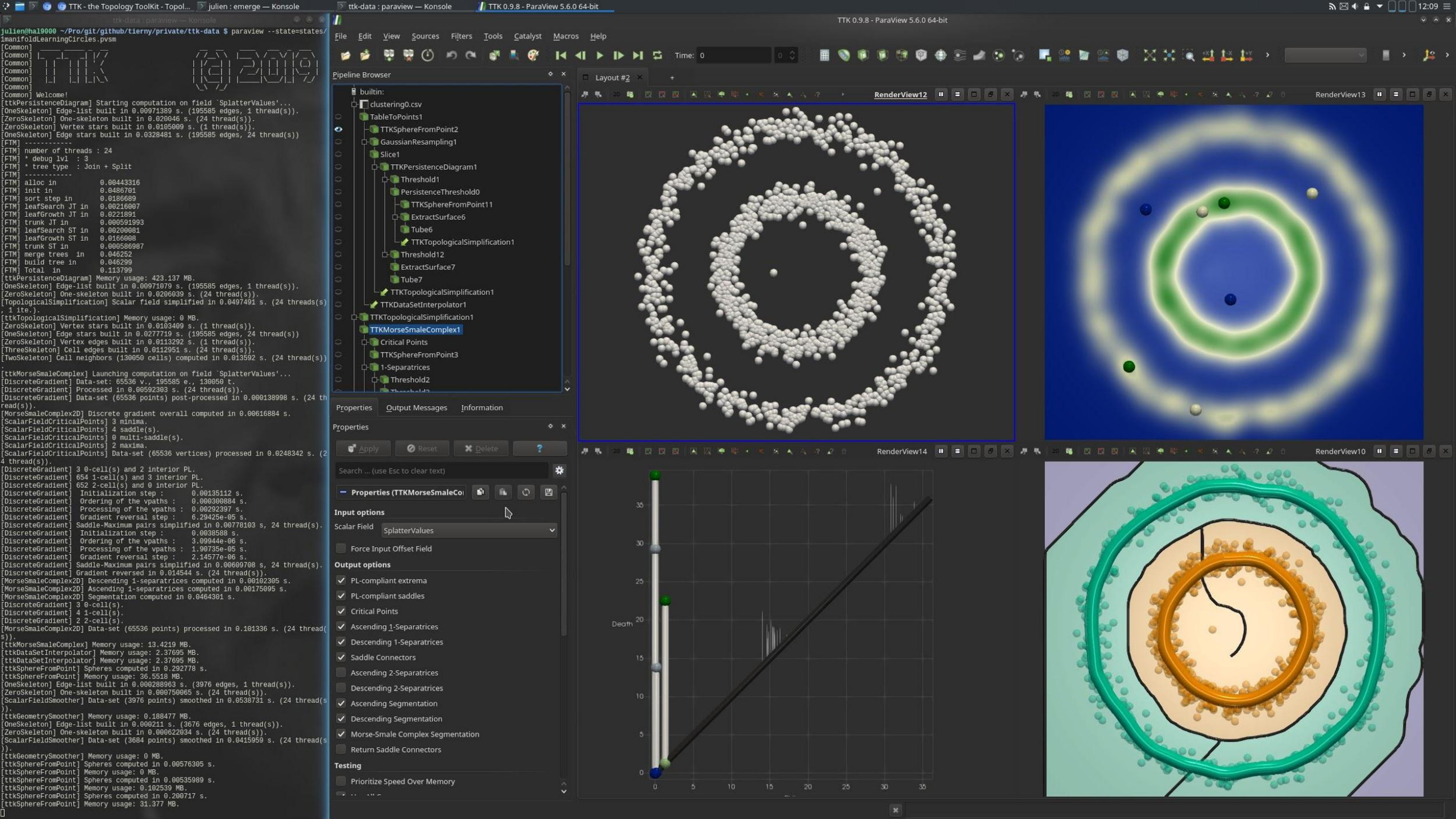
Output options
PL-compliant extrema
PL-compliant saddles
Ascending 1-Separatrices
Descending 1-Separatrices
Saddle Connectors
Ascending 2-Separatrices
Descending 2-Separatrices
Ascending Segmentation
Descending Segmentation
Morse-Smale Complex Segmentation
Testing
Use All Cores
Thread Number
1
Debug Level
3
Representation
Surface
Coloring
Solid Color
Edit
Scalar Coloring
Map Scalars
Interpolate Scalars Before Mapping
Styling
Layout #1
RenderView1
RenderView4
RenderView2
RenderView3


TTK 0.9.8 - ParaView 5.6.0 64-bit
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Macros
Help
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clustering0.csv
TableToPoints1
TTKSphereFromPoint2
GaussianResampling1
Slice1
TTKPersistenceDiagram1
Threshold1
PersistenceThreshold0
TTKSphereFromPoint11
ExtractSurface6
Tube6
TTKTopologicalSimplification1
Threshold12
ExtractSurface7
Tube7
TTKTopologicalSimplification1
TTKDataSetInterpolator1
TTKTopologicalSimplification1
TTKMorseSmaleComplex1
Critical Points
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1-Separatrices
Threshold2
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Input options
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Force Input Offset Field
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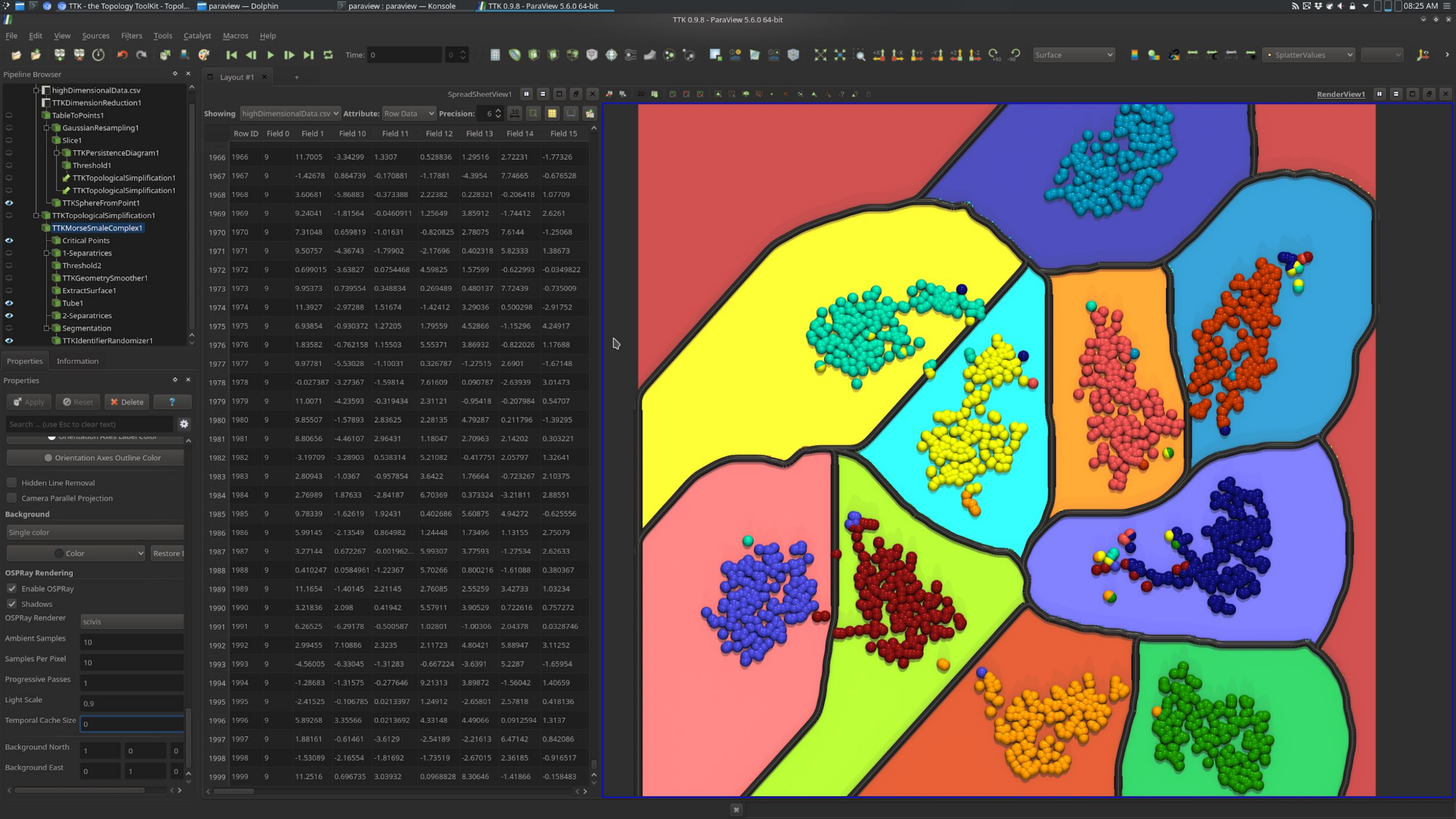
PL-compliant saddles
Critical Points
Ascending 1-Separatrices
Descending 1-Separatrices
Saddle Connectors
Ascending 2-Separatrices
Descending 2-Separatrices
Ascending Segmentation
Descending Segmentation
Morse-Smale Complex Segmentation
Return Saddle Connectors
Testing
Prioritize Speed Over Memory
Layout #2
RenderView12
RenderView13
RenderView14
RenderView10
Death
julien@hal9000 ~/Pro/git/github/tierny/private/ttk-data $ paraview --state=states/1manifoldLearningCircles.pvsm
[Common] Welcome!
[ttkPersistenceDiagram] Starting computation on field `SplatterValues'...
[ttkMorseSmaleComplex] Launching computation on field `SplatterValues'...


TTK 0.9.8 - ParaView 5.6.0 64-bit
File Edit View Sources Filters Tools Catalyst Macros Help
Pipeline Browser
highDimensionalData.csv
TTKDimensionReduction1
TableToPoints1
GaussianResampling1
Slice1
TTKPersistenceDiagram1
Threshold1
TTKTopologicalSimplification1
TTKTopologicalSimplification1
TTKSphereFromPoint1
TTKTopologicalSimplification1
TTKMorseSmaleComplex1
Critical Points
1-Separatrices
Threshold2
TTKGeometrySmoother1
ExtractSurface1
Tube1
2-Separatrices
Segmentation
TTKIdentifierRandomizer1
Properties Information
Apply Reset Delete
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Hidden Line Removal
Camera Parallel Projection
Background
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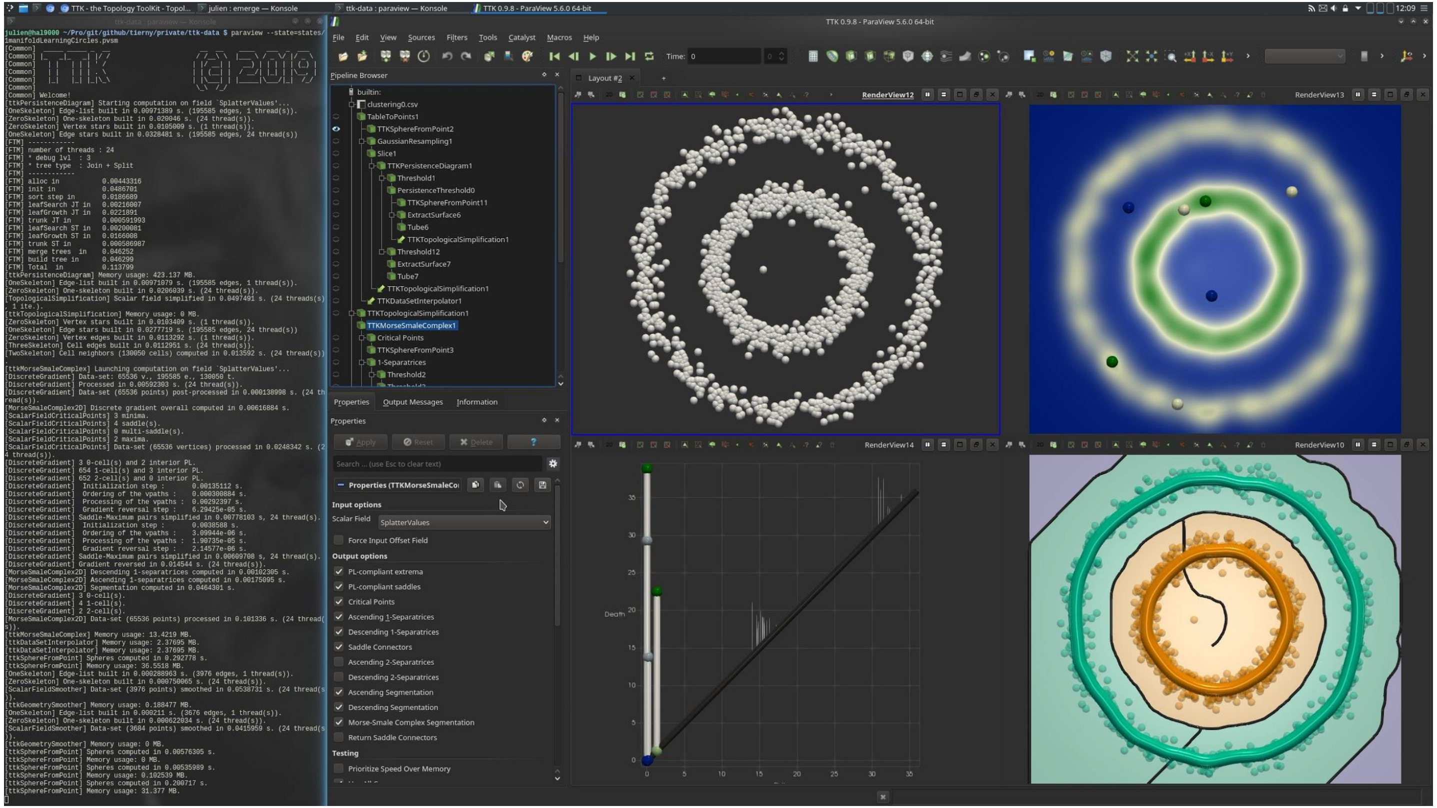
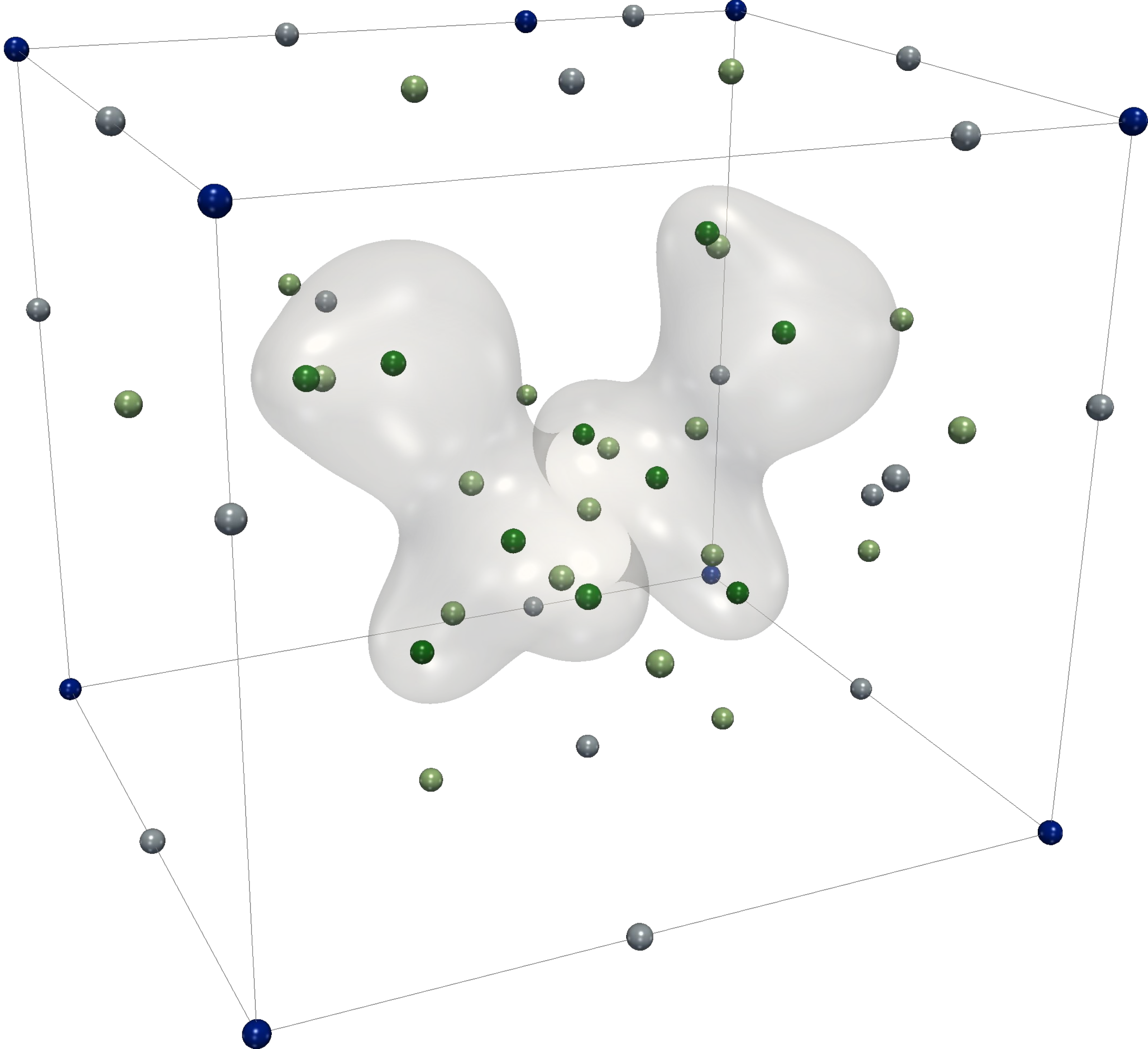
Samples Per Pixel 10
Progressive Passes 1
Light Scale 0.9
Temporal Cache Size 0
Background North
Background East
Layout #1
SpreadSheetView1
Showing highDimensionalData.csv Attribute: Row Data Precision: 6
Row ID Field 0 Field 1 Field 10 Field 11 Field 12 Field 13 Field 14 Field 15
RenderView1
Surface
SplatterValues

Applications



Valid if

- **Ascending and descending manifolds**
  - Transversal intersection



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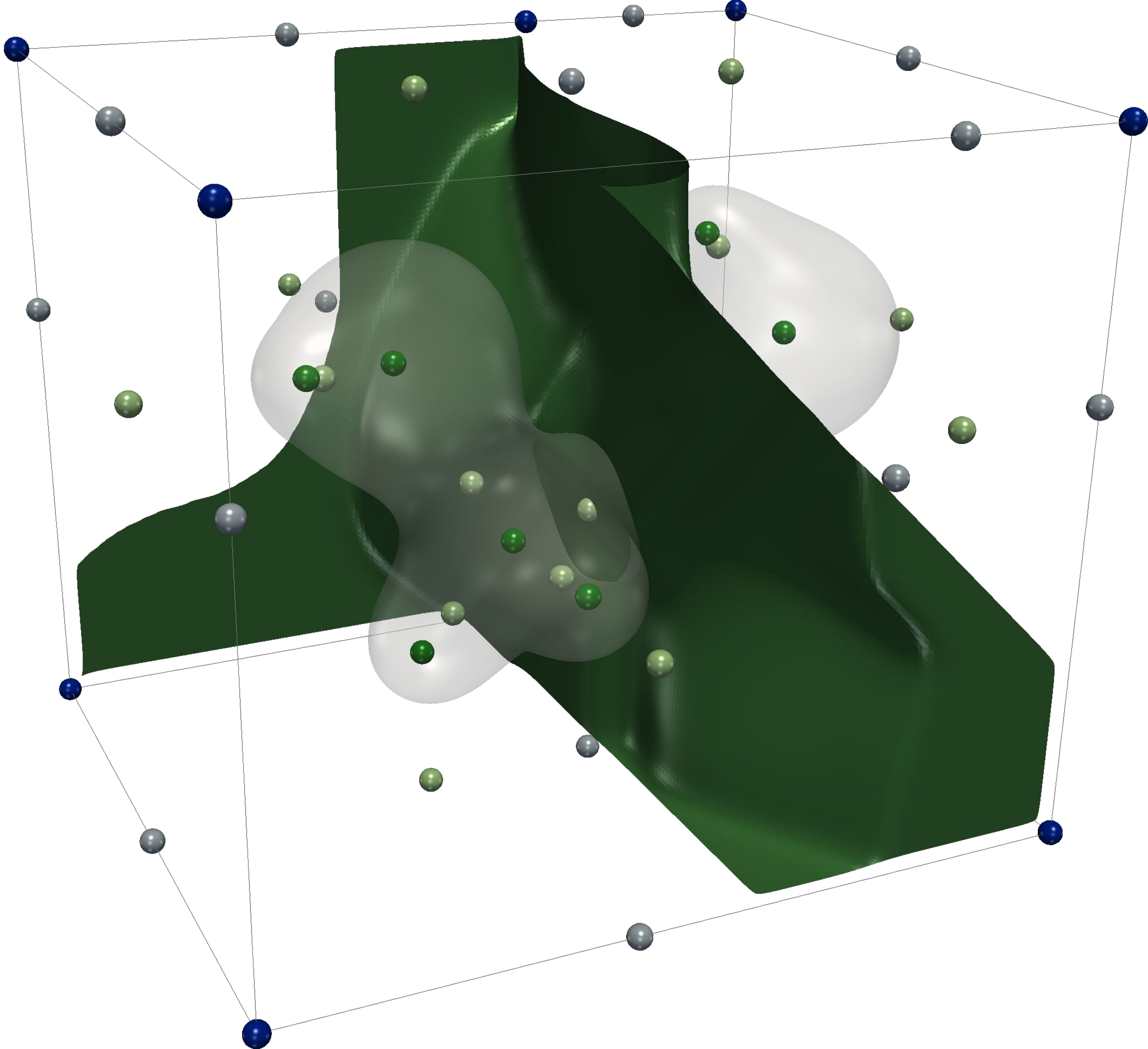

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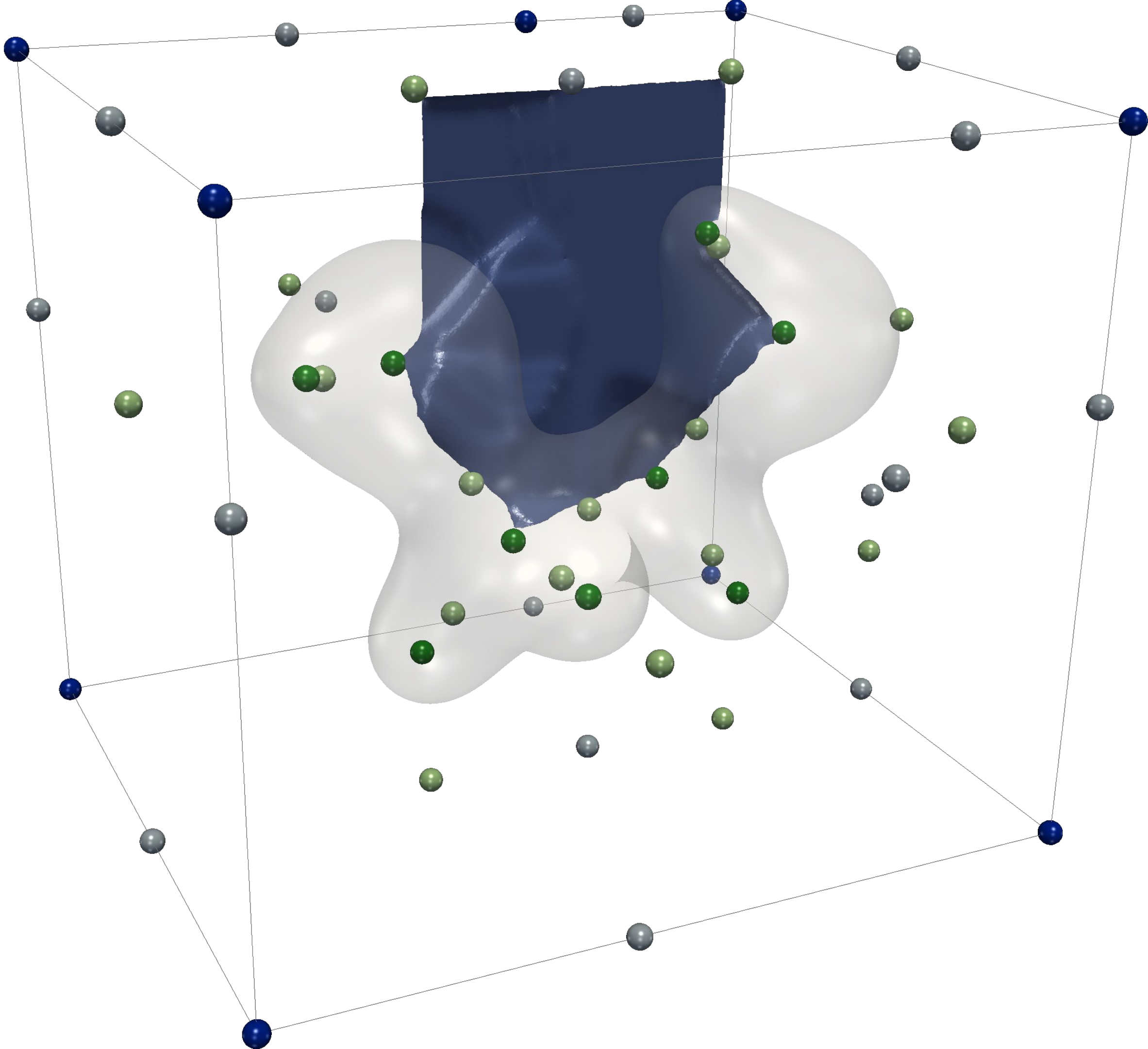
  - Transversal intersection



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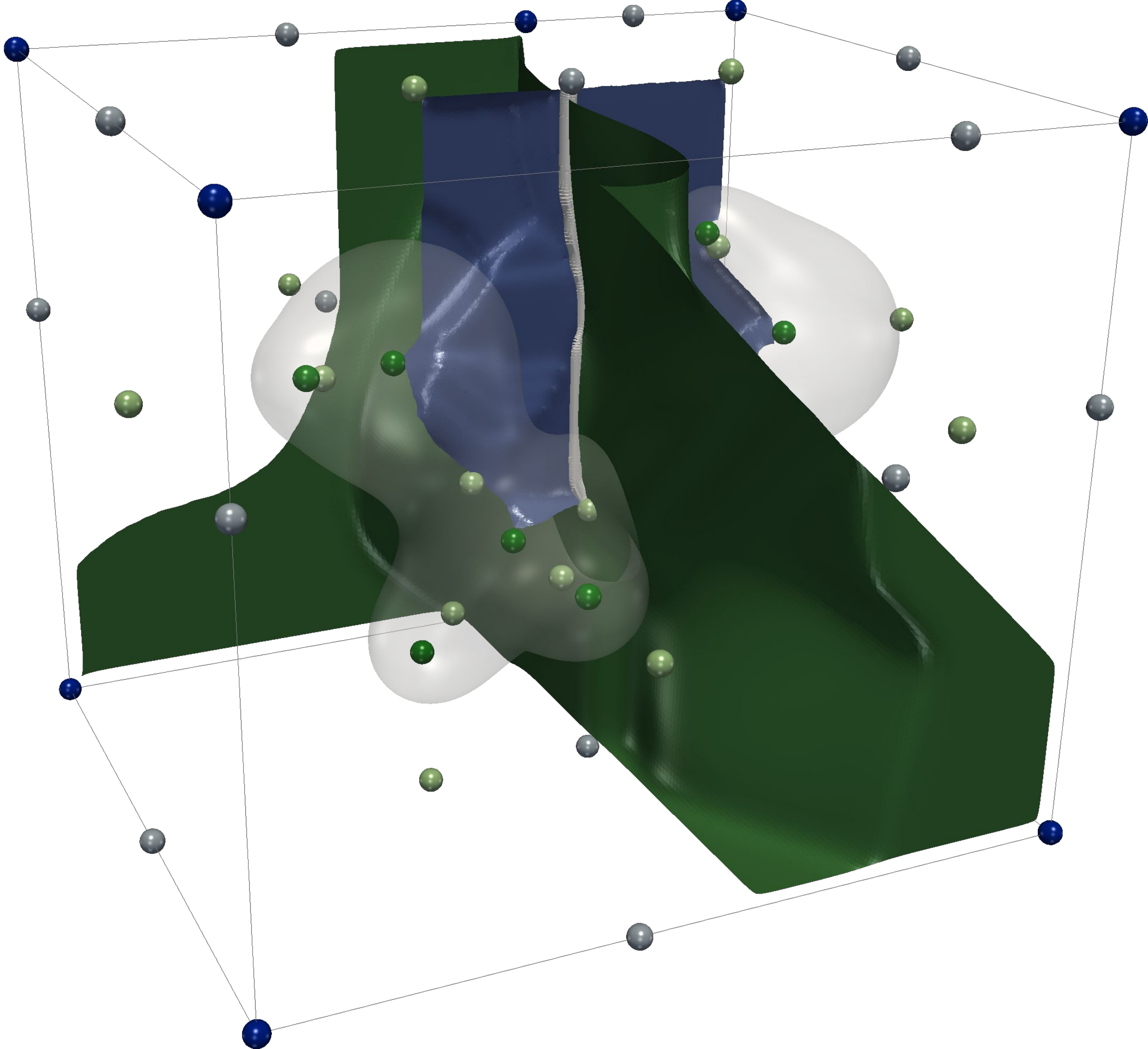
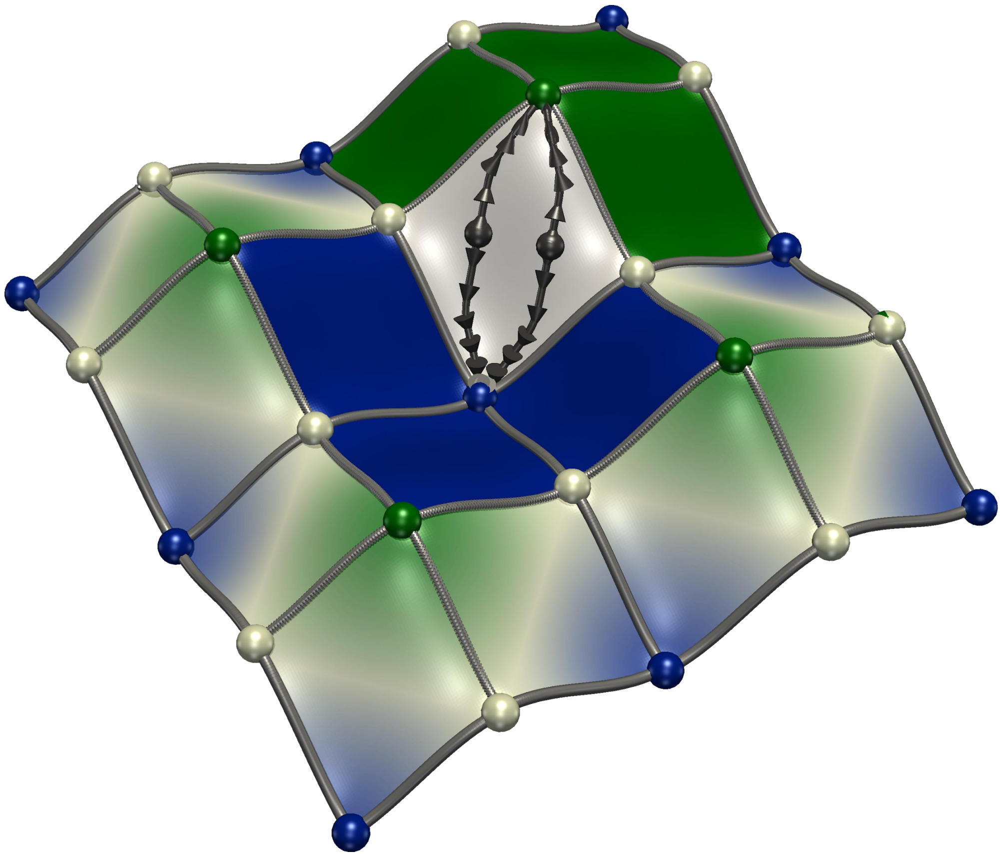


Properties

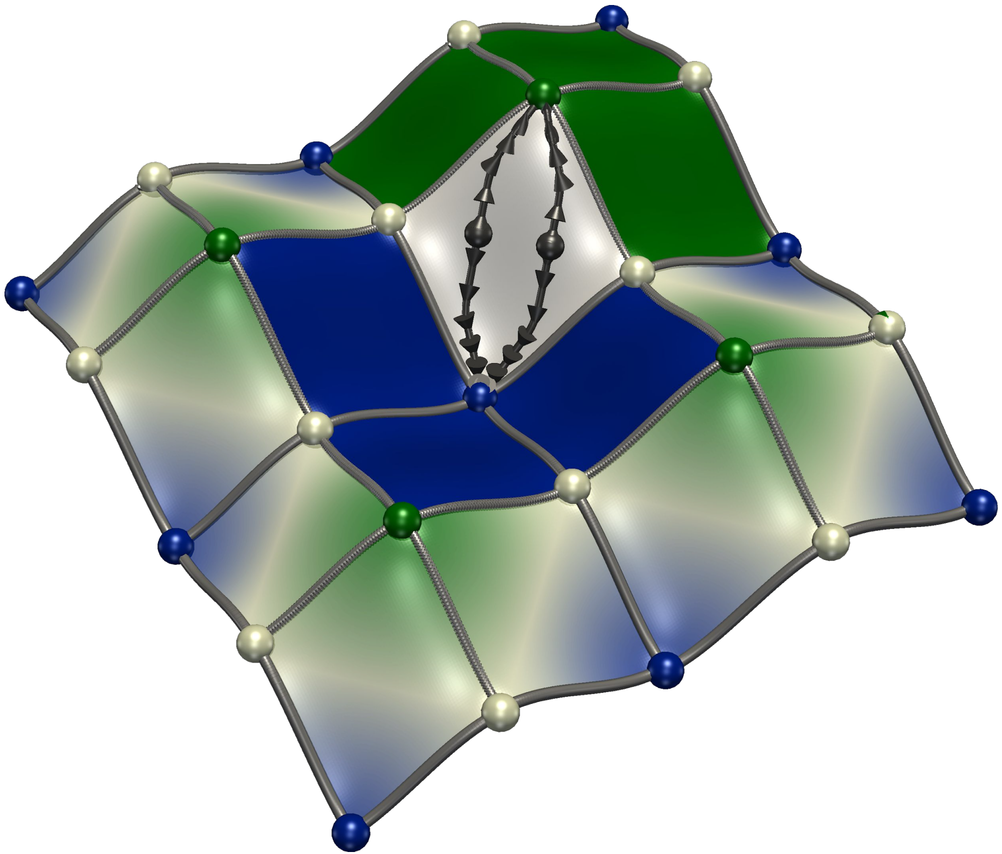
- **Generic CW complex**
  - 1D cells: index d to d+1

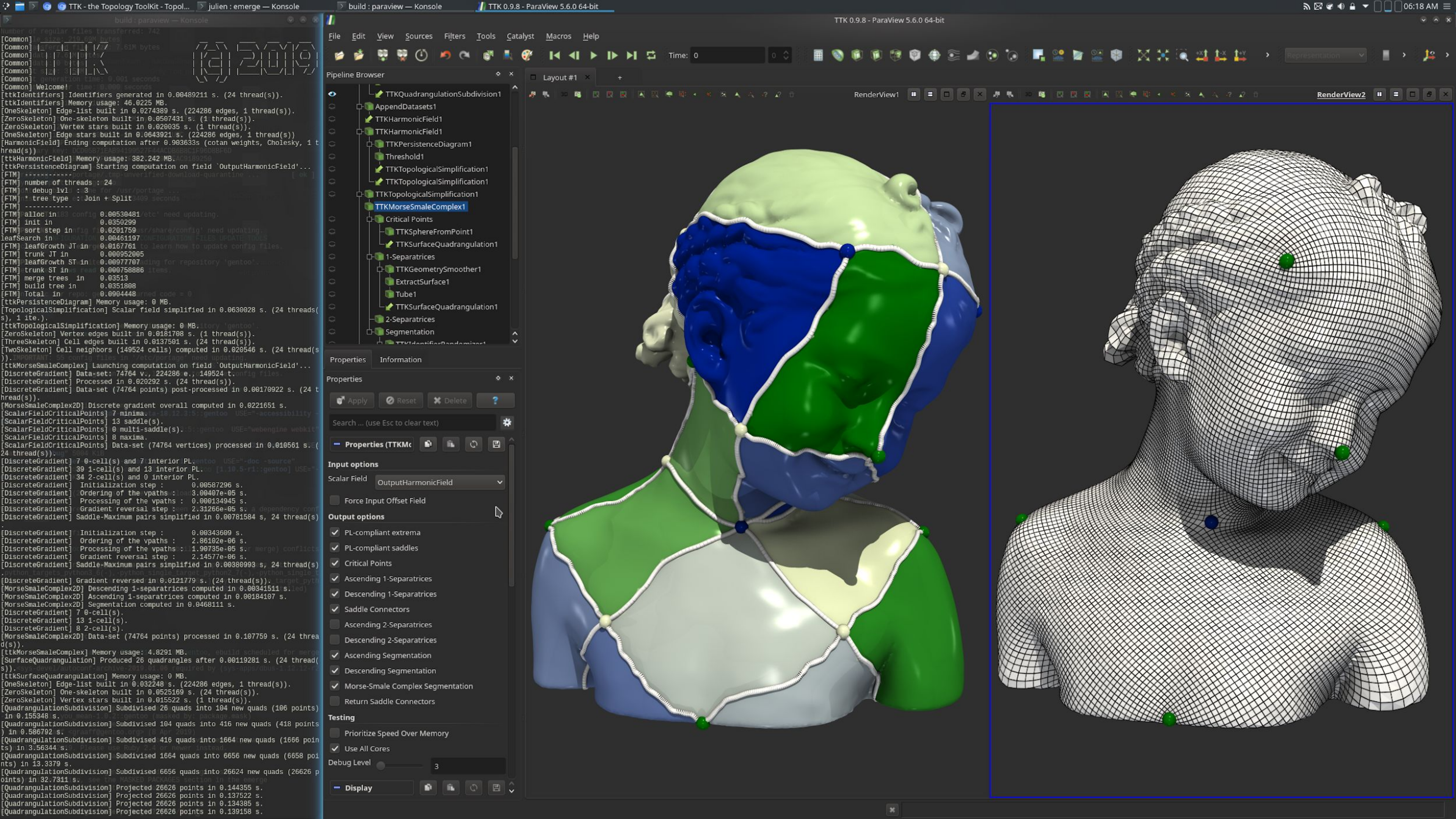


Properties

- **Generic CW complex**
  - 1D cells: index d to d+1

- **2-manifolds**
  - 2D cells: quadrangles
  - Simple saddles: valence 4
  - Extrema: arbitrary valence

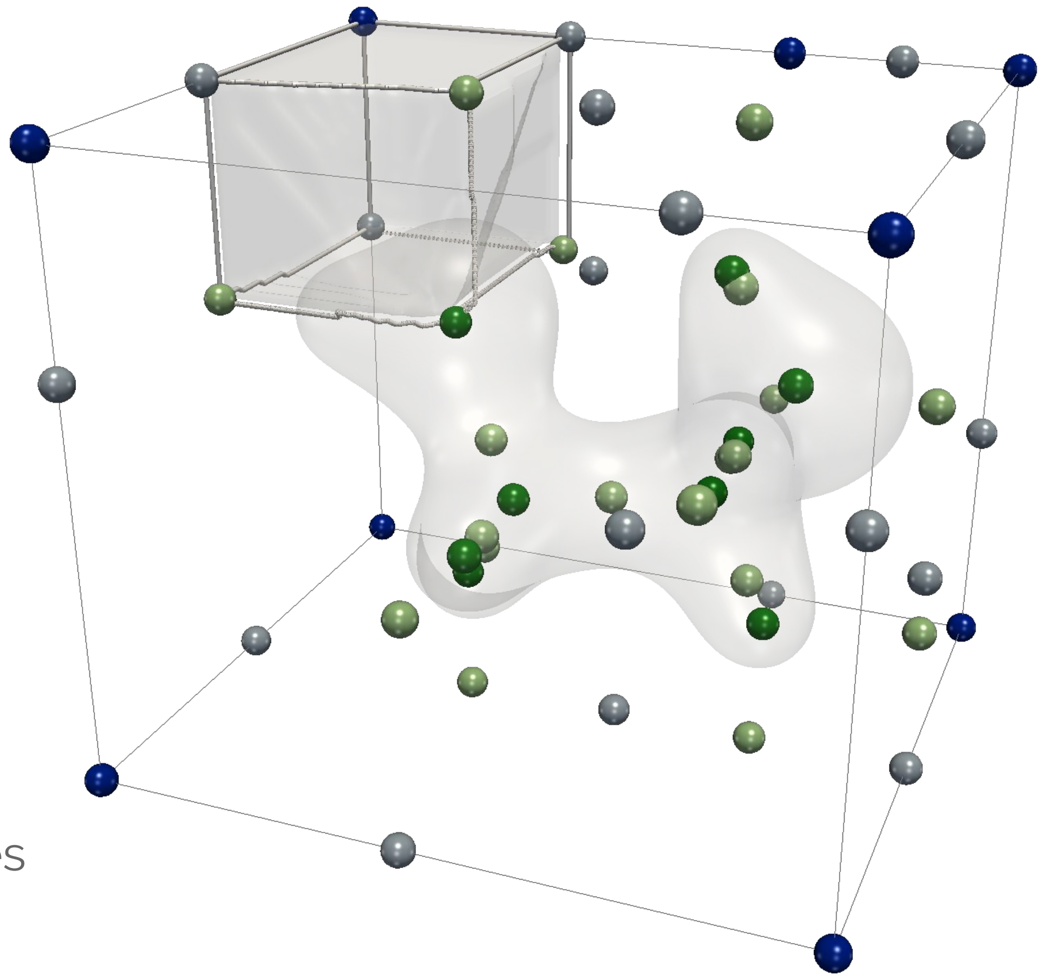




TTK - the Topology ToolKit - Topol...
julien : emerge — Konsole
build : paraview — Konsole
TTK 0.9.8 - ParaView 5.6.0 64-bit
06:18 AM
build : paraview — Konsole
[Common] Welcome!
[ttkIdentifiers] Identifiers generated in 0.00489211 s. (24 thread(s)).
[ttkIdentifiers] Memory usage: 46.0225 MB.
[OneSkeleton] Edge-list built in 0.0274389 s. (224286 edges, 1 thread(s)).
[ZeroSkeleton] One-skeleton built in 0.0507431 s. (1 thread(s)).
[ZeroSkeleton] Vertex stars built in 0.020035 s. (1 thread(s)).
[OneSkeleton] Edge stars built in 0.0643921 s. (224286 edges, 1 thread(s))
[HarmonicField] Ending computation after 0.903633s (cotan weights, Cholesky, 1 thread(s))
[ttkHarmonicField] Memory usage: 382.242 MB.
[ttkPersistenceDiagram] Starting computation on field `OutputHarmonicField'...
[FTM] ------------
[FTM] number of threads : 24
[FTM] * debug lvl : 3
[FTM] * tree type : Join + Split
[FTM] ------------
[FTM] alloc in 0.00530481
[FTM] init in 0.0350299
[FTM] sort step in 0.0201759
leafSearch in 0.00461197
[FTM] leafGrowth JT in 0.0167761
[FTM] trunk JT in 0.000952005
[FTM] leafGrowth ST in 0.00977707
[FTM] trunk ST in 0.000758886
[FTM] merge trees in 0.03513
[FTM] build tree in 0.0351808
[FTM] Total in 0.0904448
[ttkPersistenceDiagram] Memory usage: 0 MB.
[TopologicalSimplification] Scalar field simplified in 0.0630028 s. (24 threads(s), 1 ite.)
[ttkTopologicalSimplification] Memory usage: 0 MB.
[ZeroSkeleton] Vertex edges built in 0.0181708 s. (1 thread(s)).
[ThreeSkeleton] Cell edges built in 0.0137501 s. (24 thread(s)).
[TwoSkeleton] Cell neighbors (149524 cells) computed in 0.020546 s. (24 thread(s)).
[ttkMorseSmaleComplex] Launching computation on field `OutputHarmonicField'...
[DiscreteGradient] Data-set: 74764 v., 224286 e., 149524 t.
[DiscreteGradient] Processed in 0.020292 s. (24 thread(s)).
[DiscreteGradient] Data-set (74764 points) post-processed in 0.00170922 s. (24 thread(s)).
[MorseSmaleComplex2D] Discrete gradient overall computed in 0.0221651 s.
[ScalarFieldCriticalPoints] 7 minima.
[ScalarFieldCriticalPoints] 13 saddle(s).
[ScalarFieldCriticalPoints] 0 multi-saddle(s).
[ScalarFieldCriticalPoints] 8 maxima.
[ScalarFieldCriticalPoints] Data-set (74764 vertices) processed in 0.010561 s. (24 thread(s)).
[DiscreteGradient] 7 0-cell(s) and 7 interior PL.
[DiscreteGradient] 39 1-cell(s) and 13 interior PL.
[DiscreteGradient] 34 2-cell(s) and 0 interior PL.
[DiscreteGradient] Initialization step : 0.00587296 s.
[DiscreteGradient] Ordering of the vpaths : 3.00407e-05 s.
[DiscreteGradient] Processing of the vpaths : 0.000134945 s.
[DiscreteGradient] Gradient reversal step : 2.31266e-05 s.
[DiscreteGradient] Saddle-Maximum pairs simplified in 0.00781584 s, 24 thread(s).
[DiscreteGradient] Initialization step : 0.00343609 s.
[DiscreteGradient] Ordering of the vpaths : 2.86102e-06 s.
[DiscreteGradient] Processing of the vpaths : 1.90735e-05 s.
[DiscreteGradient] Gradient reversal step : 2.14577e-06 s.
[DiscreteGradient] Saddle-Maximum pairs simplified in 0.00380993 s, 24 thread(s).
[DiscreteGradient] Gradient reversed in 0.0121779 s. (24 thread(s)).
[MorseSmaleComplex2D] Descending 1-separatrices computed in 0.00341511 s.
[MorseSmaleComplex2D] Ascending 1-separatrices computed in 0.00184107 s.
[MorseSmaleComplex2D] Segmentation computed in 0.0468111 s.
[DiscreteGradient] 7 0-cell(s).
[DiscreteGradient] 13 1-cell(s).
[DiscreteGradient] 8 2-cell(s).
[MorseSmaleComplex2D] Data-set (74764 points) processed in 0.107759 s. (24 thread(s)).
[ttkMorseSmaleComplex] Memory usage: 4.8291 MB.
[SurfaceQuadrangulation] Produced 26 quadrangles after 0.00119281 s. (24 thread(s)).
[ttkSurfaceQuadrangulation] Memory usage: 0 MB.
[OneSkeleton] Edge-list built in 0.032248 s. (224286 edges, 1 thread(s)).
[ZeroSkeleton] One-skeleton built in 0.0525169 s. (24 thread(s)).
[ZeroSkeleton] Vertex stars built in 0.015522 s. (1 thread(s)).
[QuadrangulationSubdivision] Subdivised 26 quads into 104 new quads (106 points) in 0.155348 s.
[QuadrangulationSubdivision] Subdivised 104 quads into 416 new quads (418 points) in 0.586792 s.
[QuadrangulationSubdivision] Subdivised 416 quads into 1664 new quads (1666 points) in 3.56344 s.
[QuadrangulationSubdivision] Subdivised 1664 quads into 6656 new quads (6658 points) in 13.3379 s.
[QuadrangulationSubdivision] Subdivised 6656 quads into 26624 new quads (26626 points) in 32.7311 s.
[QuadrangulationSubdivision] Projected 26626 points in 0.144355 s.
[QuadrangulationSubdivision] Projected 26626 points in 0.137522 s.
[QuadrangulationSubdivision] Projected 26626 points in 0.134385 s.
[QuadrangulationSubdivision] Projected 26626 points in 0.139158 s.
TTK 0.9.8 - ParaView 5.6.0 64-bit
File Edit View Sources Filters Tools Catalyst Macros Help
Time: 0
Pipeline Browser
TTKQuadrangulationSubdivision1
AppendDatasets1
TTKHarmonicField1
TTKHarmonicField1
TTKPersistenceDiagram1
Threshold1
TTKTopologicalSimplification1
TTKTopologicalSimplification1
TTKTopologicalSimplification1
TTKMorseSmaleComplex1
Critical Points
TTKSphereFromPoint1
TTKSurfaceQuadrangulation1
1-Separatrices
TTKGeometrySmoother1
ExtractSurface1
Tube1
TTKSurfaceQuadrangulation1
2-Separatrices
Segmentation
Properties
Information
Properties
Apply
Reset
Delete
Search ... (use Esc to clear text)
Properties (TTKMc
Input options
Scalar Field
OutputHarmonicField
Force Input Offset Field
Output options
PL-compliant extrema
PL-compliant saddles
Critical Points
Ascending 1-Separatrices
Descending 1-Separatrices
Saddle Connectors
Ascending 2-Separatrices
Descending 2-Separatrices
Ascending Segmentation
Descending Segmentation
Morse-Smale Complex Segmentation
Return Saddle Connectors
Testing
Prioritize Speed Over Memory
Use All Cores
Debug Level
3
Display
Layout #1
RenderView1
RenderView2

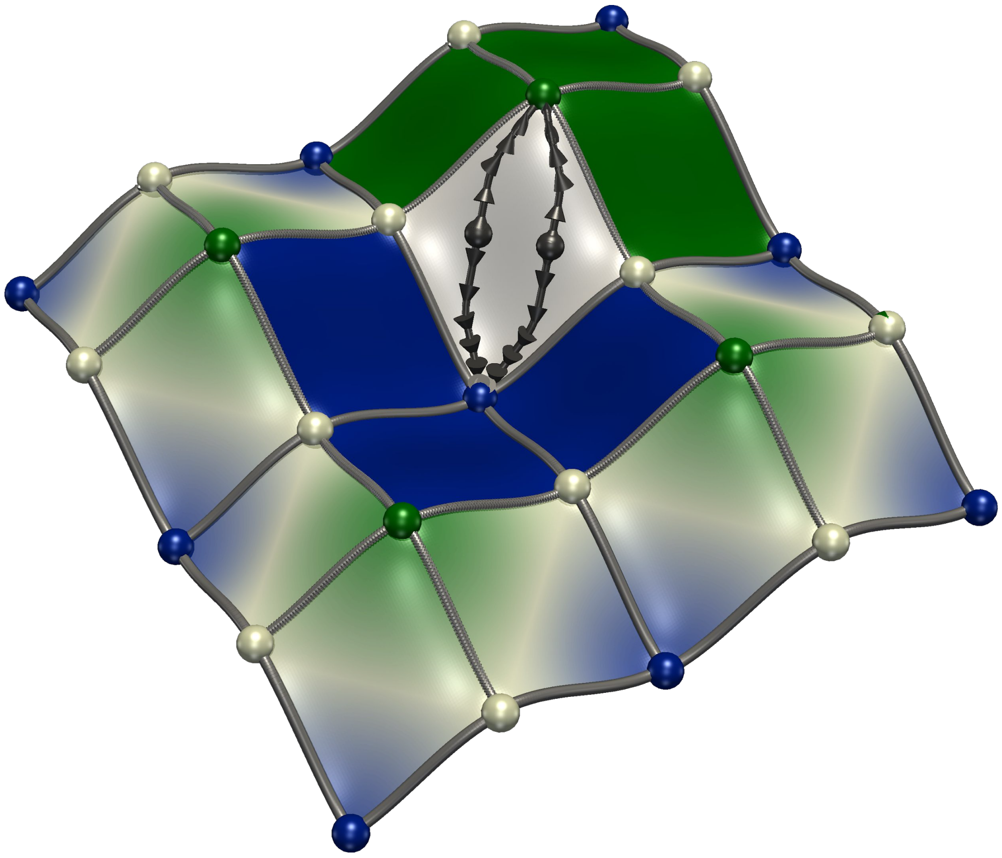
Properties

- **Generic CW complex**
  - 1D cells: index d to d+1
- **2-manifolds**
  - 2D cells: quadrangles
  - Simple saddles: valence 4
  - Extrema: arbitrary valence
- **3-manifolds**
  - 3D cells: arbitrary number of faces
    - Polygonal faces: quadrangles



Algorithms

- **PL setting**
  - 2-manifolds
    - Edelsbrunner et al. 2000
    - Bremer et al. 2003



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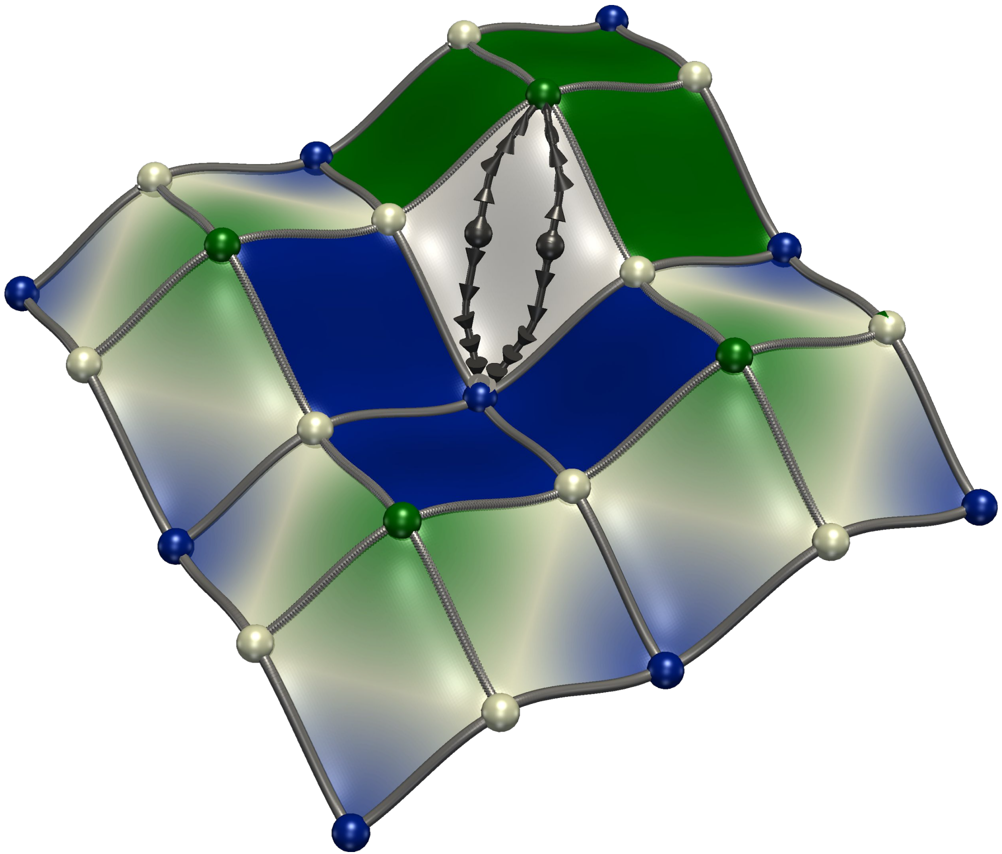

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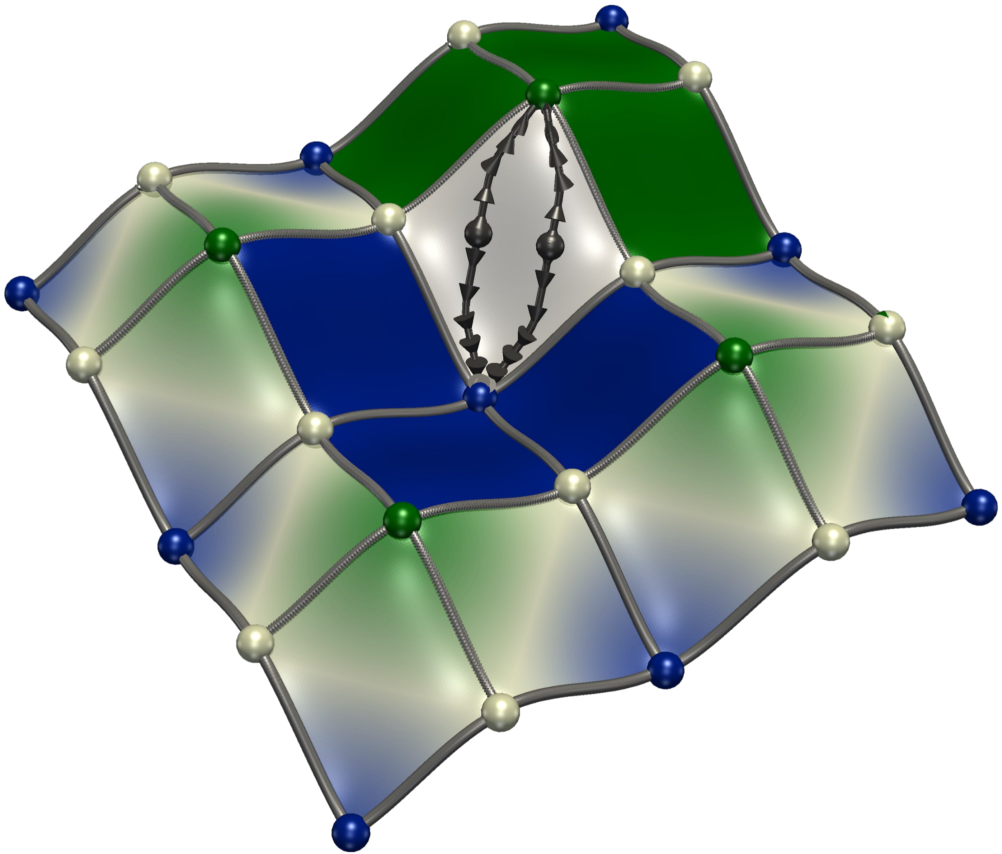
    - Transversal intersection
    - No known robust implementation



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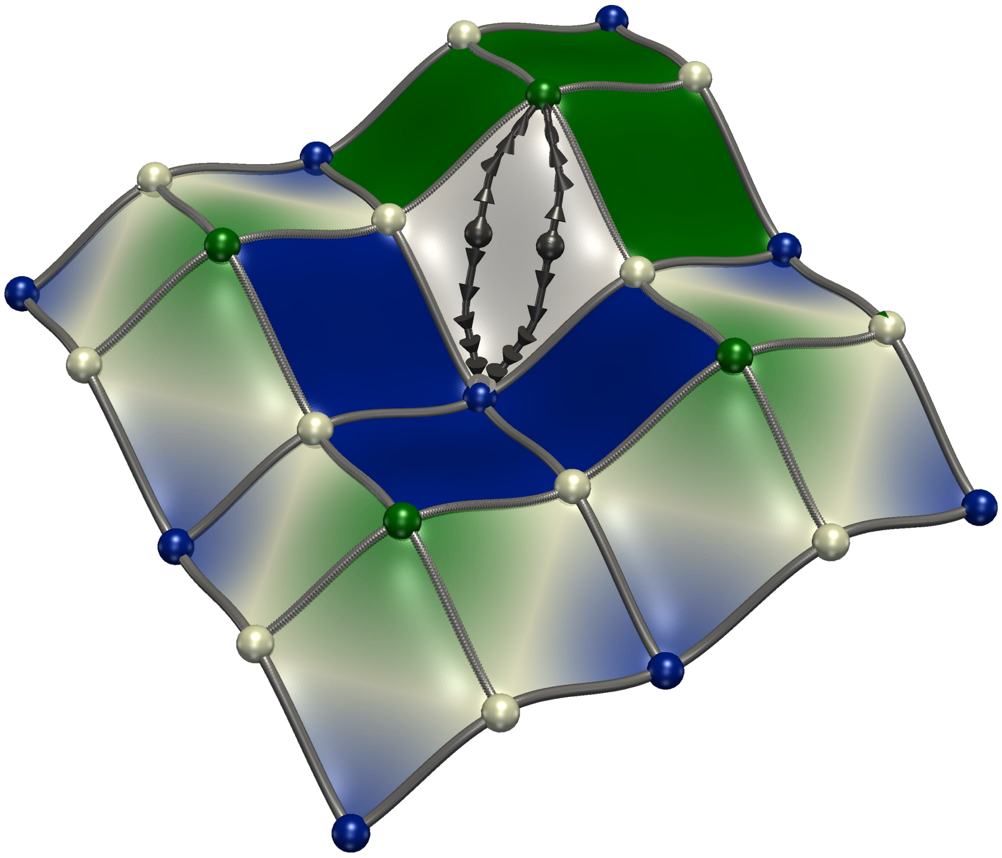
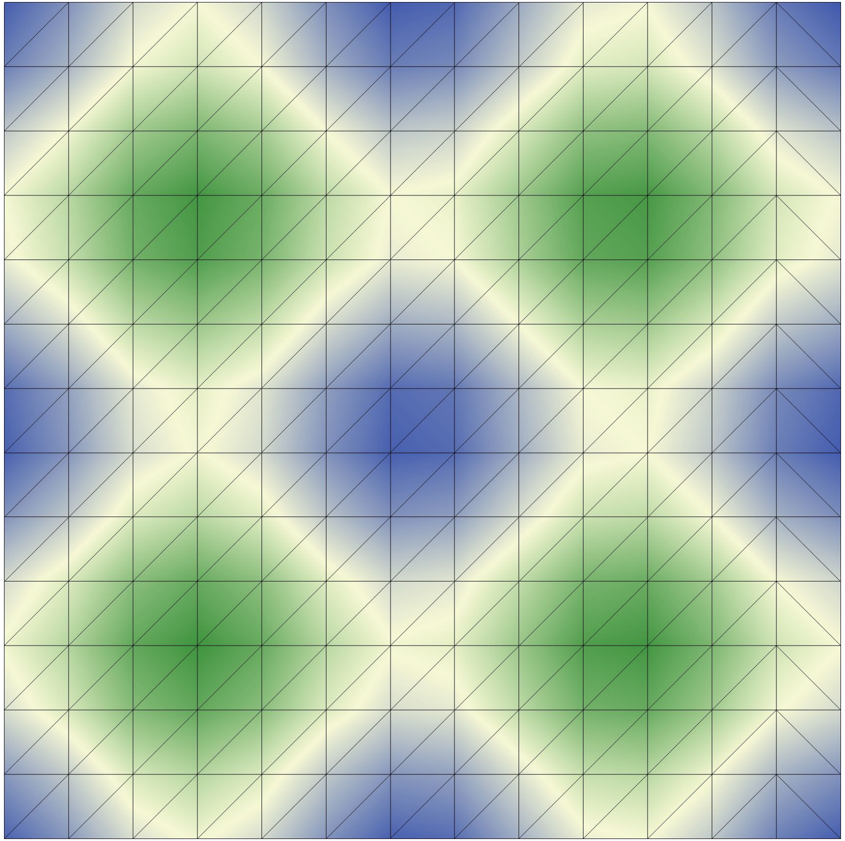


Discrete Morse Theory

- **Robin Forman**
  - "A user's guide to DMT" 2002

- Discrete Morse Function
  - Maps each simplex σ_i to $f(\sigma_i) \in \mathbb{R}$



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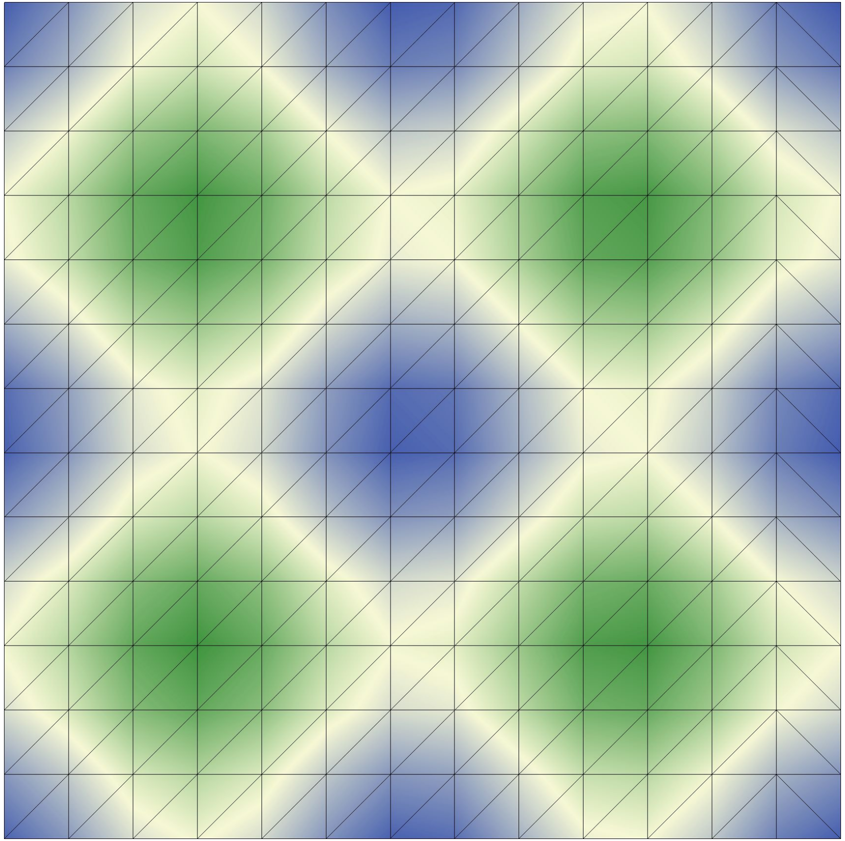

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  - Critical simplices
    - $|\{\sigma_{i-1} < \sigma_i \mid f(\sigma_{i-1}) \geq f(\sigma_i)\}| = 0$
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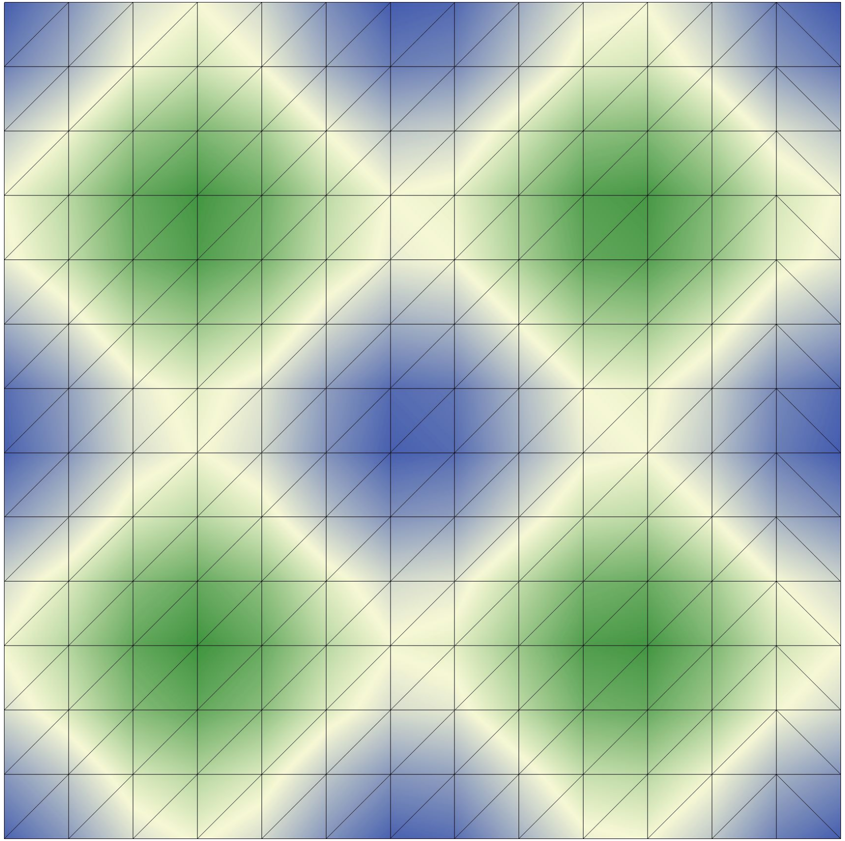
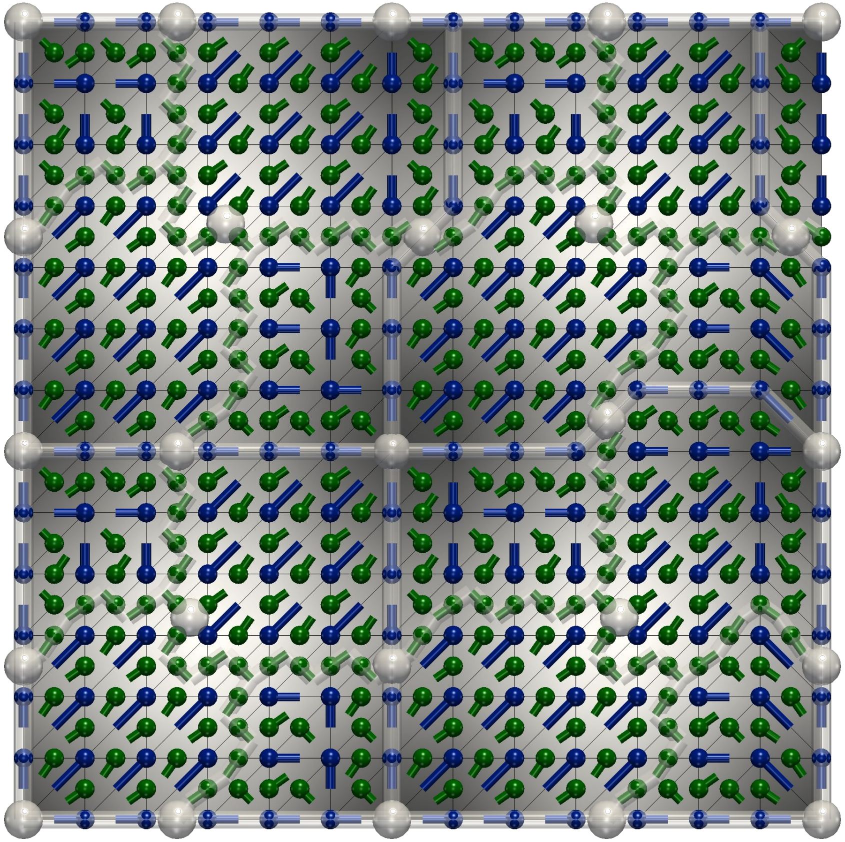


Discrete Gradient Field

- **Discrete vector**
  - Pair $\{\sigma_i < \sigma_{i+1}\}$



Discrete Gradient Field

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- **Discrete vector field**
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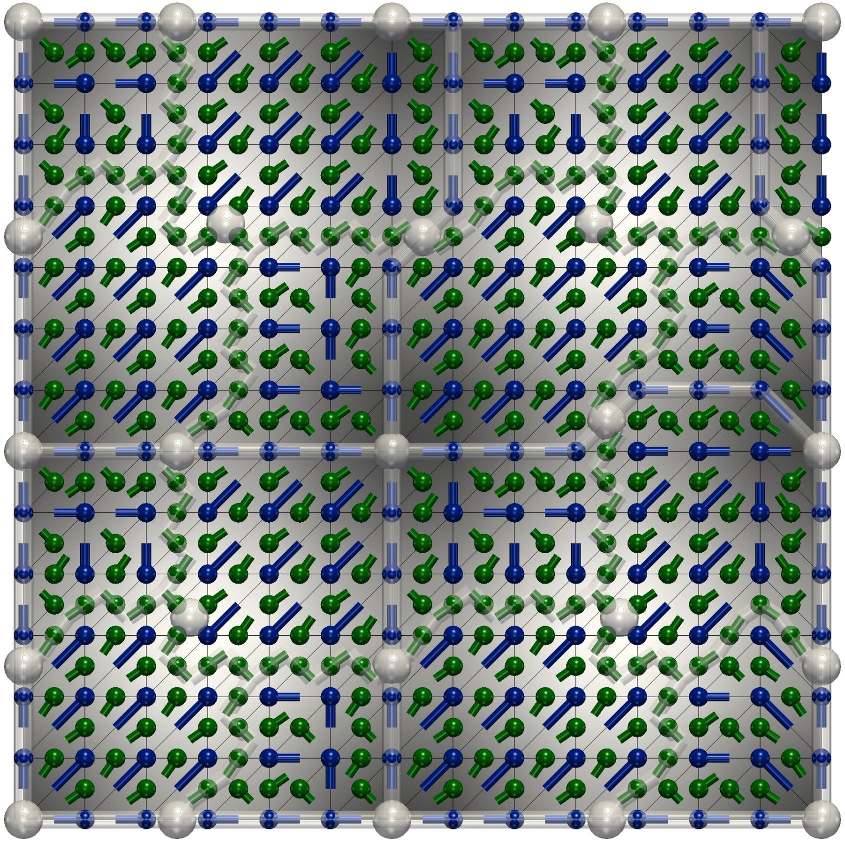
  - Each simplex in **at most** 1 pair



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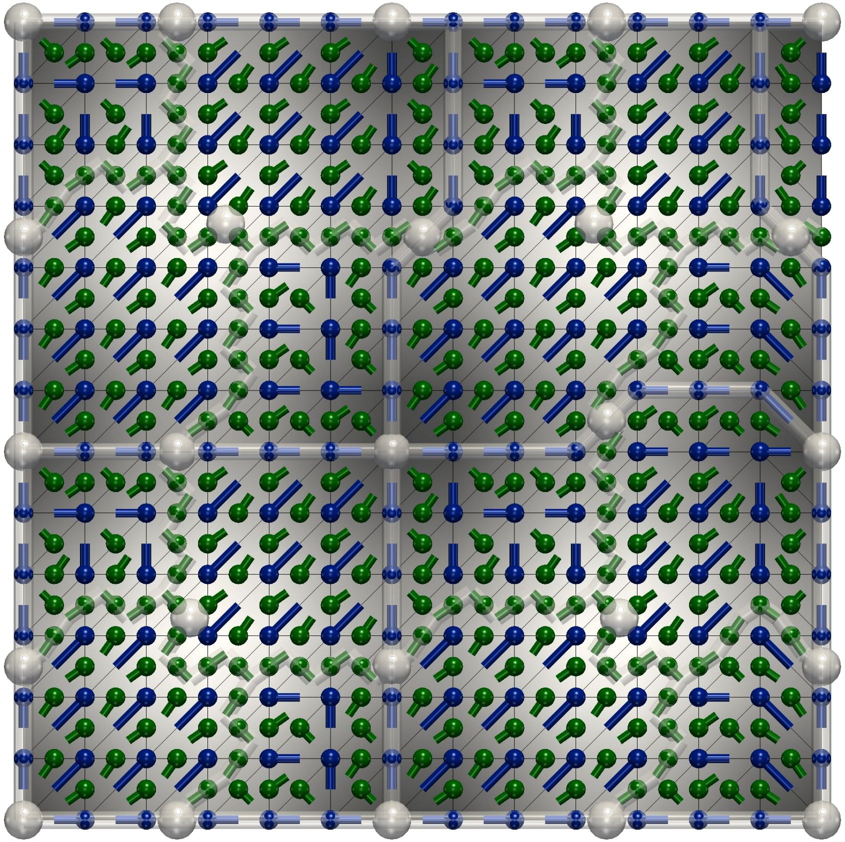
  - Sequence of vectors
    - $\{\sigma_i^0 < \sigma_{i+1}^0\} \ldots \{\sigma_i^n < \sigma_{i+1}^n\}$



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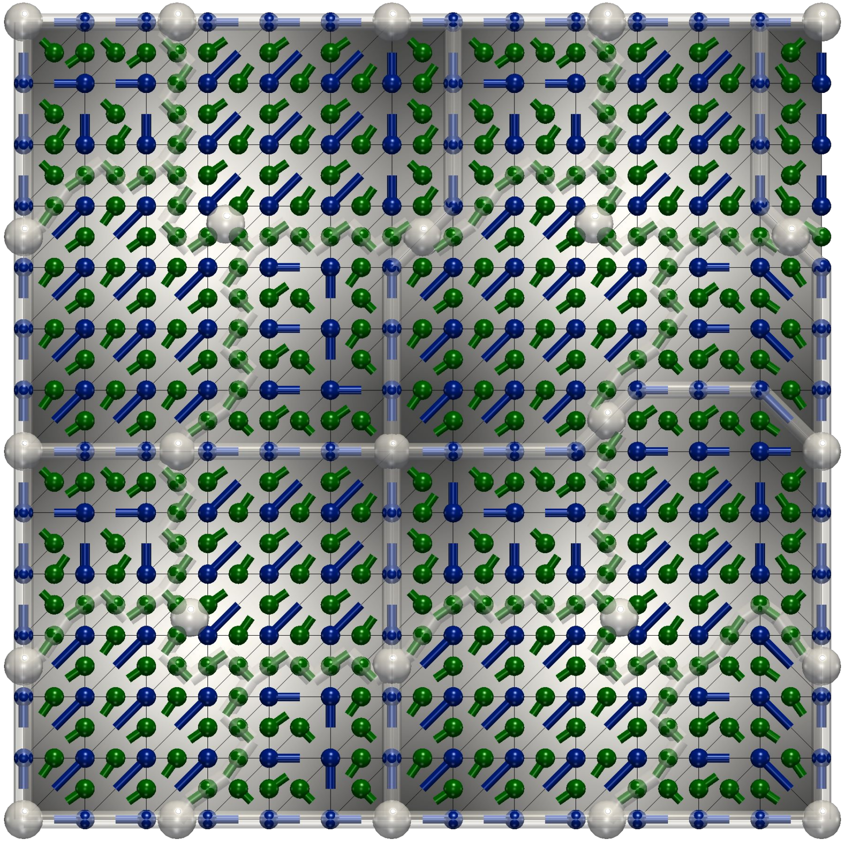

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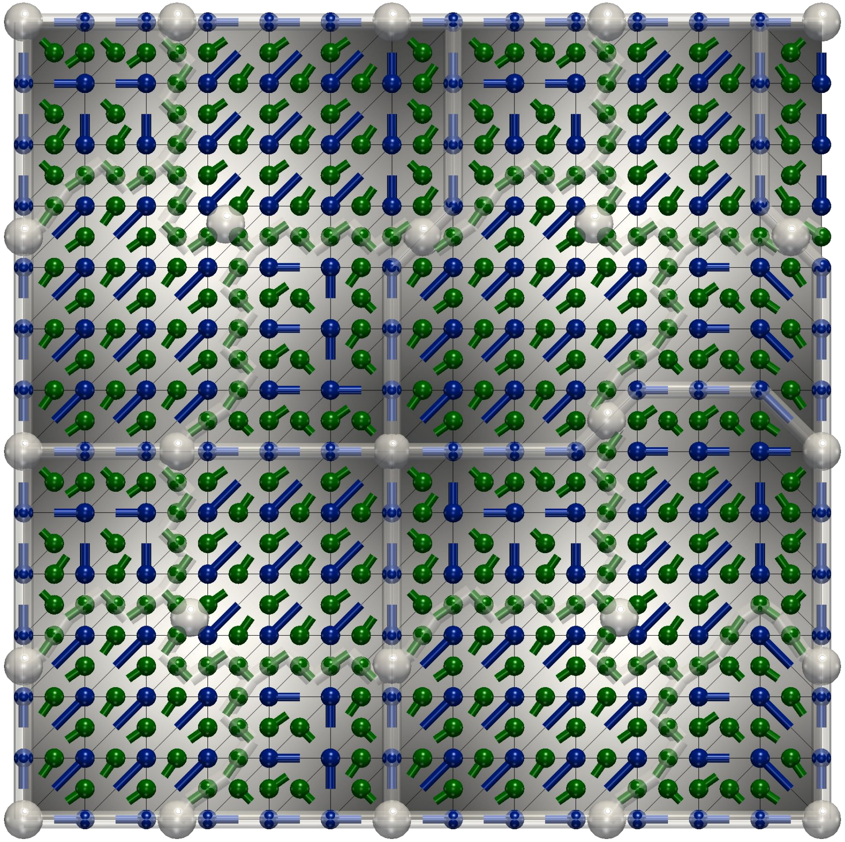
- **Discrete gradient field**
  - All v-paths are loop free



Algorithm example

- **Discrete gradient from PL data**
  - Shrivashankar et al. 2012

Algorithm example

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  - For each dimension i

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Algorithm example

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      - Co-faces maximized by σ_i

$$C^-(\sigma_i) = \{\sigma_{i+1} > \sigma_i \ \mid \sigma_i = argmax_{\sigma_i' < \sigma_{i+1}} f(\sigma_i')$$

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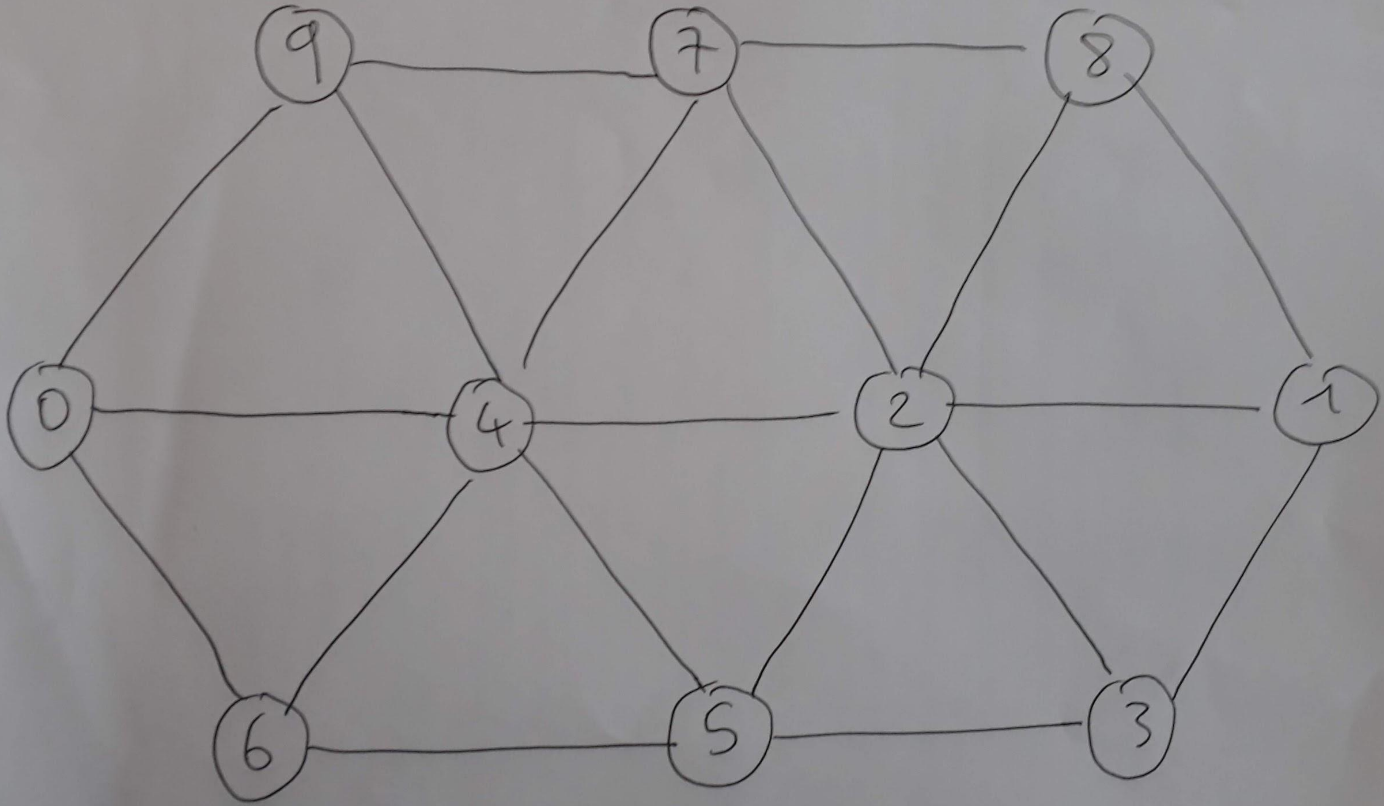
  - Trivially parallelizable
- **Comparing simplices**
  - Based on max vertices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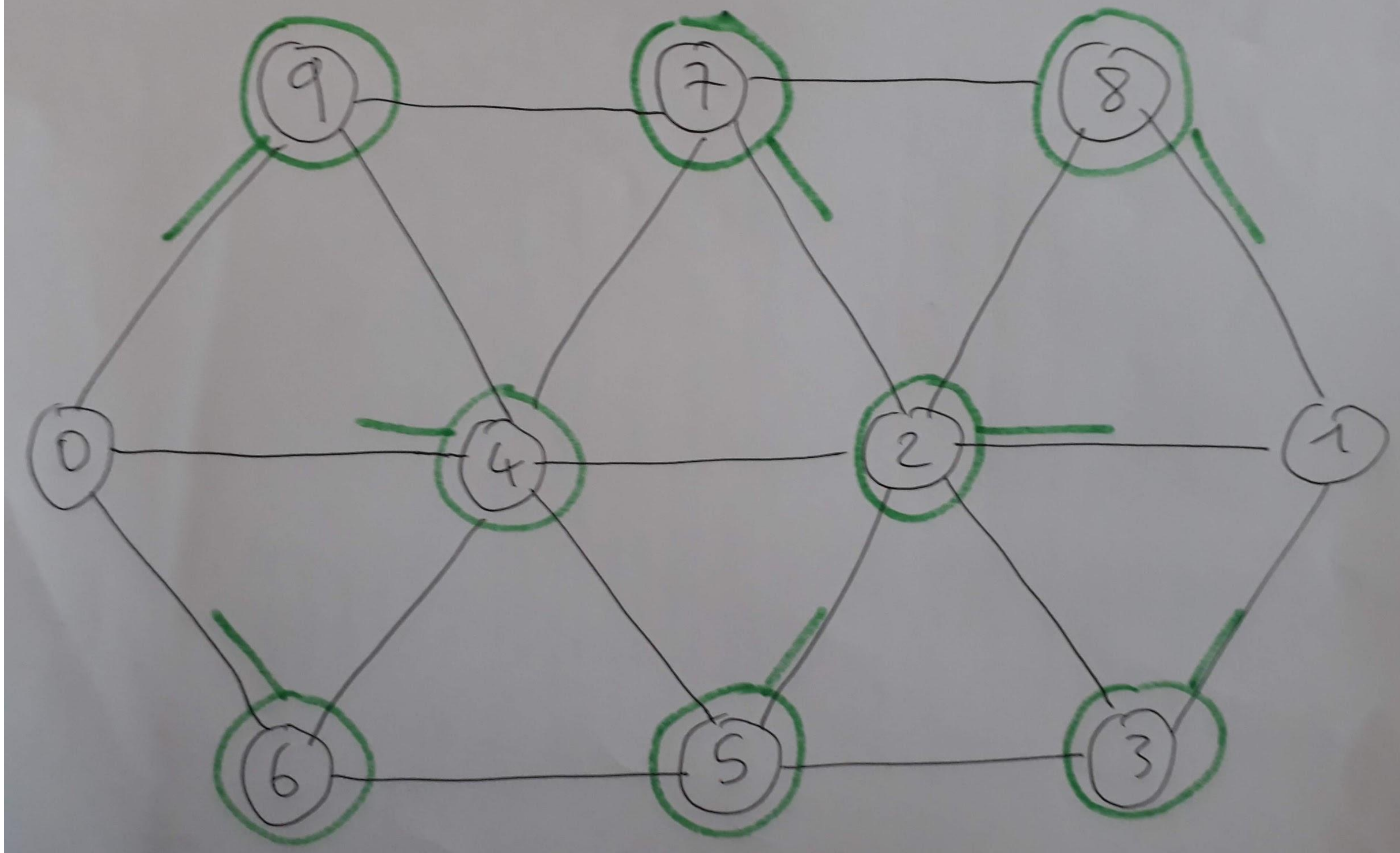
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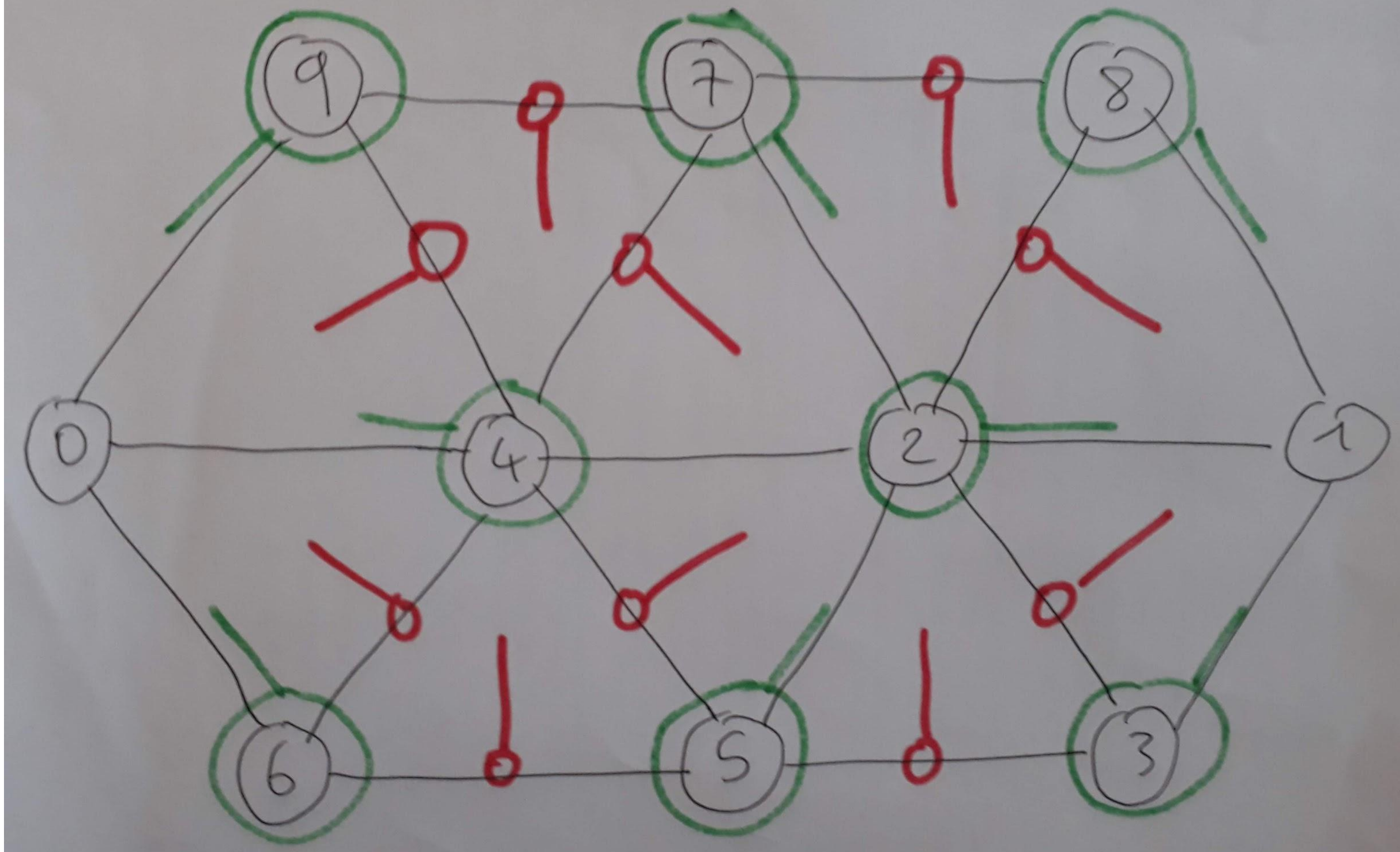
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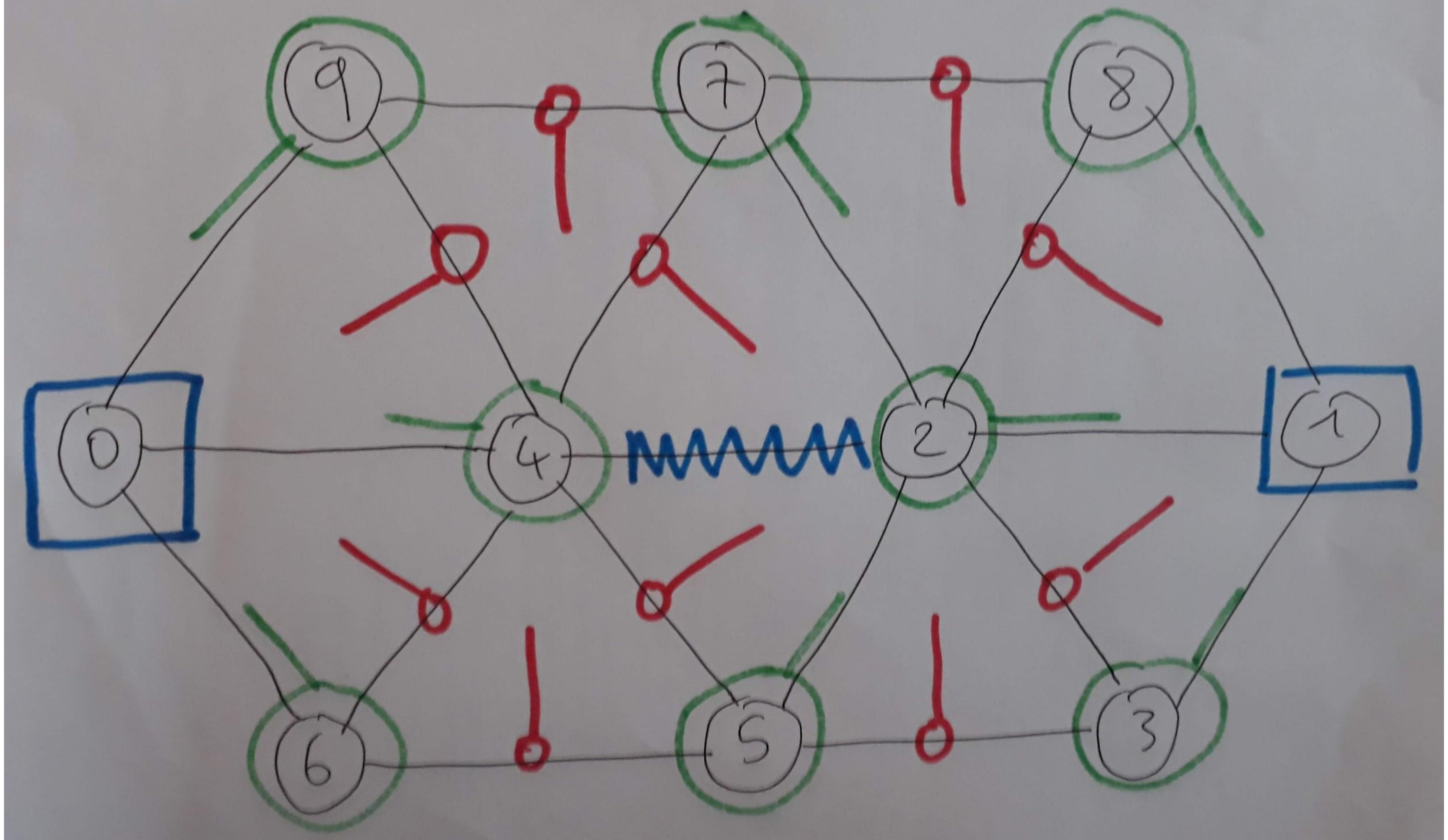

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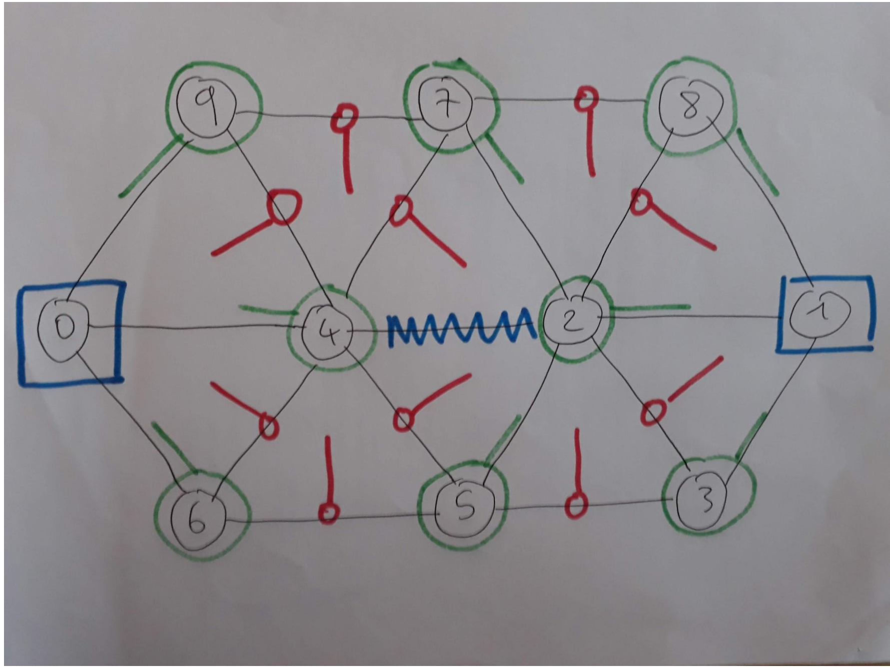
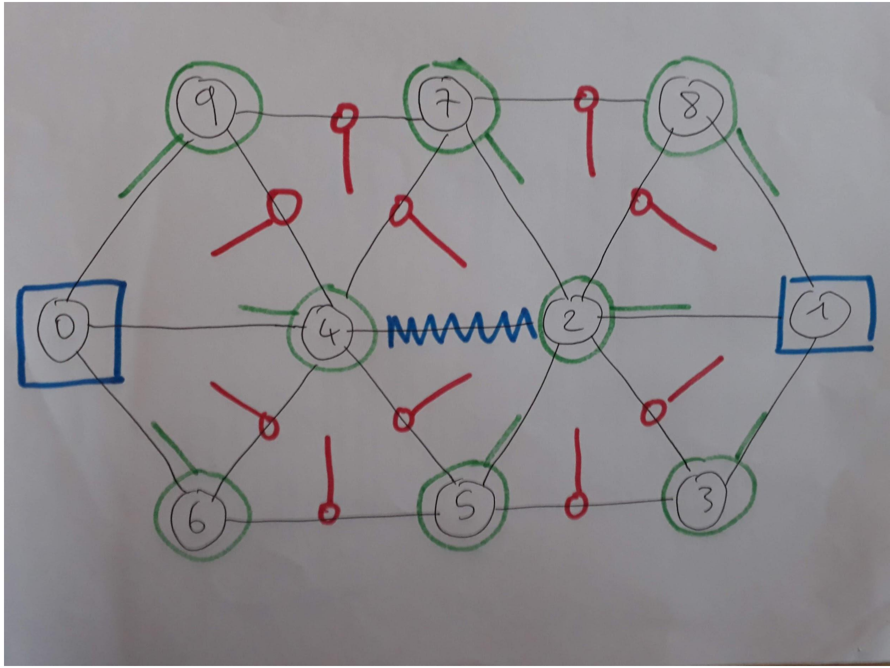
Properties

- **Unpaired simplices**
  - Dead-ends for v-paths



Properties

- **Unpaired simplices**
  - Dead-ends for v-paths
    - Notion of **critical simplex**



Properties

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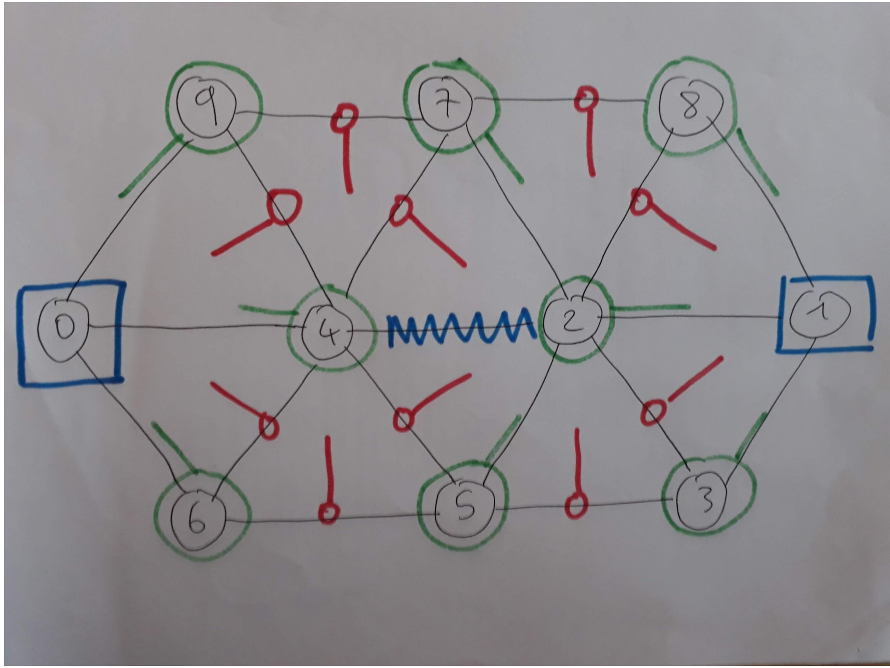
    - Dimension: index

\o/



Properties

- **Unpaired simplices**
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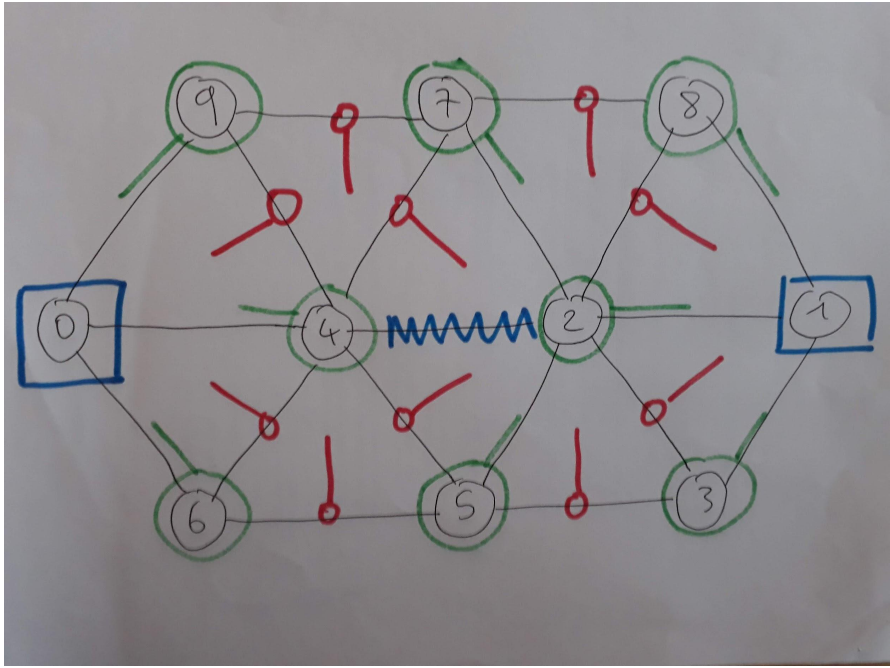
  - Manifold domain
    - No degenerate critical point! **\o/**



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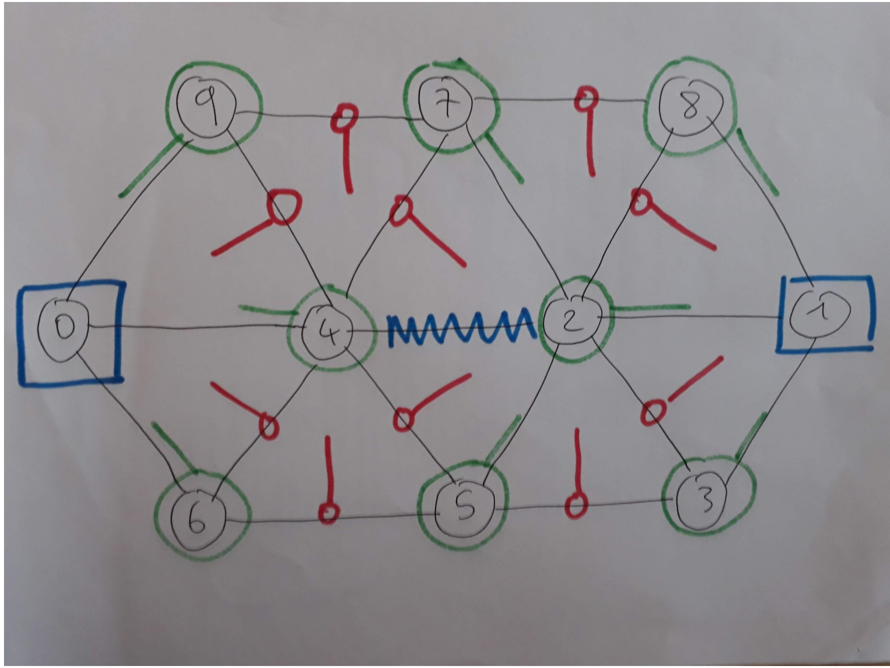
    - Depends on critical index
    - 2D: ascending and descending manifolds on different dimensions



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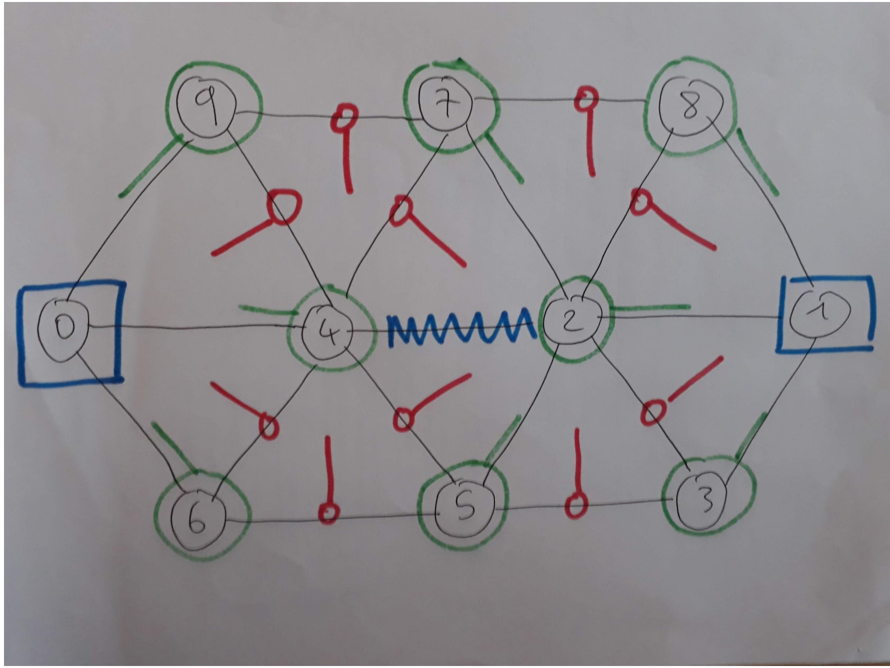
  - Descending manifolds: primal complex
  - Ascending manifolds: dual complex



Discrete Morse-Smale complex

- **Algorithm**
  - Collect the ascending and descending manifolds of each critical cell

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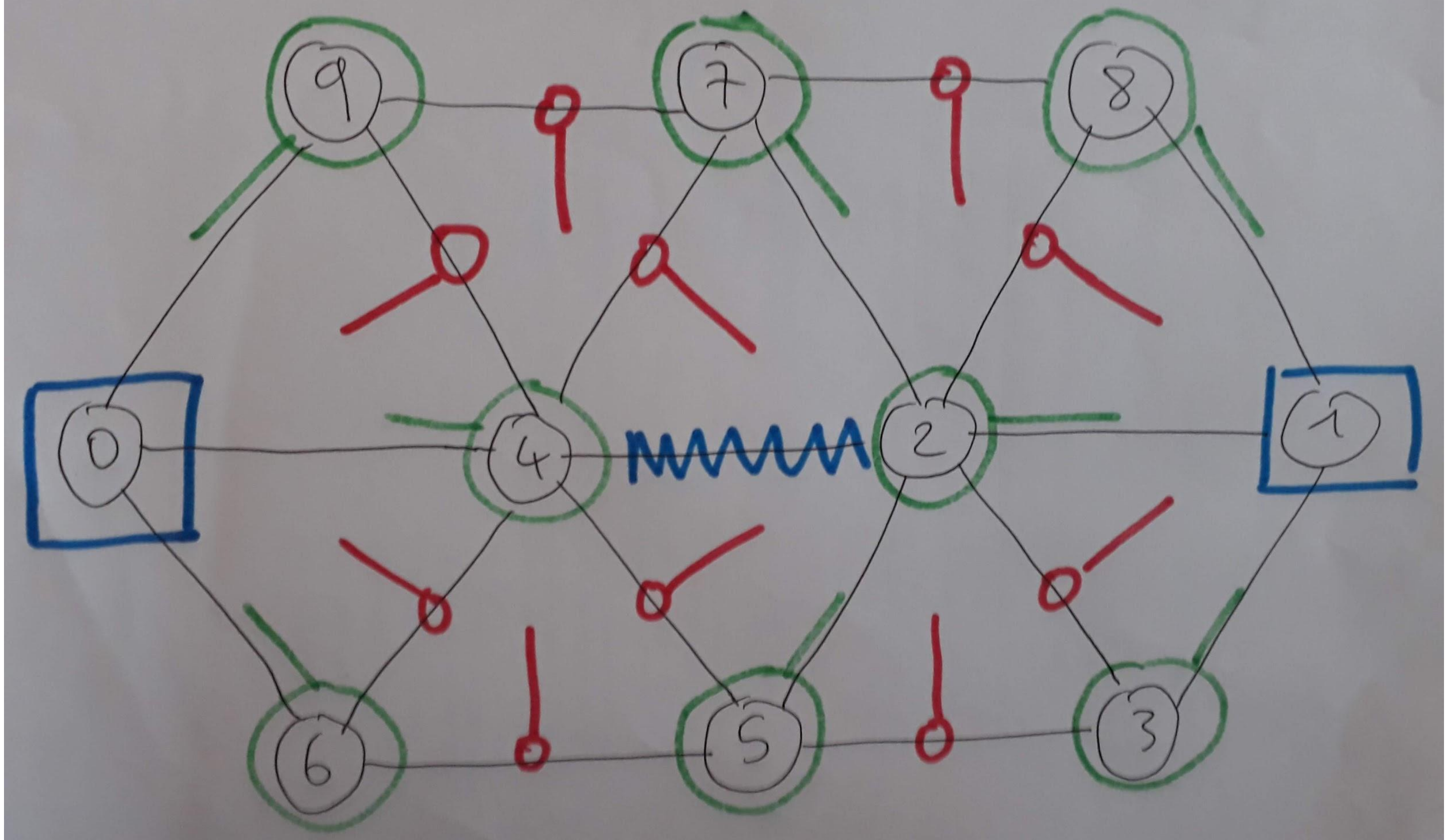
    - Descending manifolds of (d-1)-saddles
    - Ascending manifolds of 1-saddles

  - Saddle connectors (3D)
    - V-path connecting a 1-saddle to a 2-saddle (intersection)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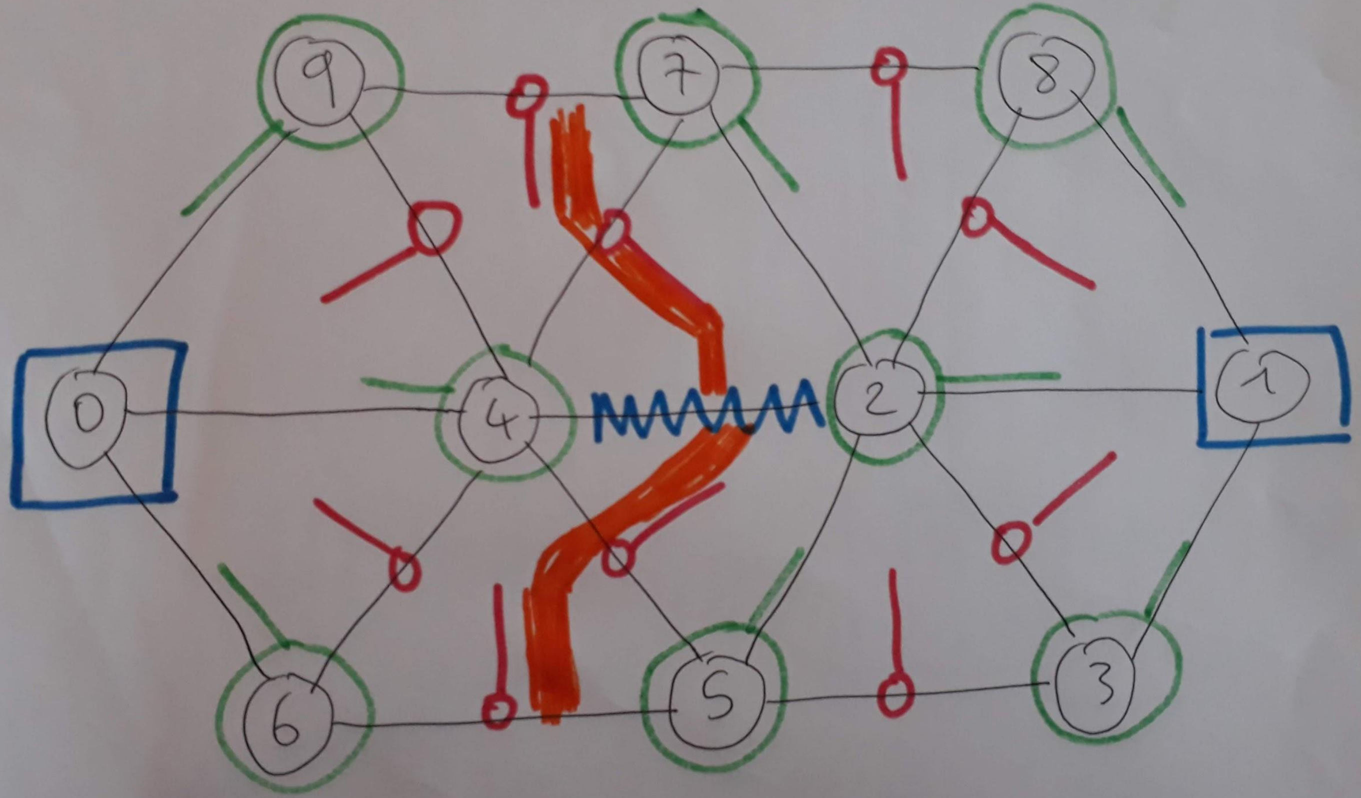

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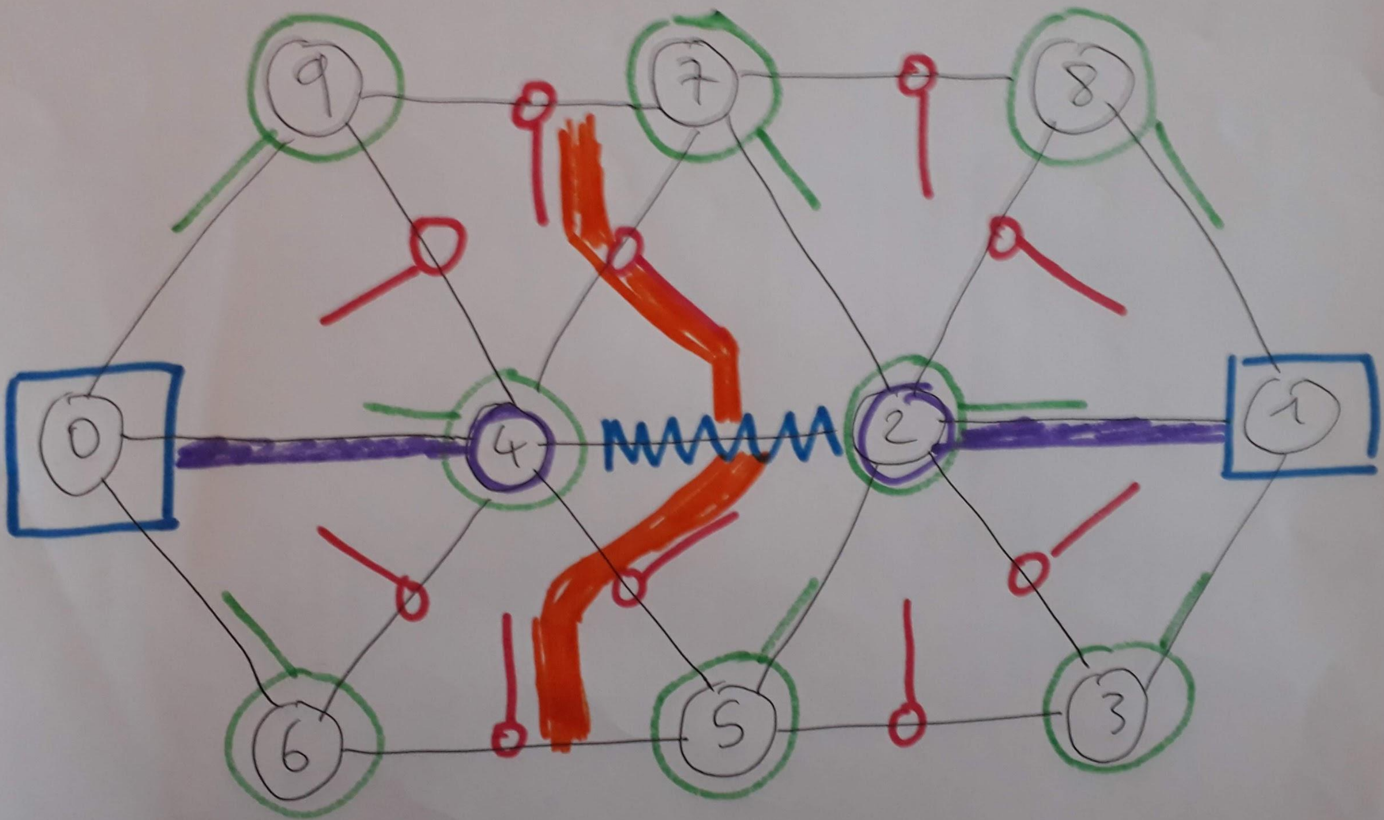

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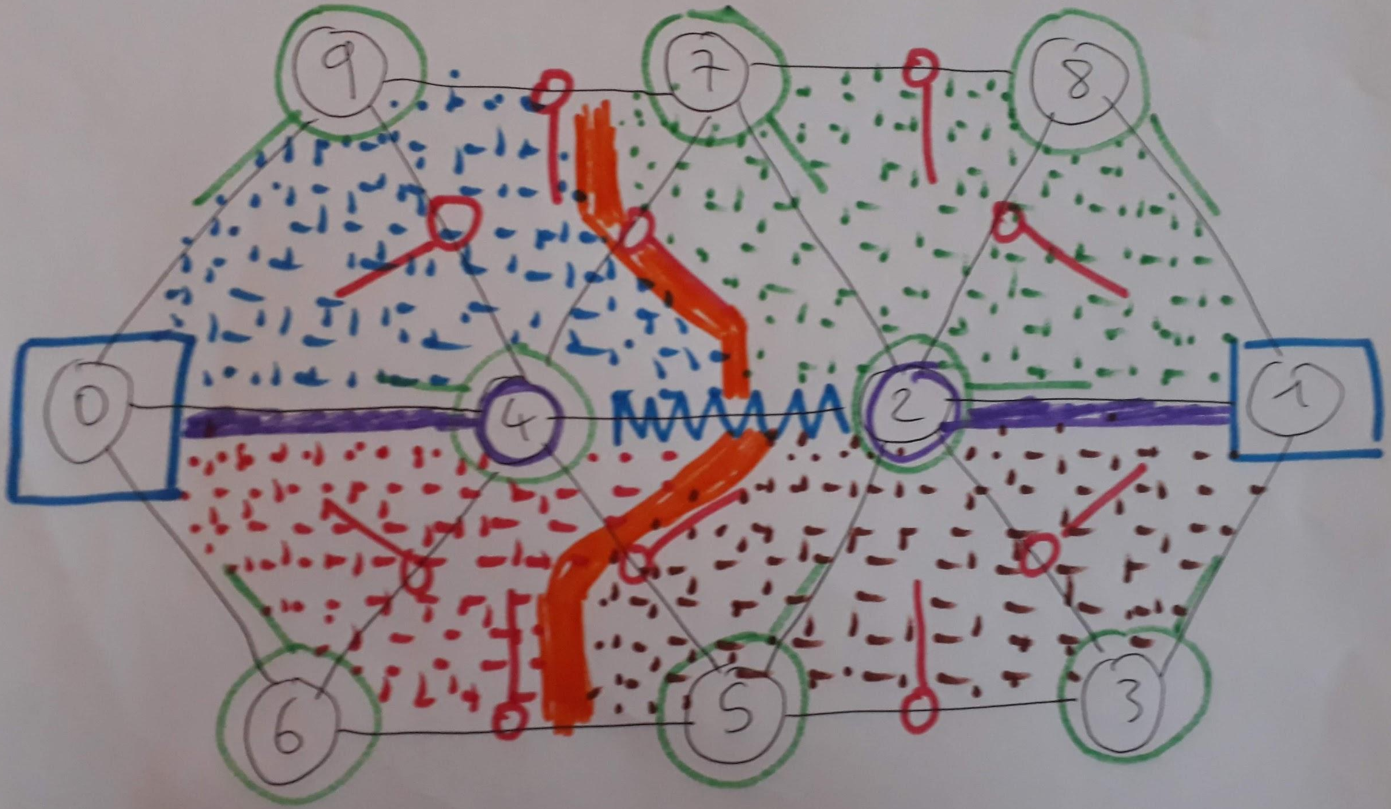
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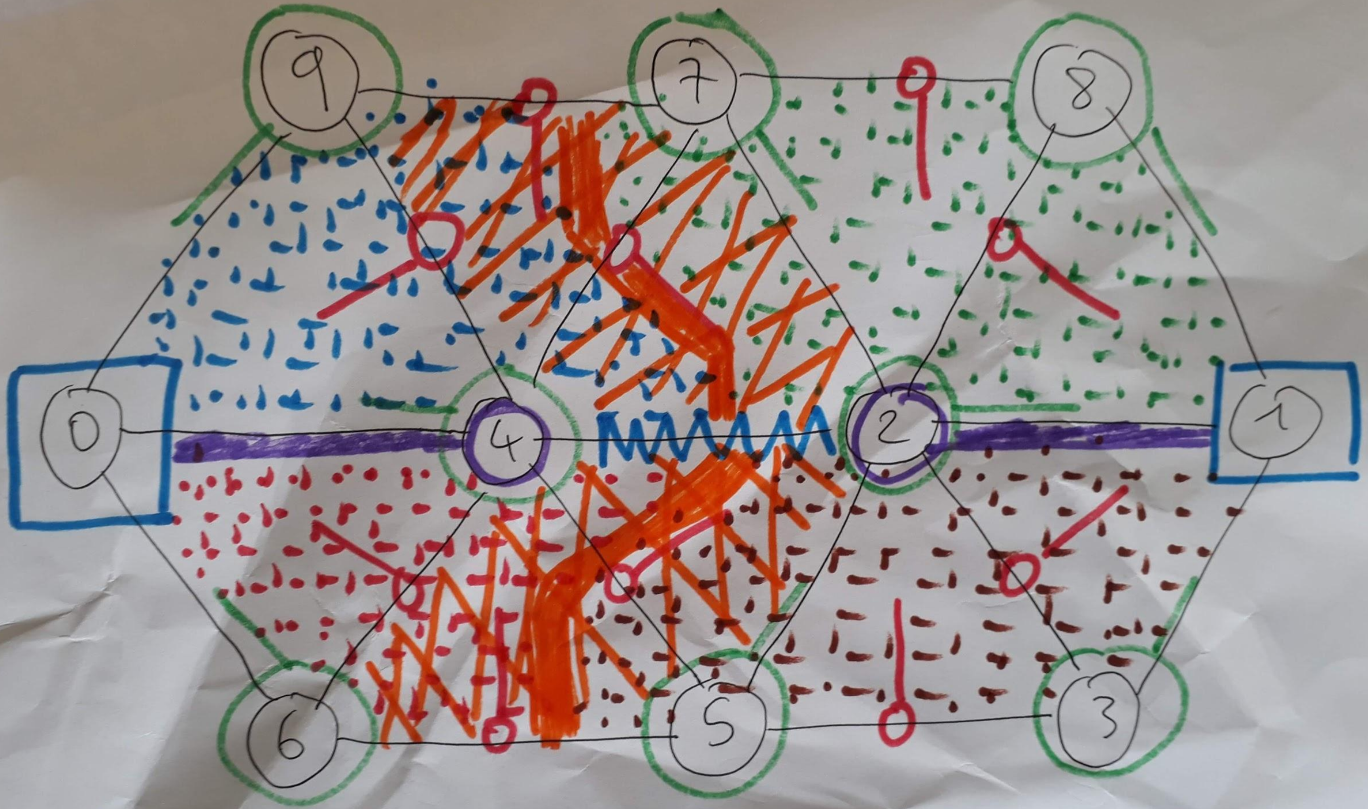
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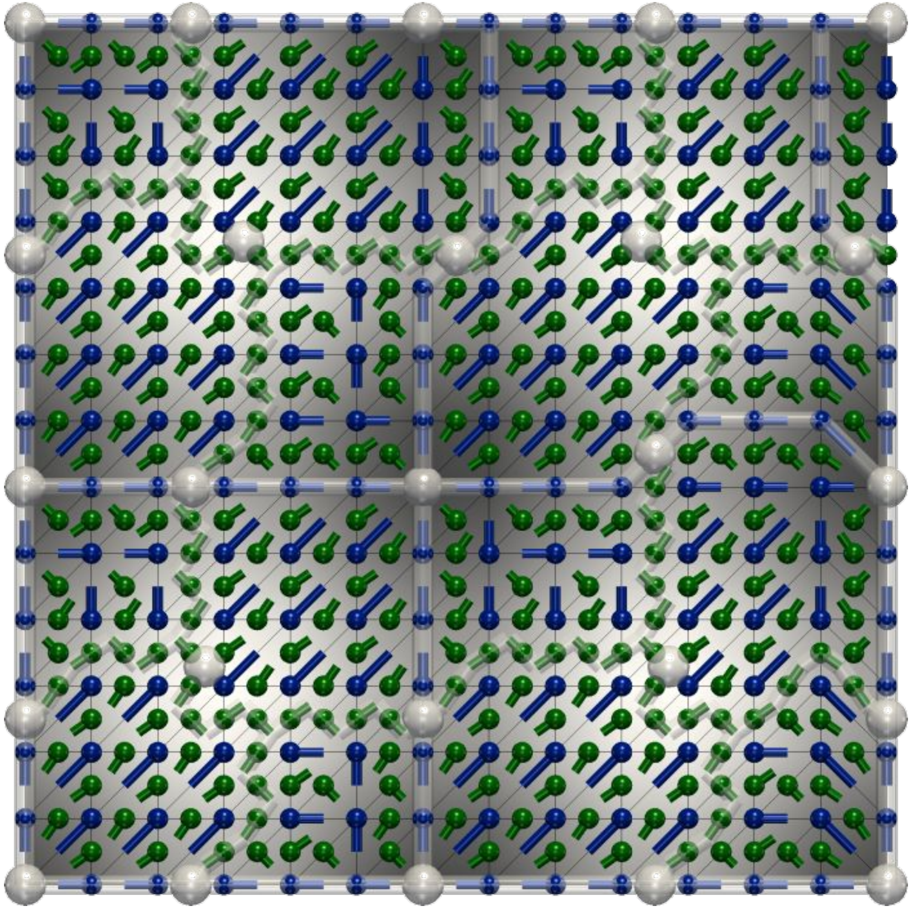
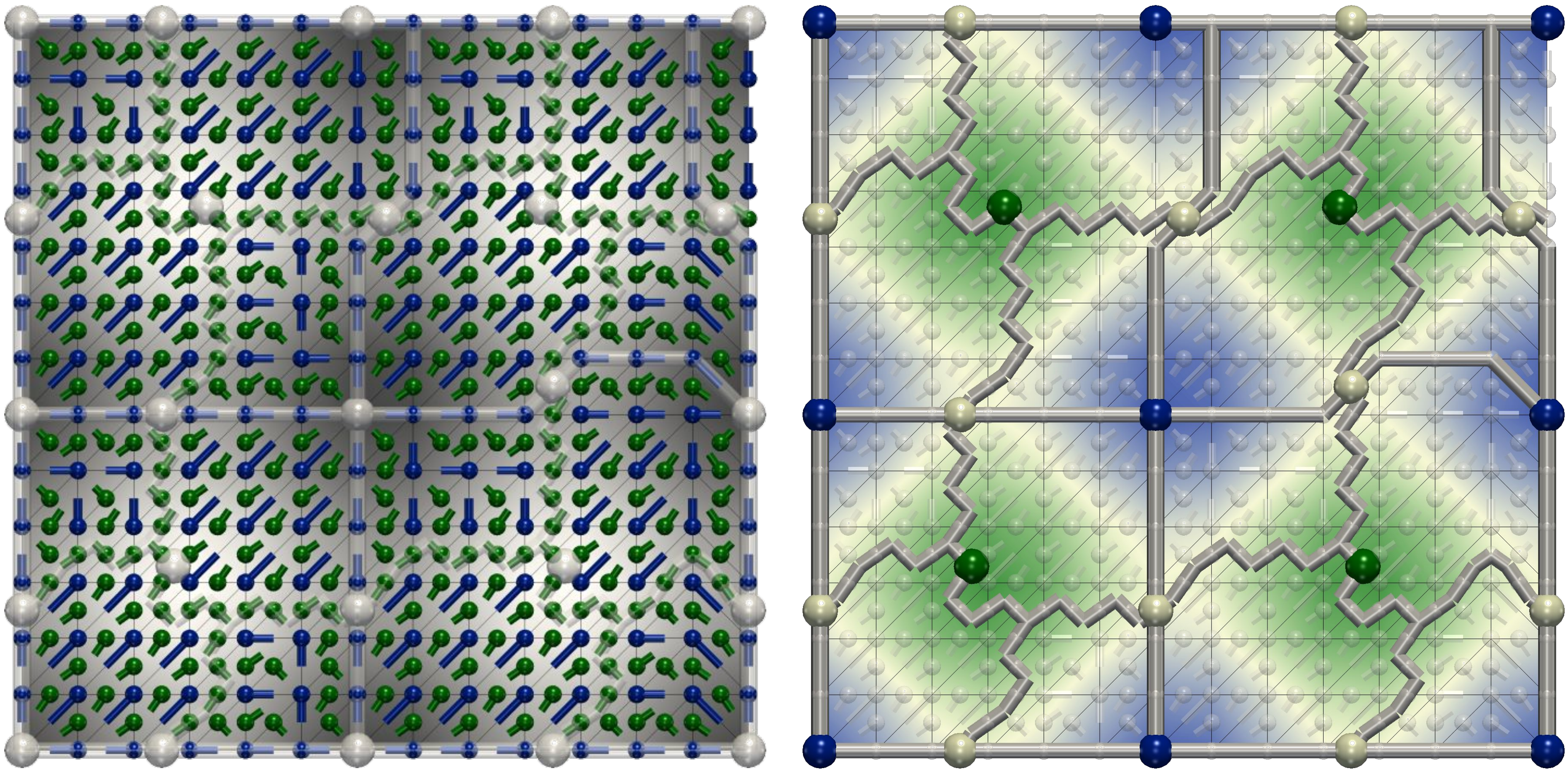
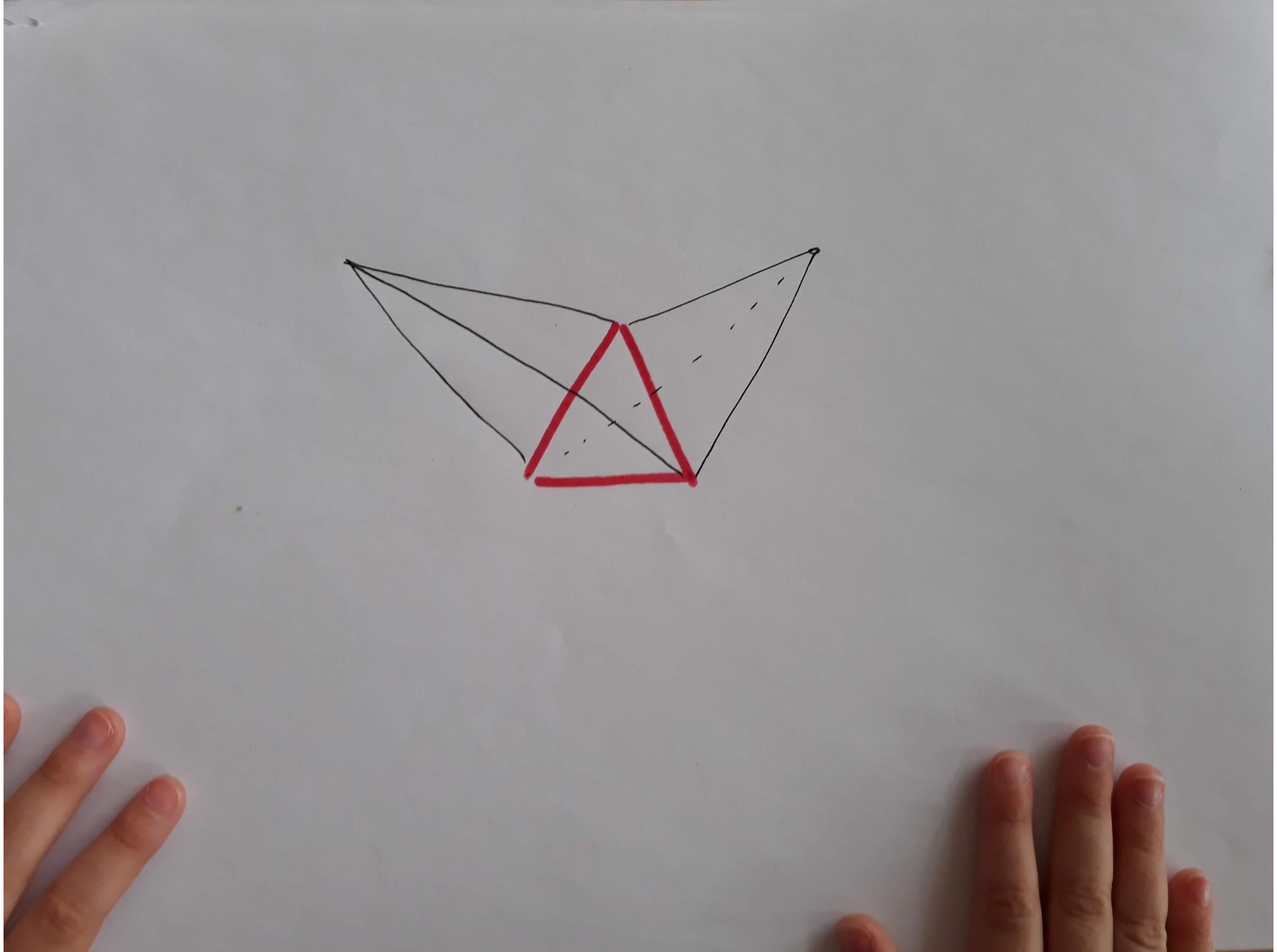
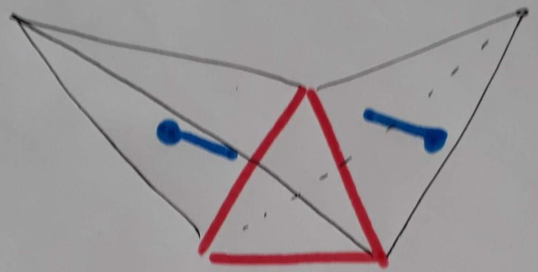
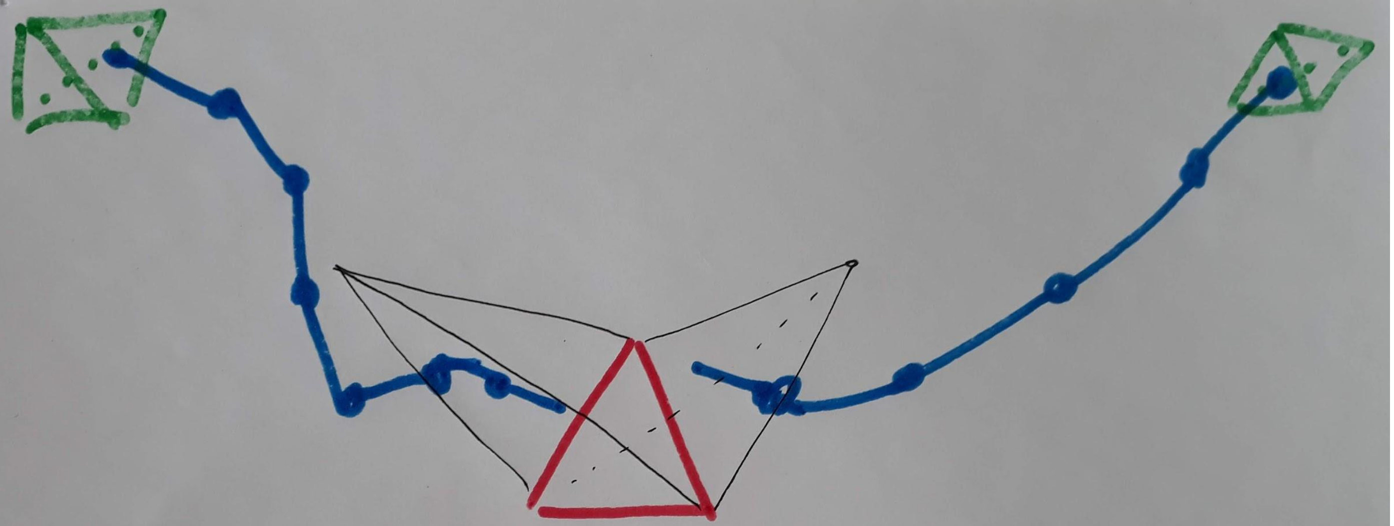
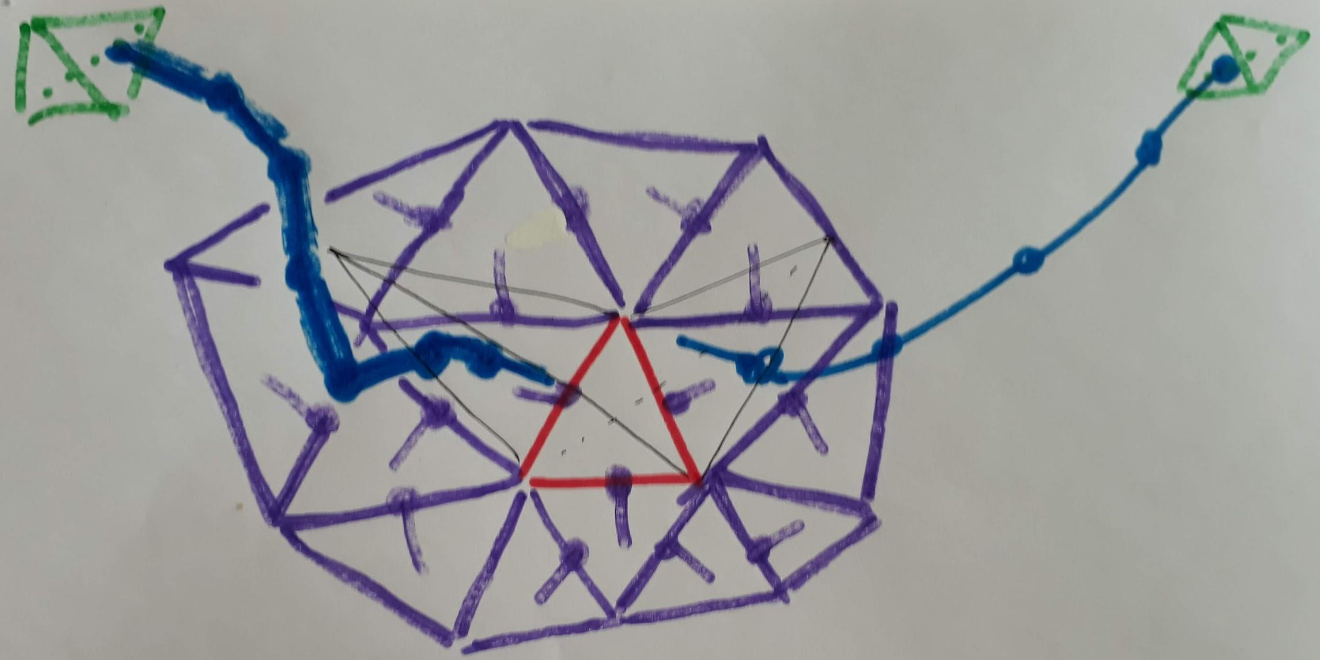


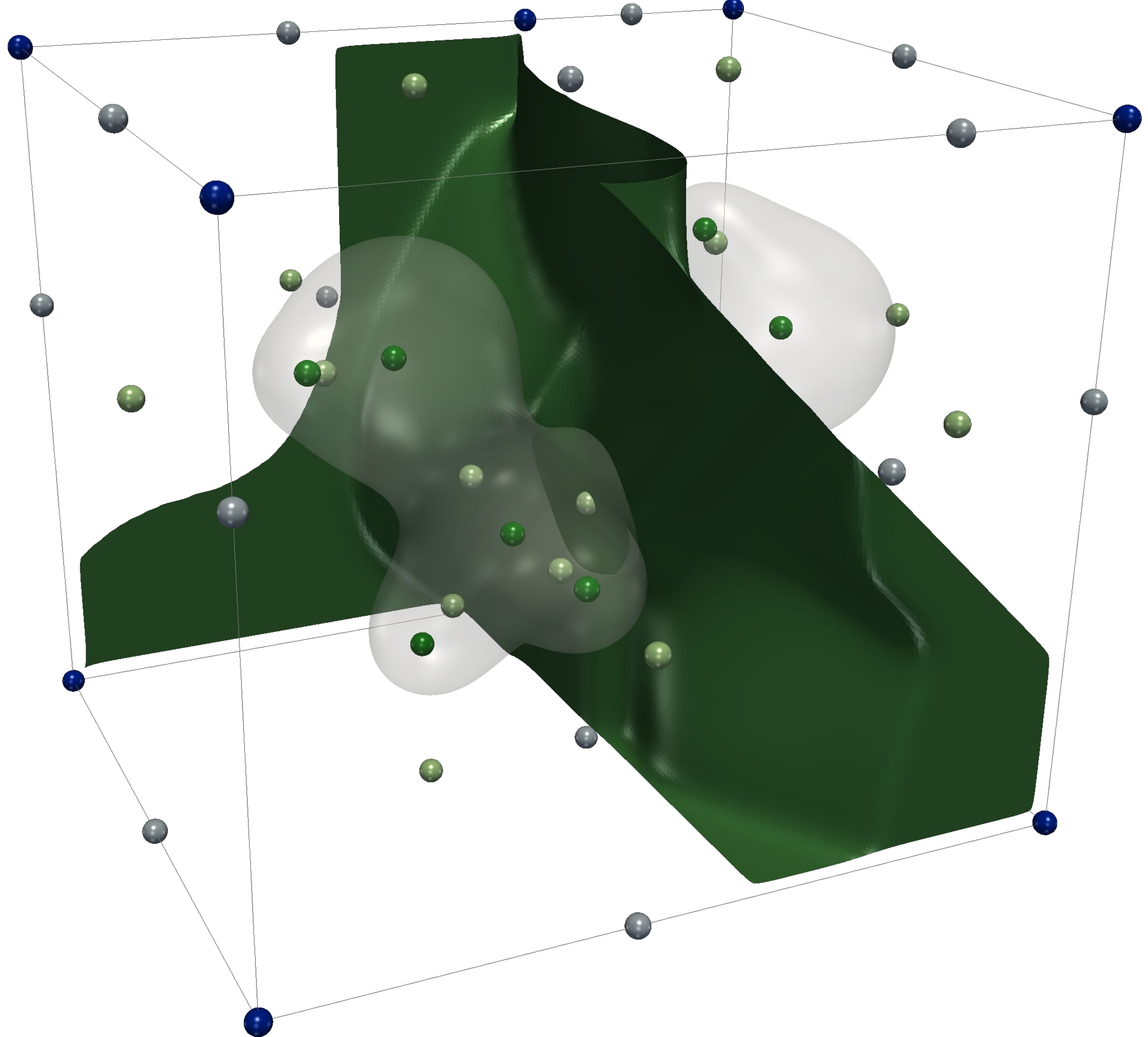


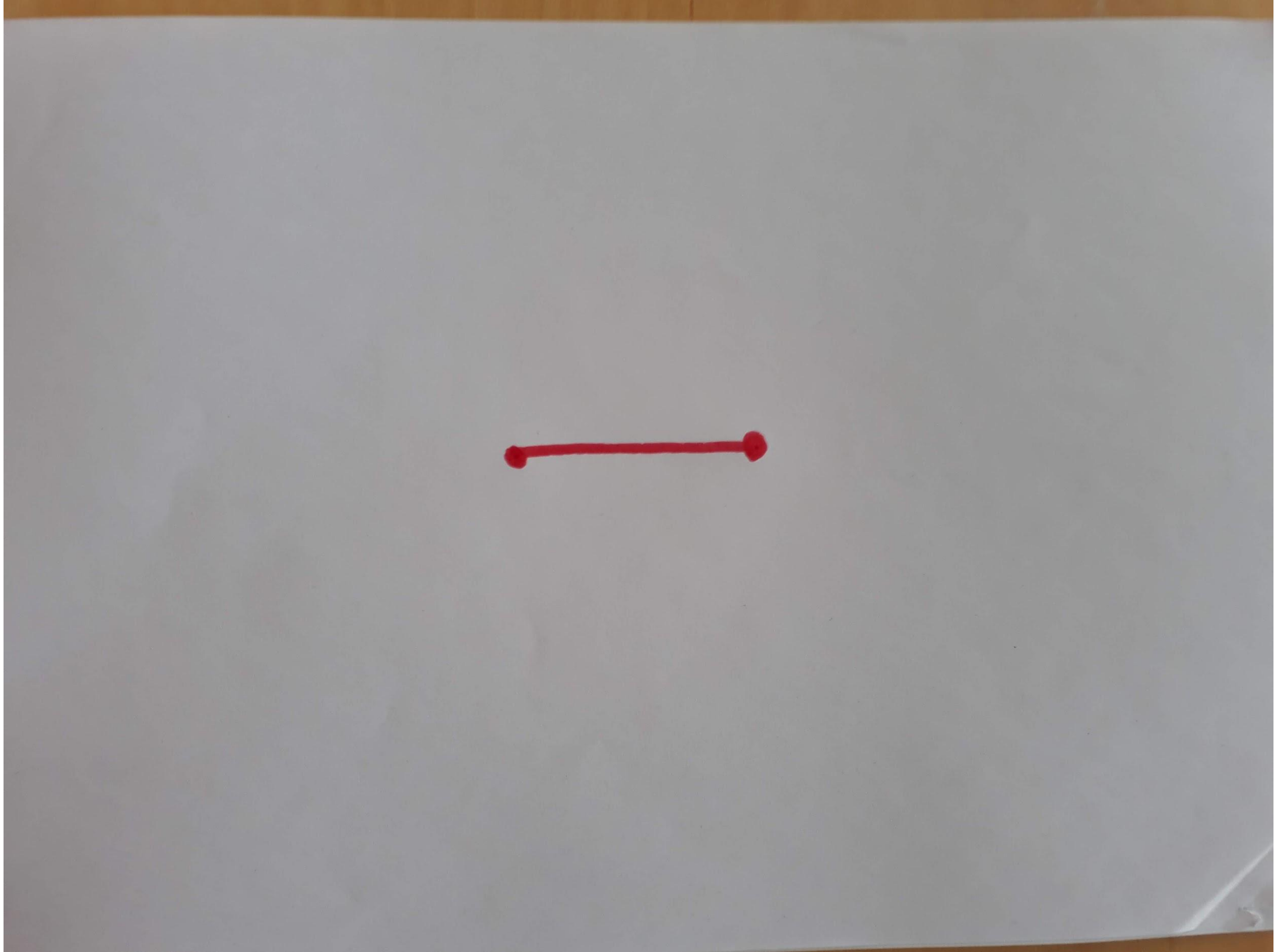


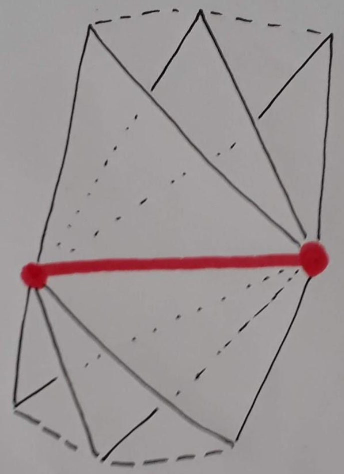


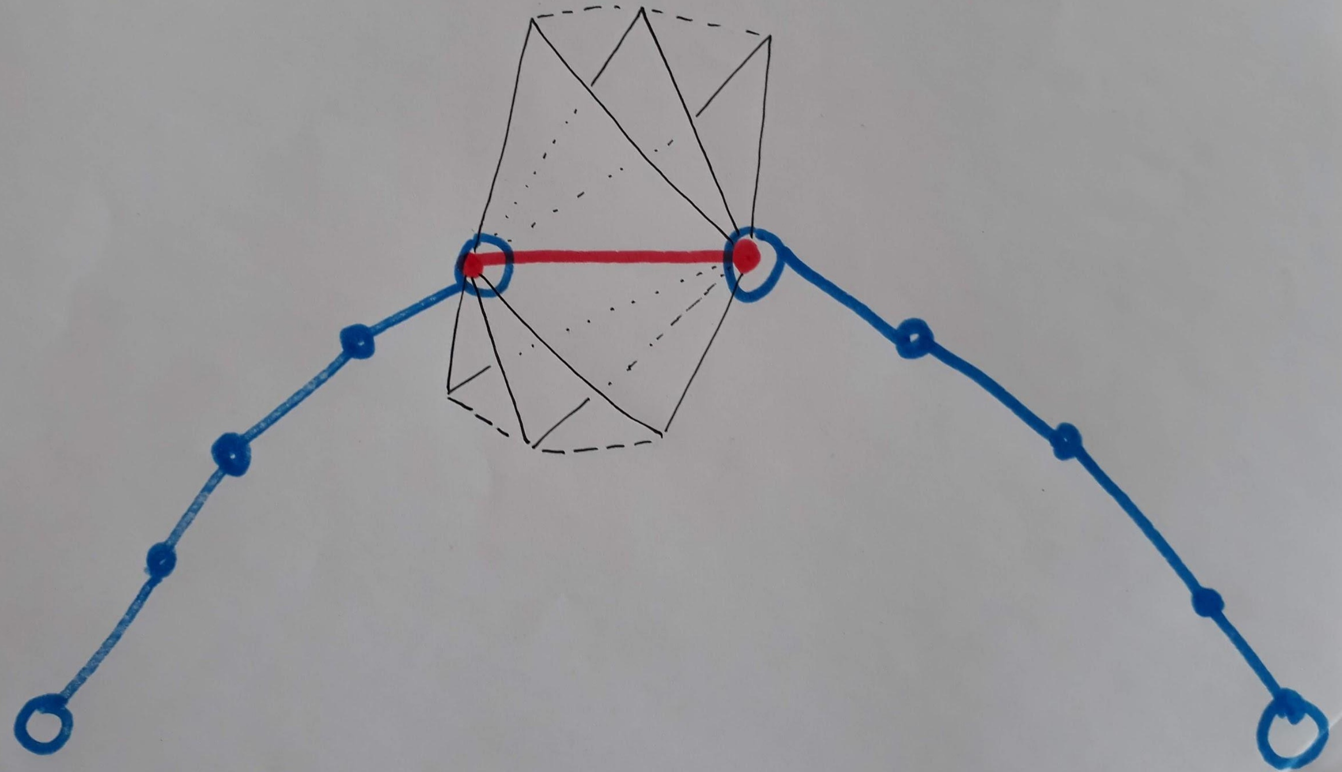


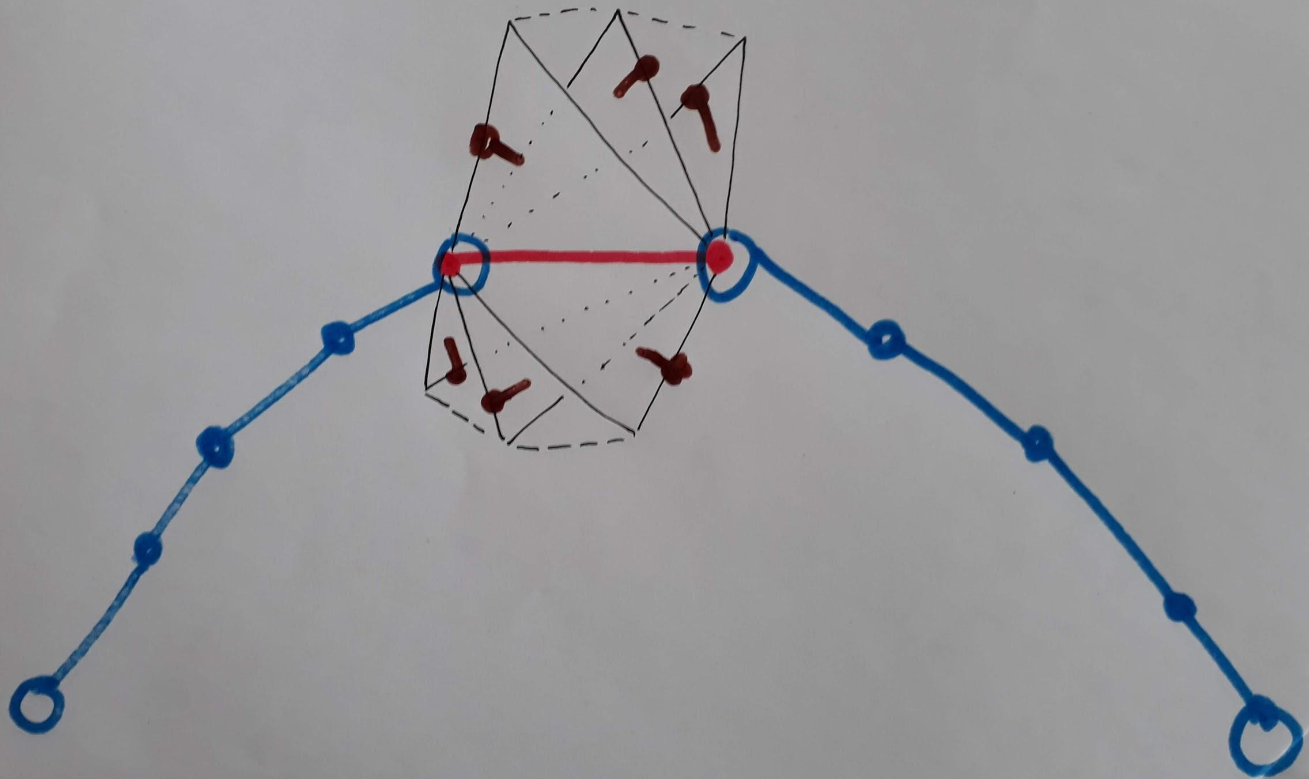


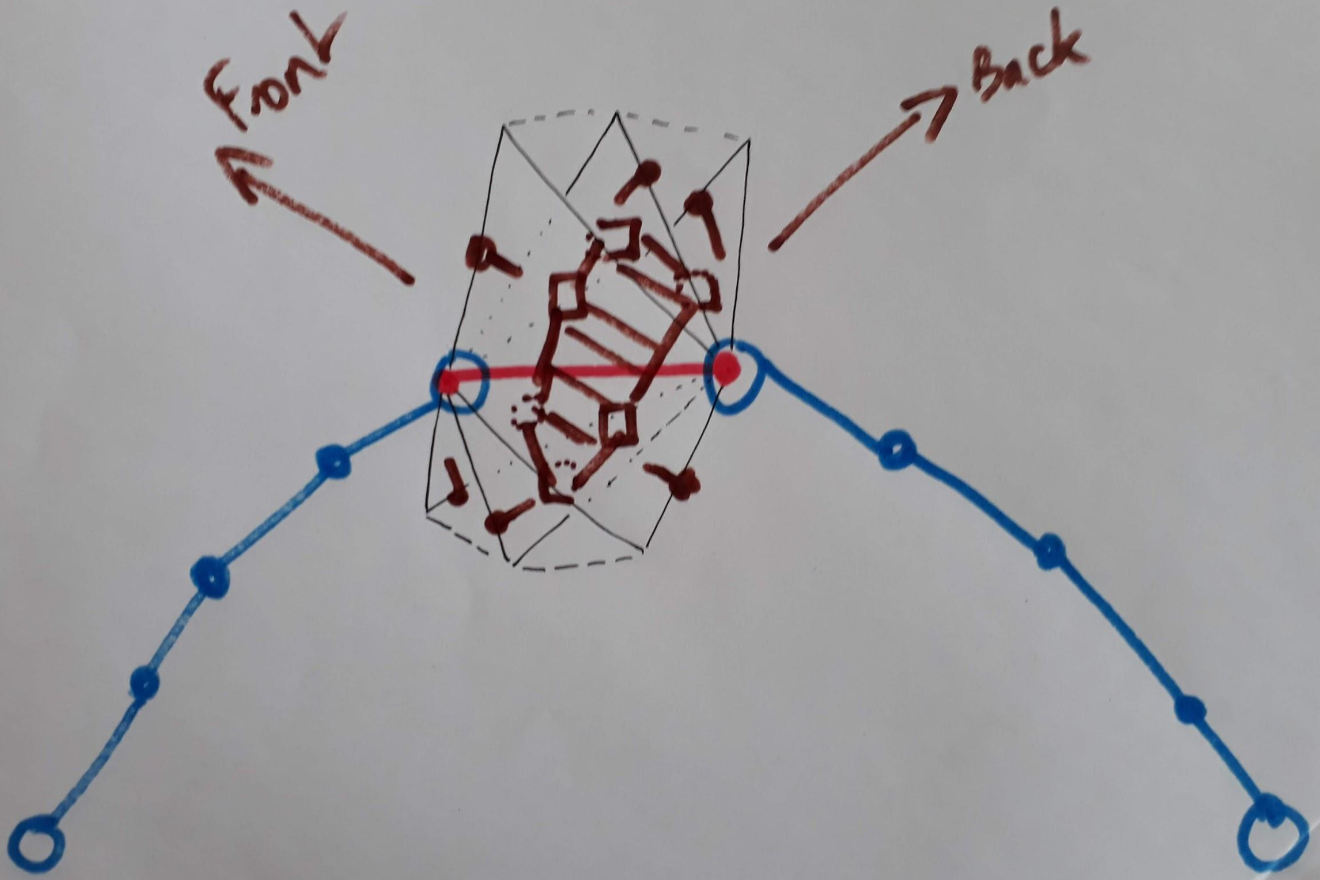




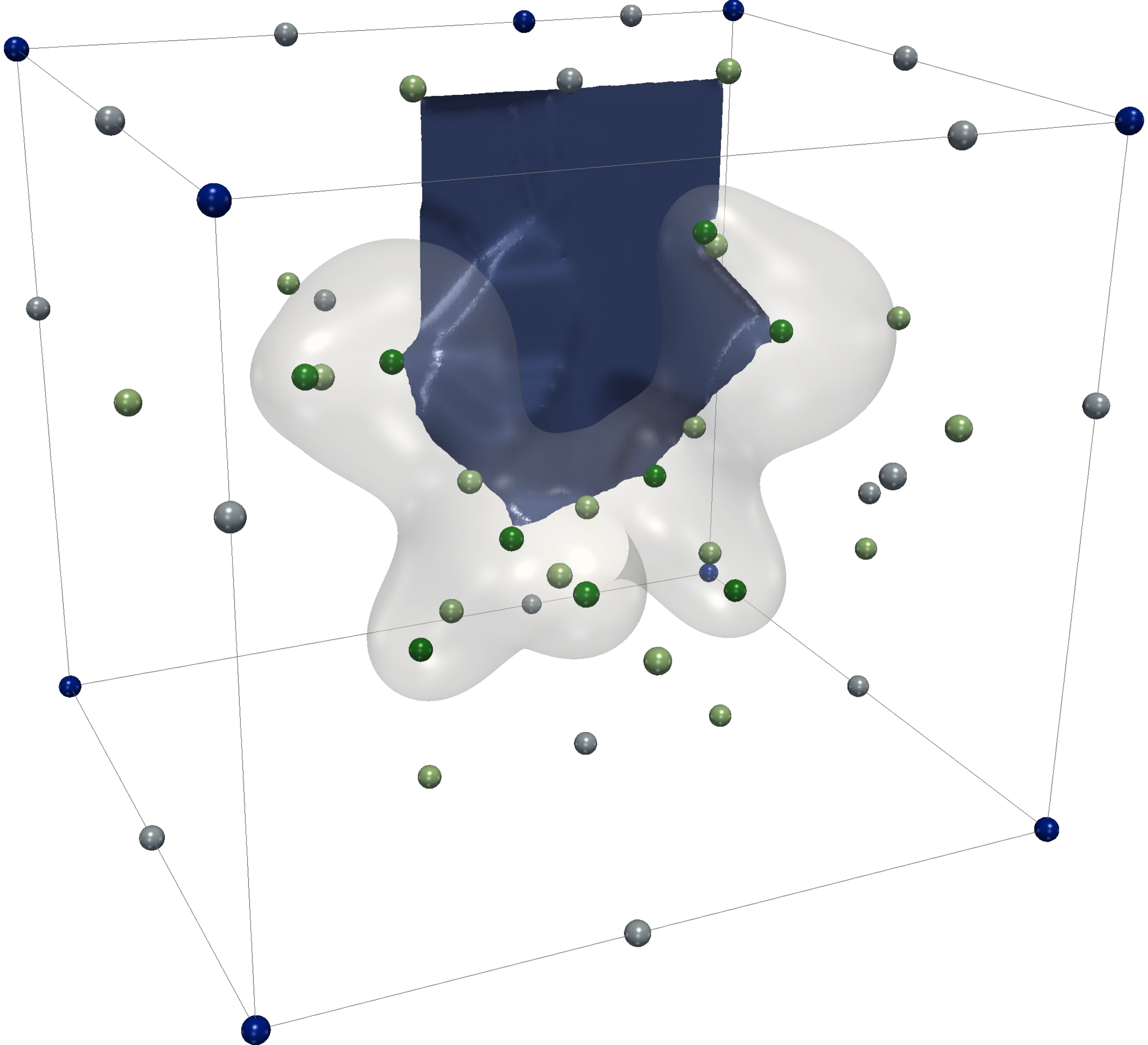


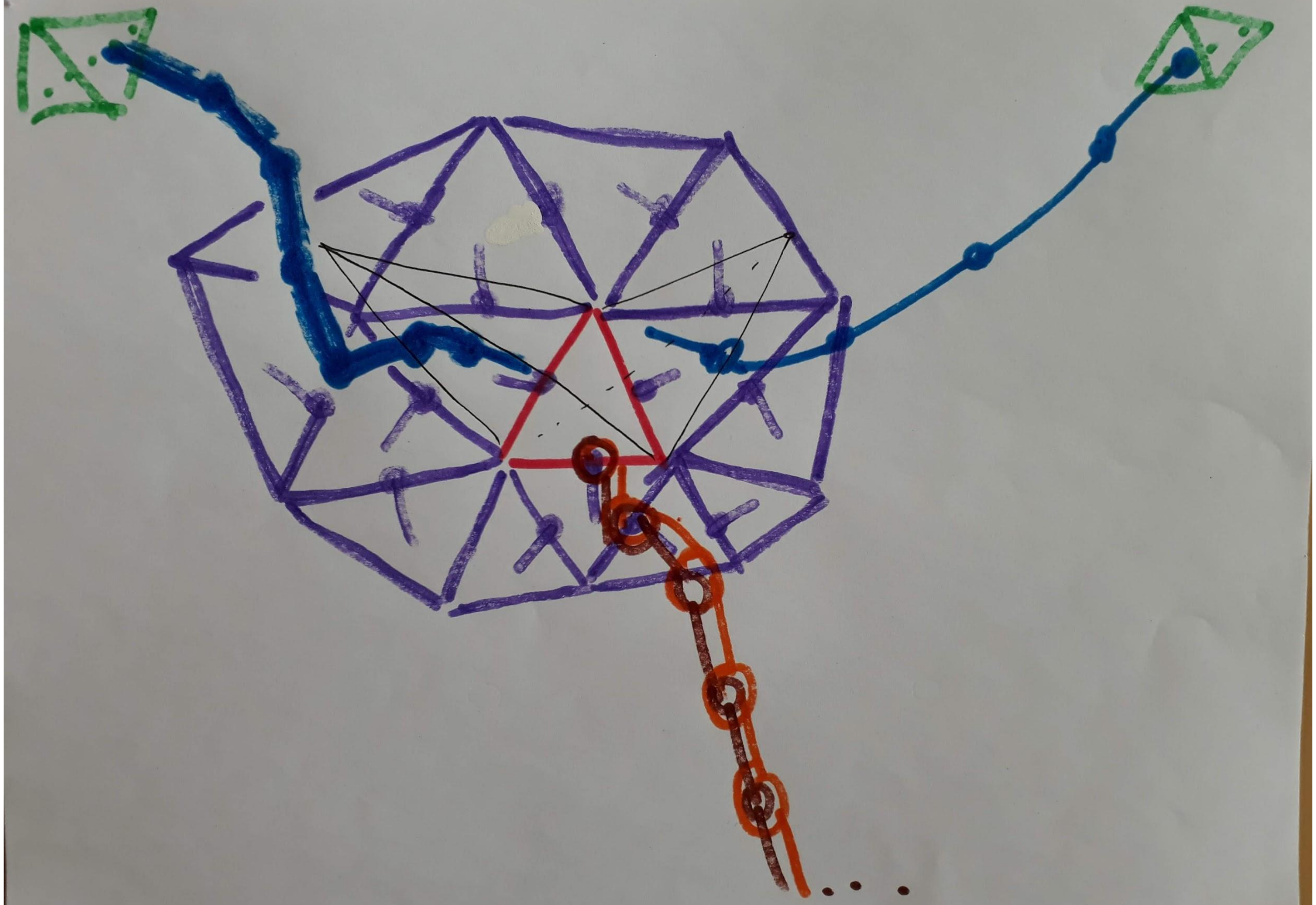


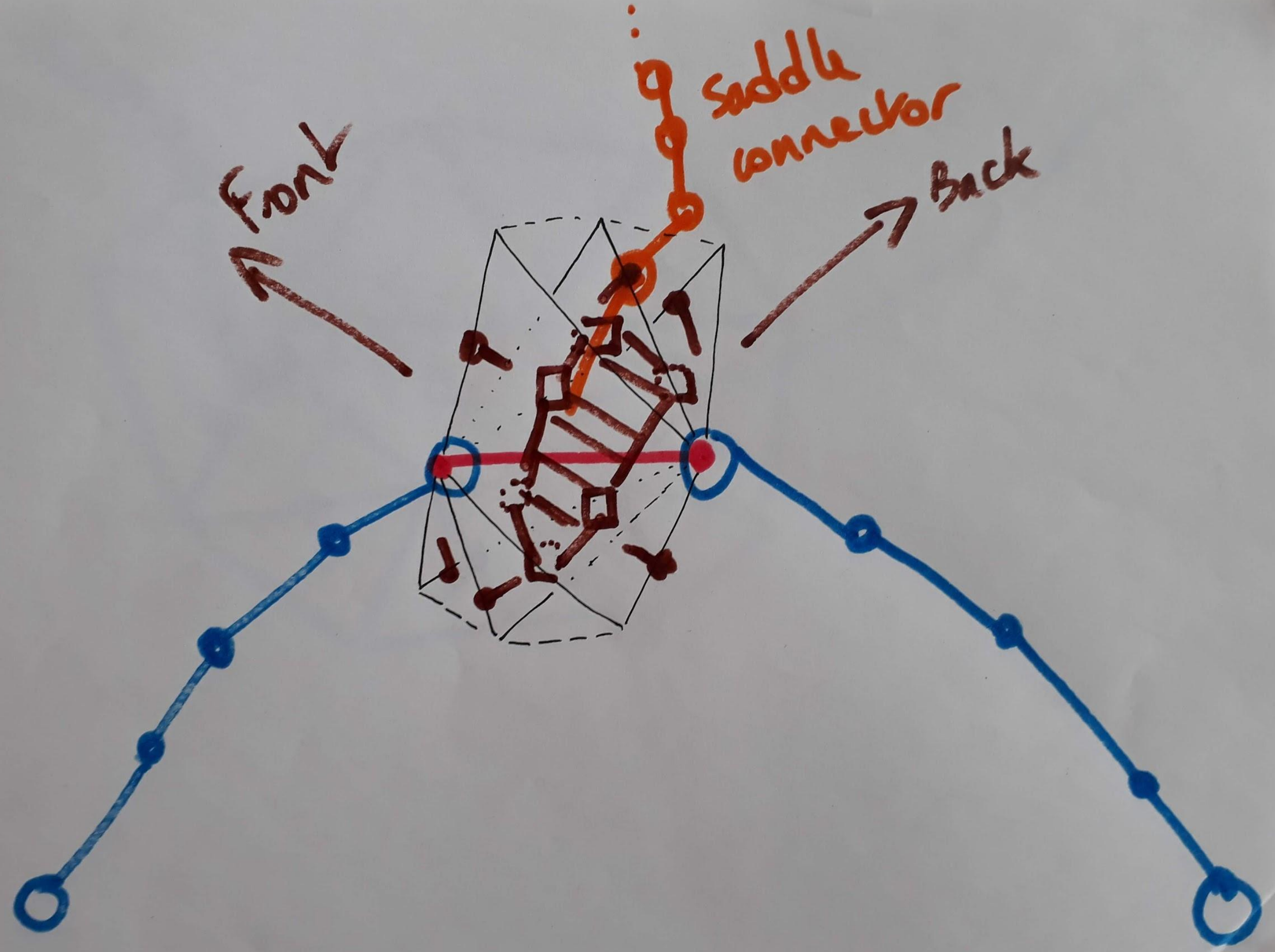


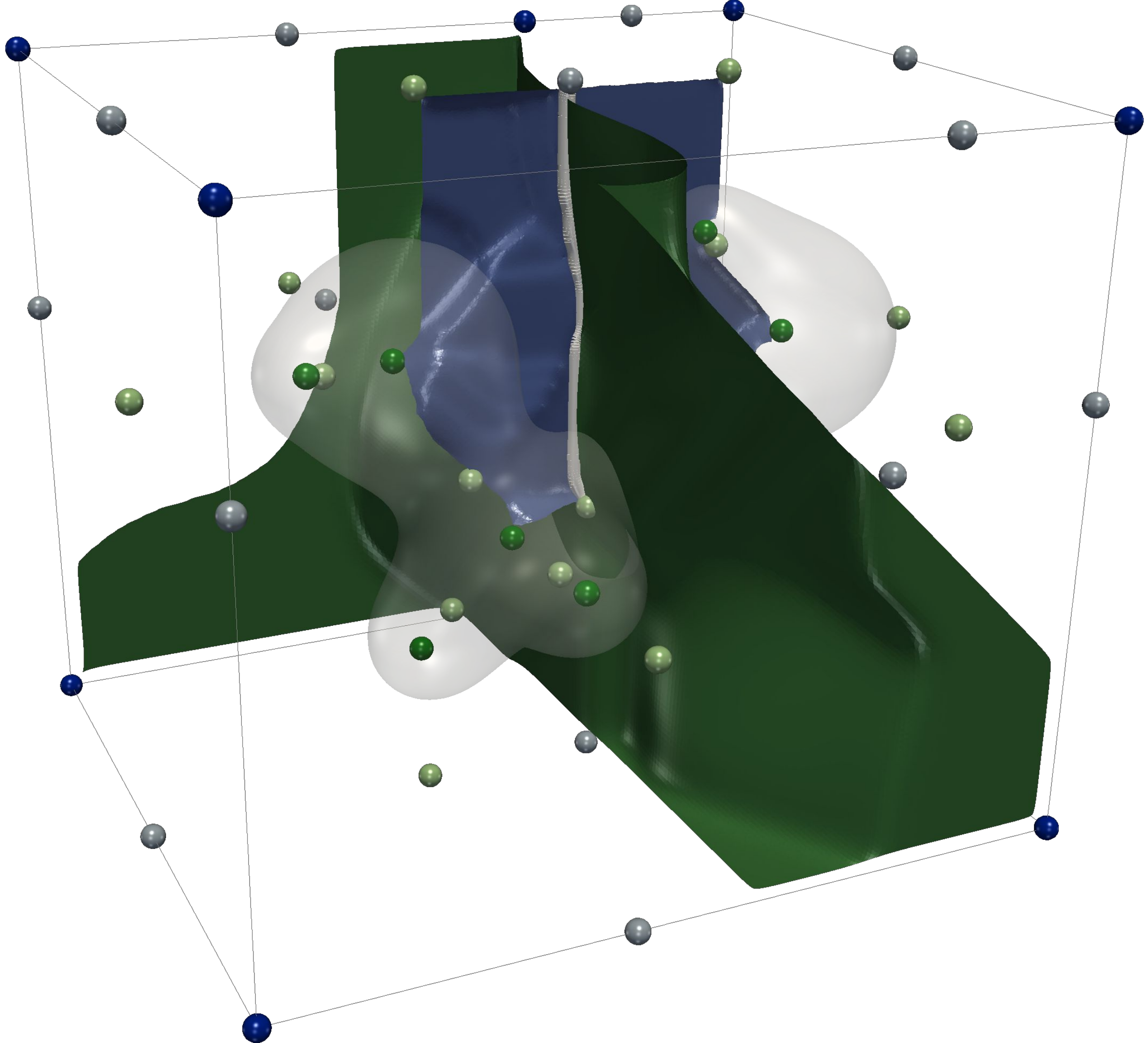


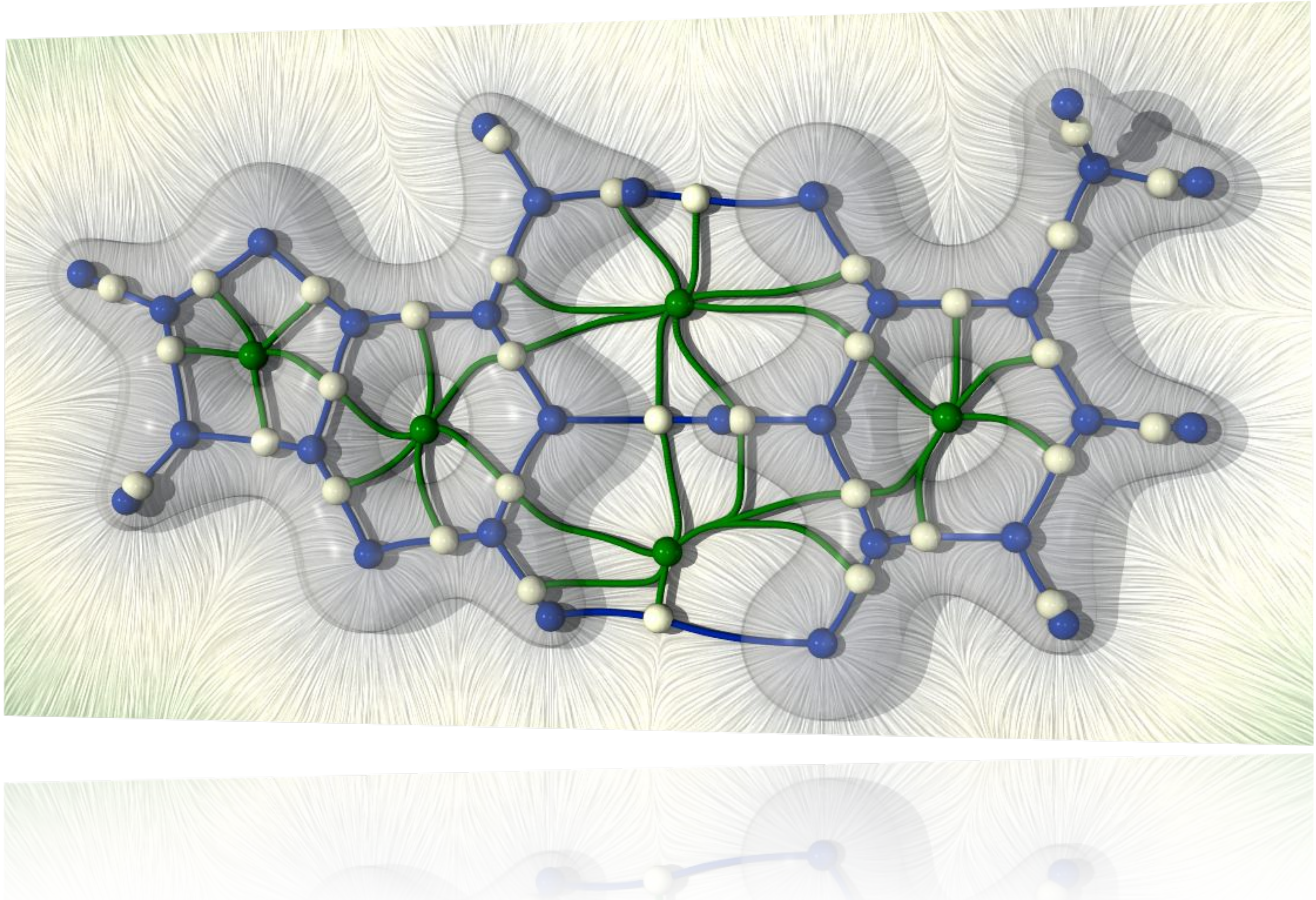
Front
Back

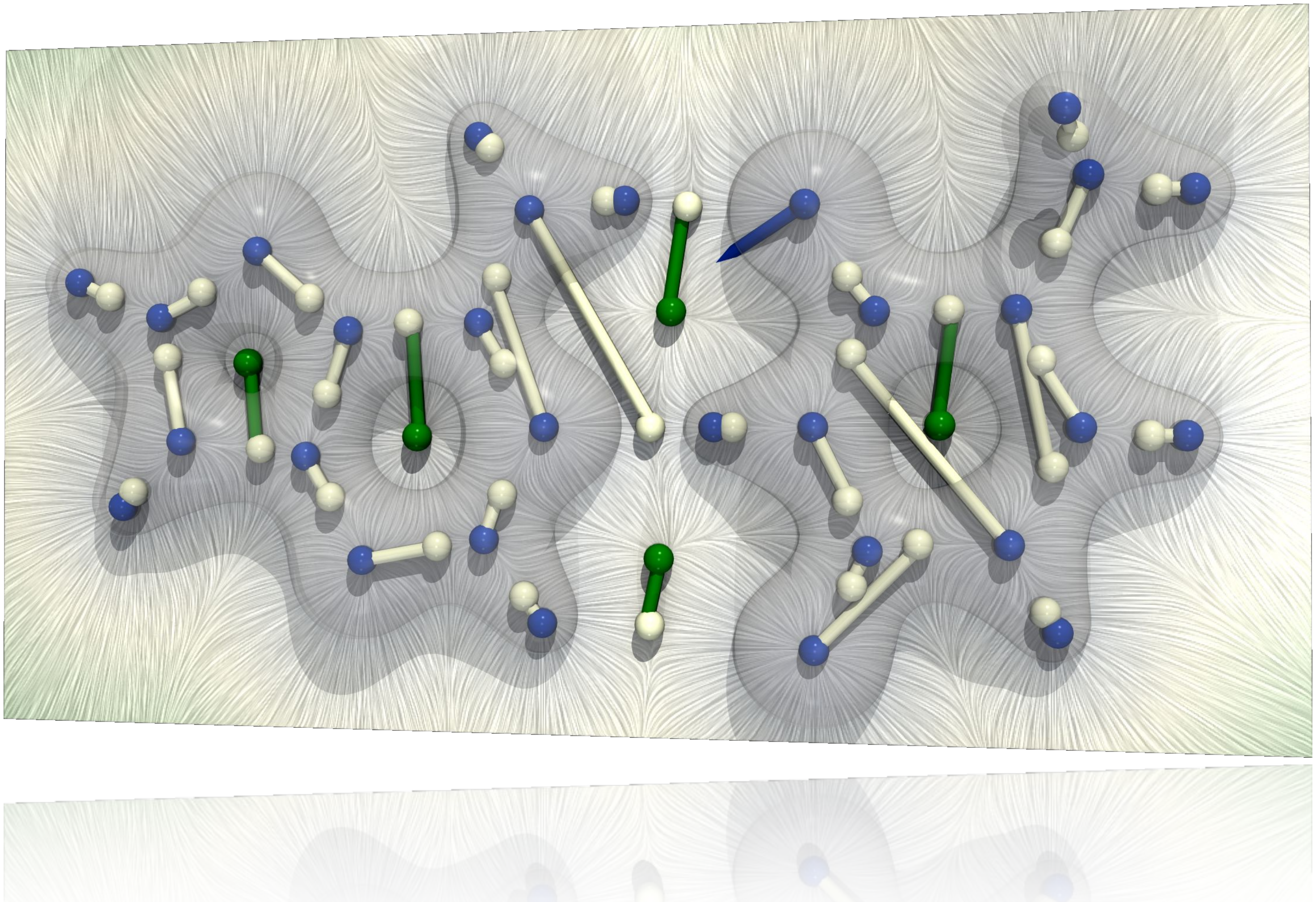


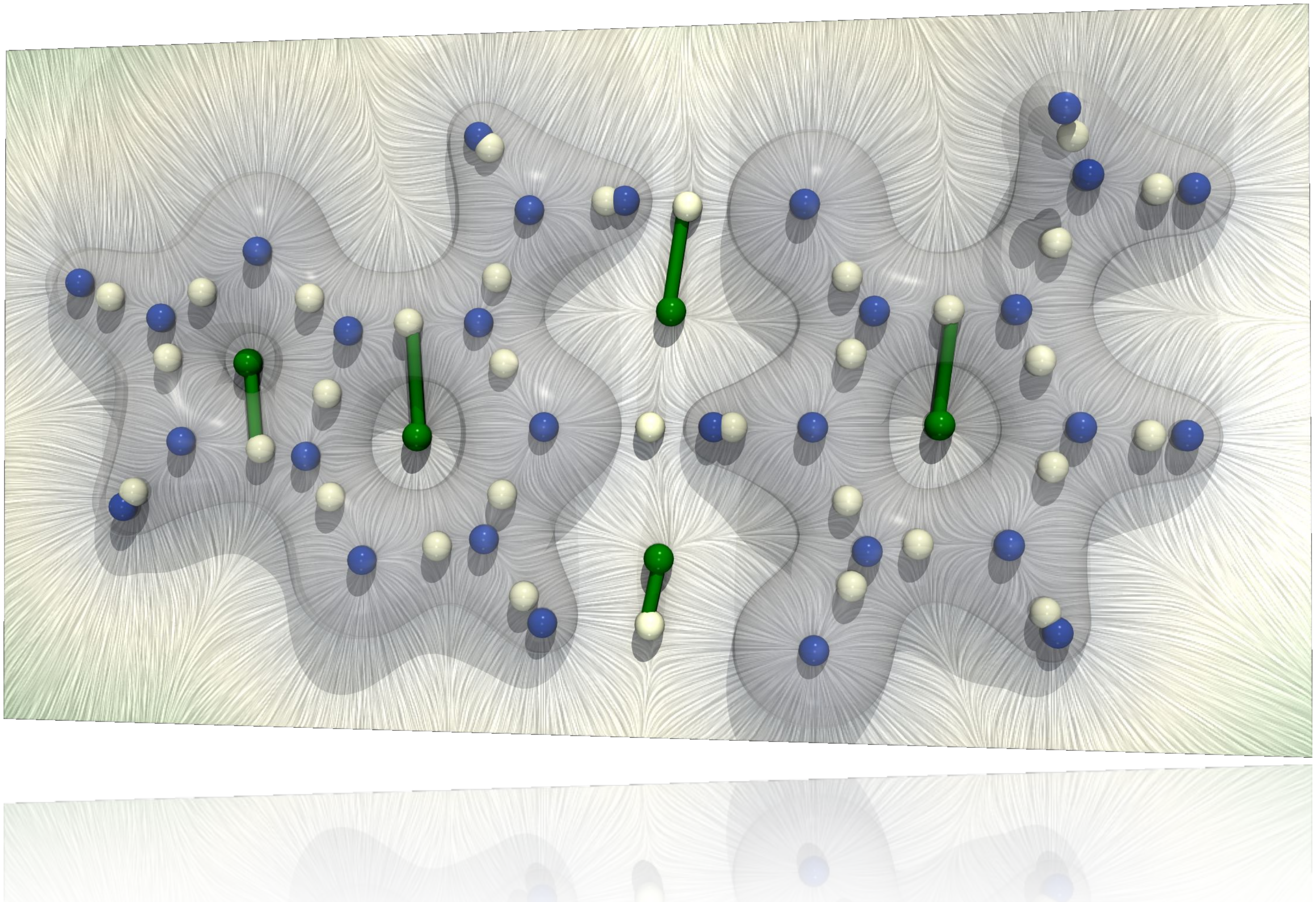


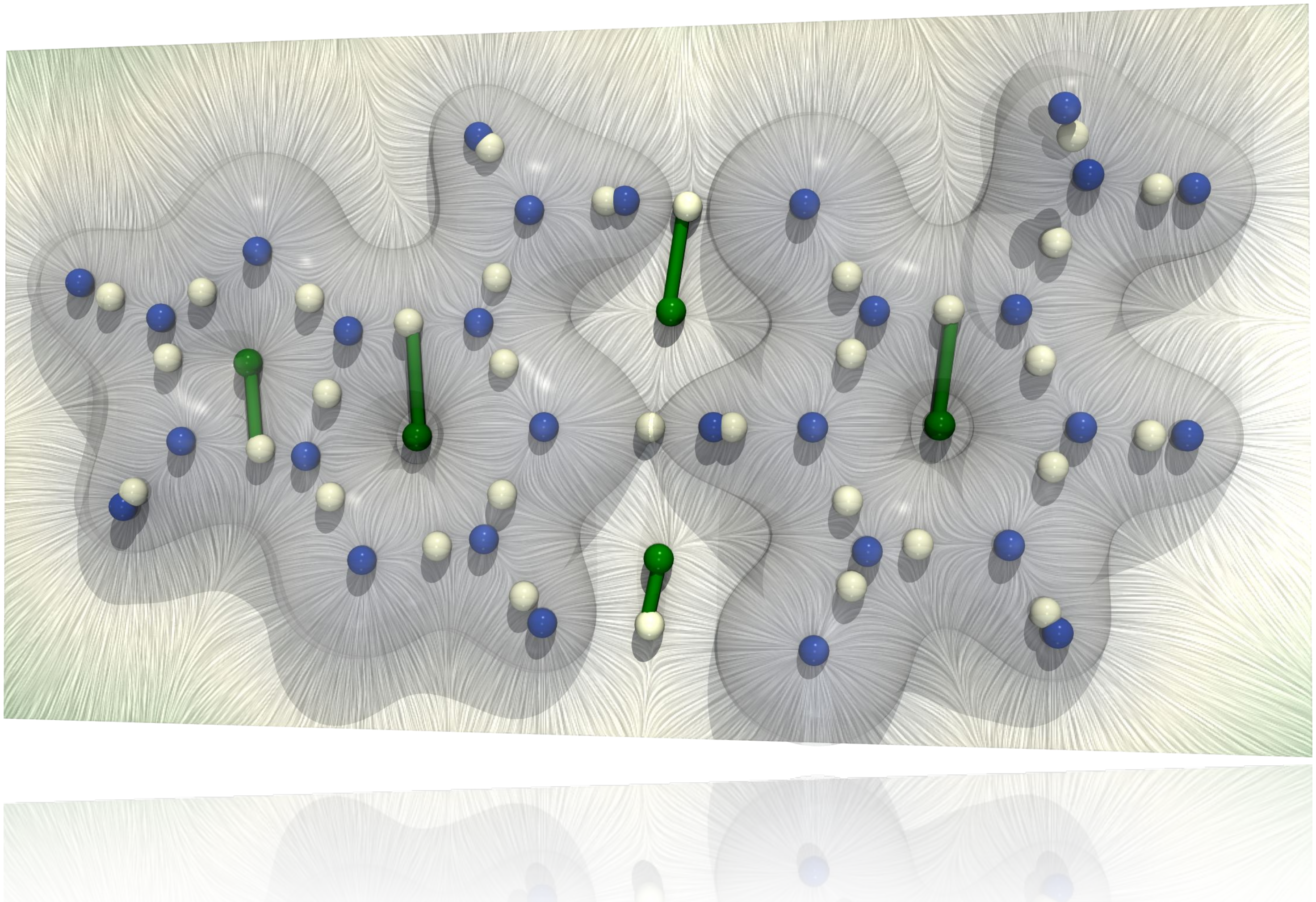

saddle
connector
Front
Back

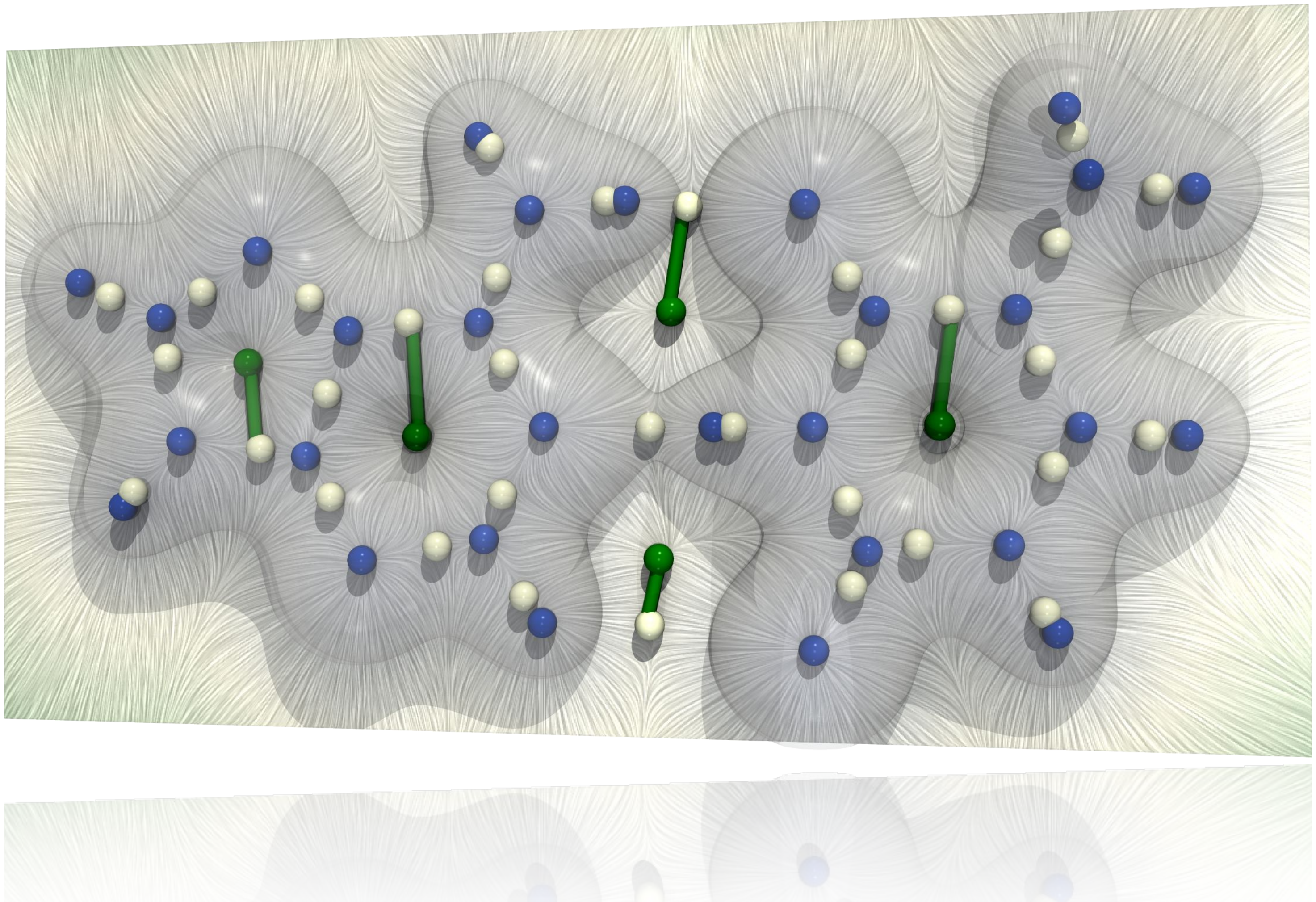


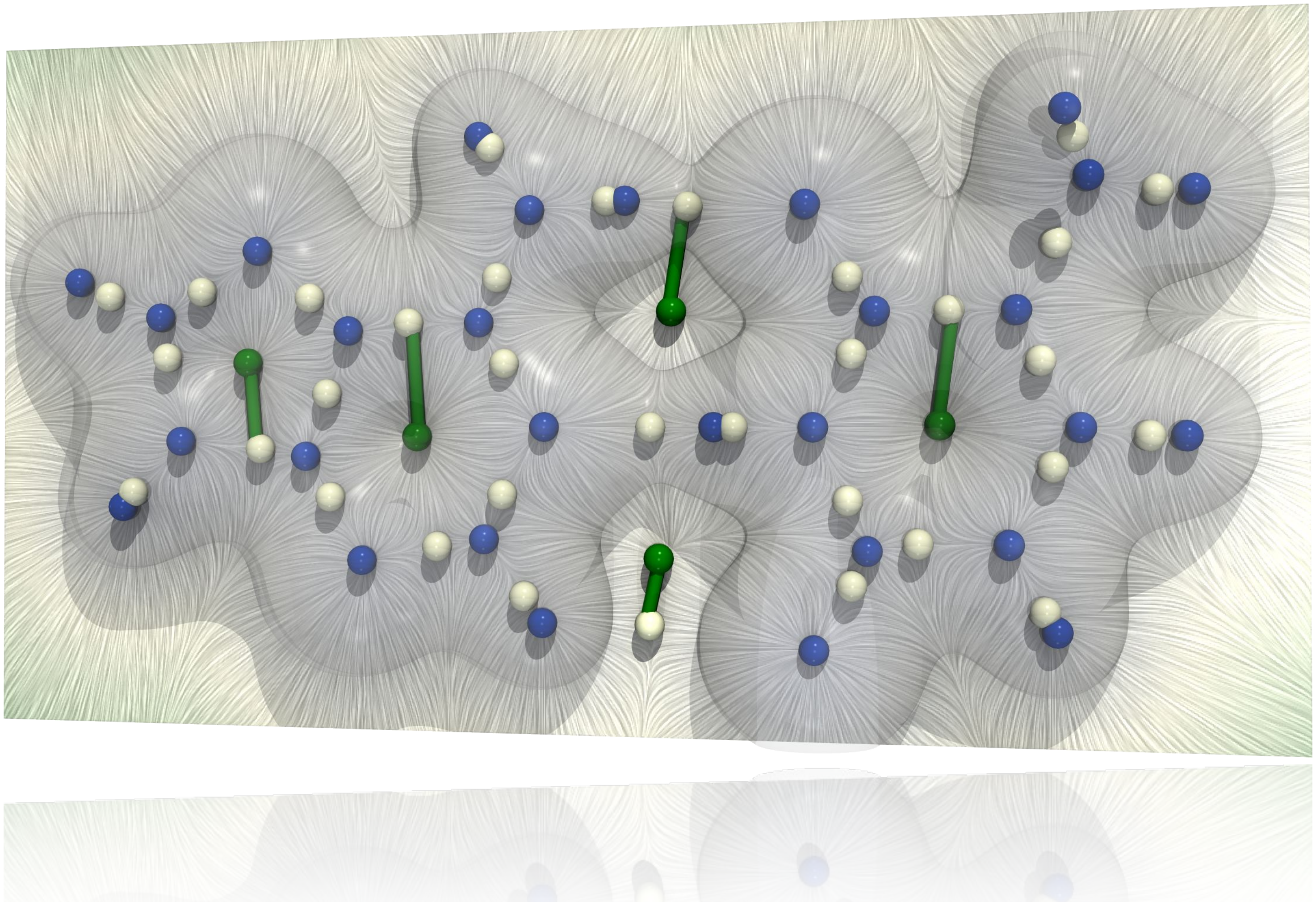


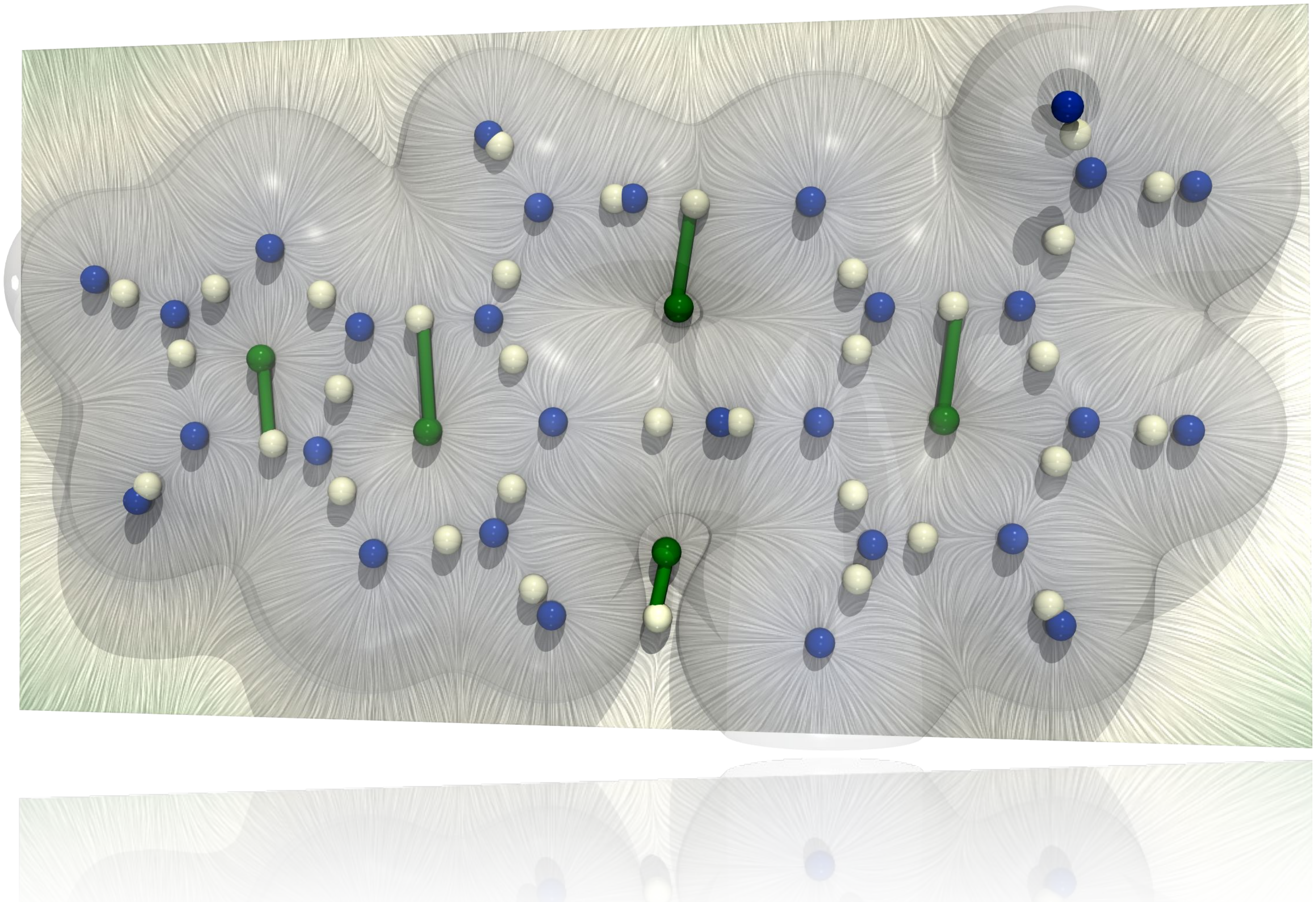




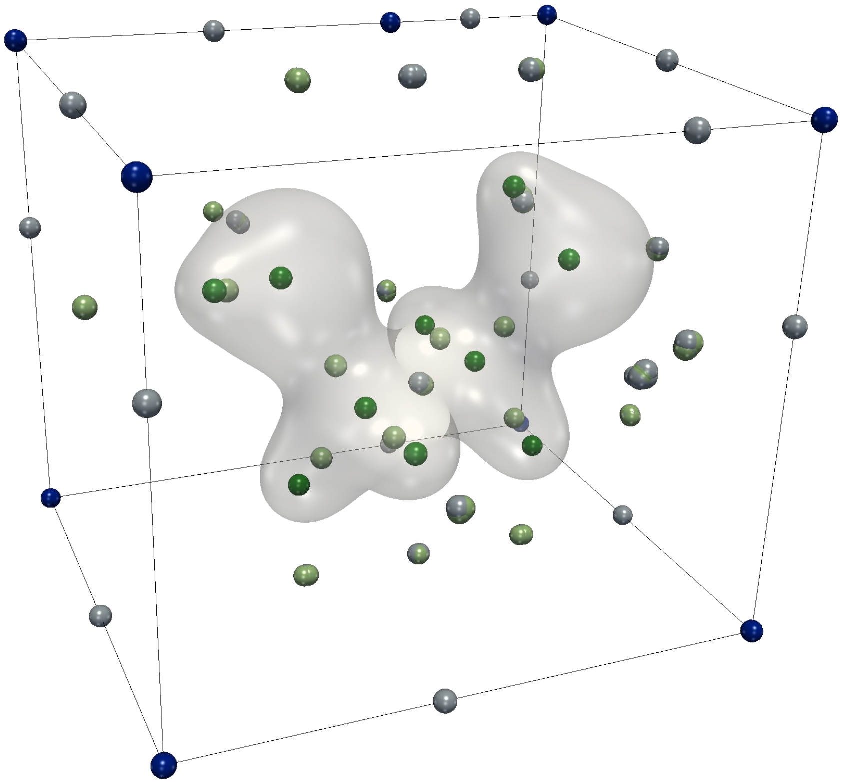




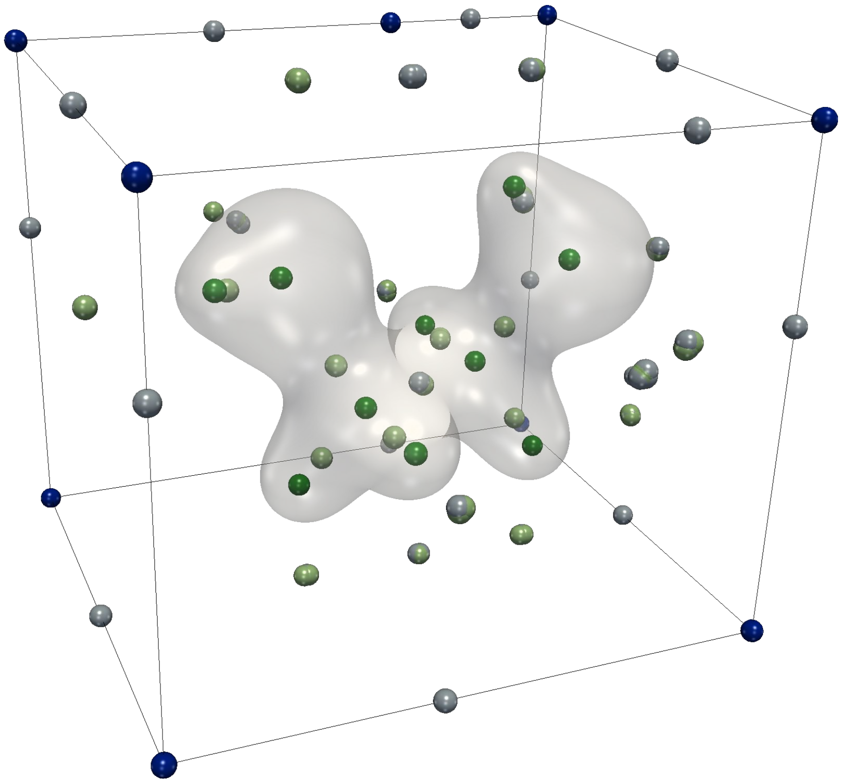


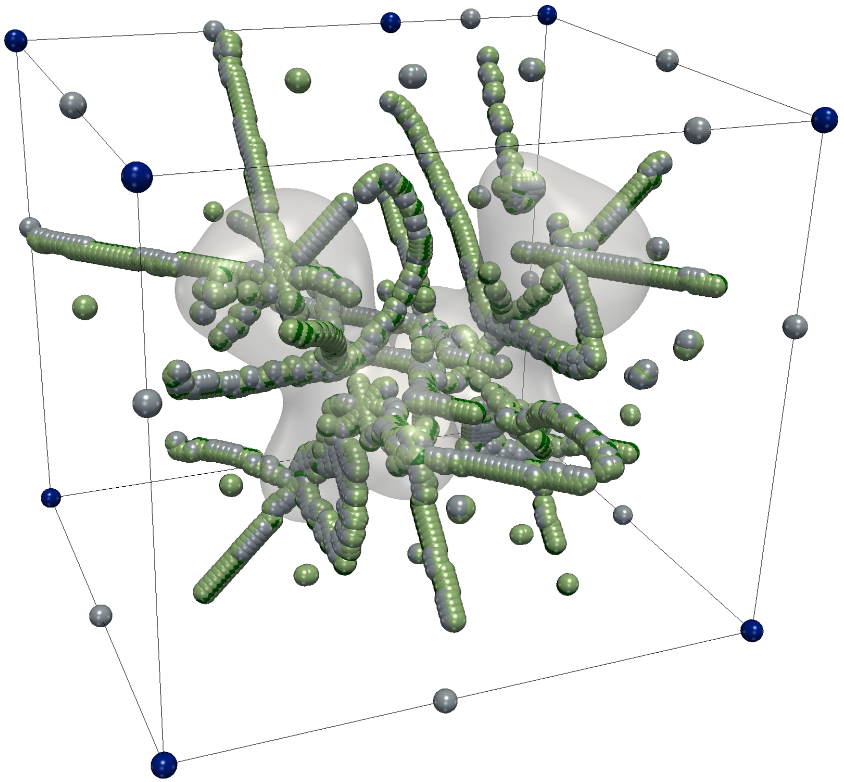


Relation to the PL setting

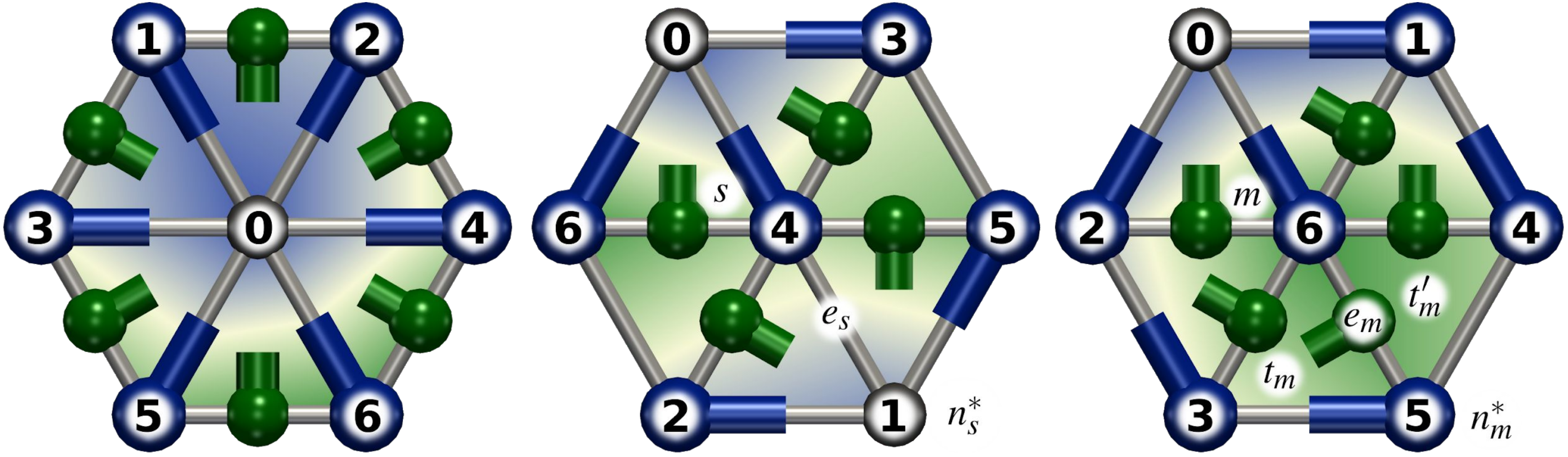


Relation to the PL setting

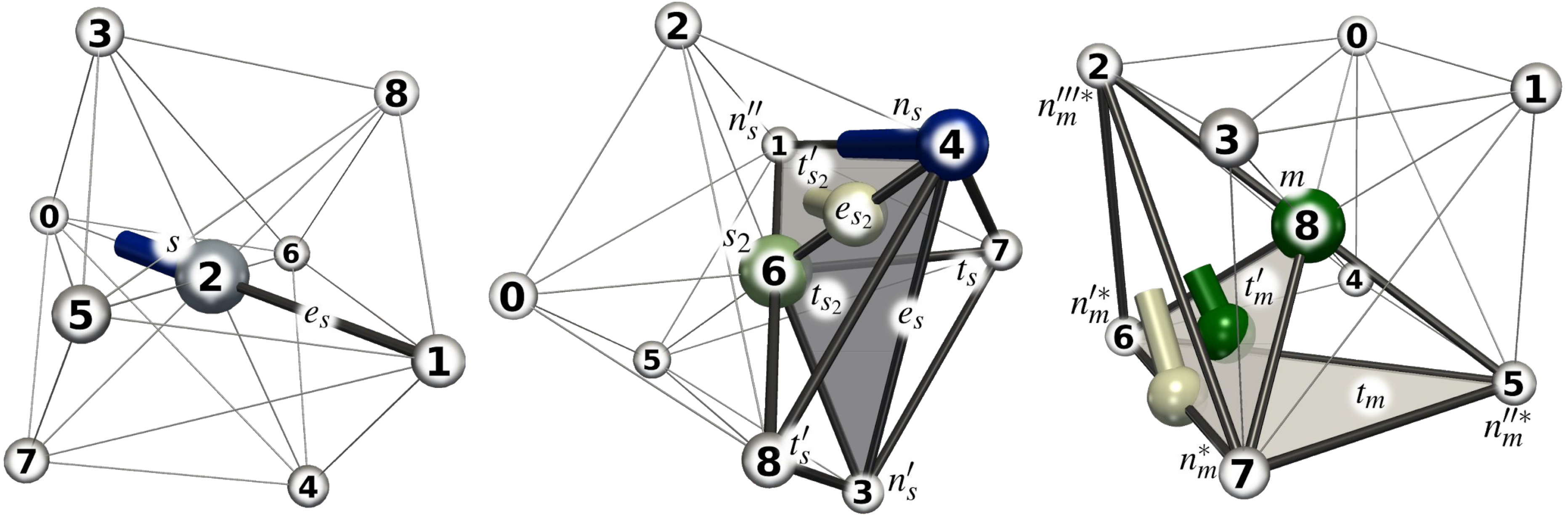




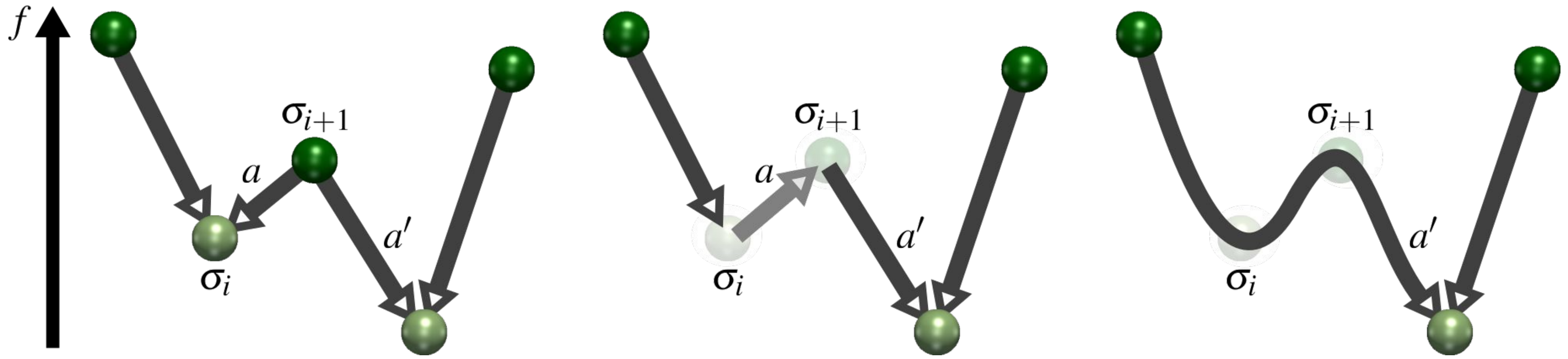
PL matching property



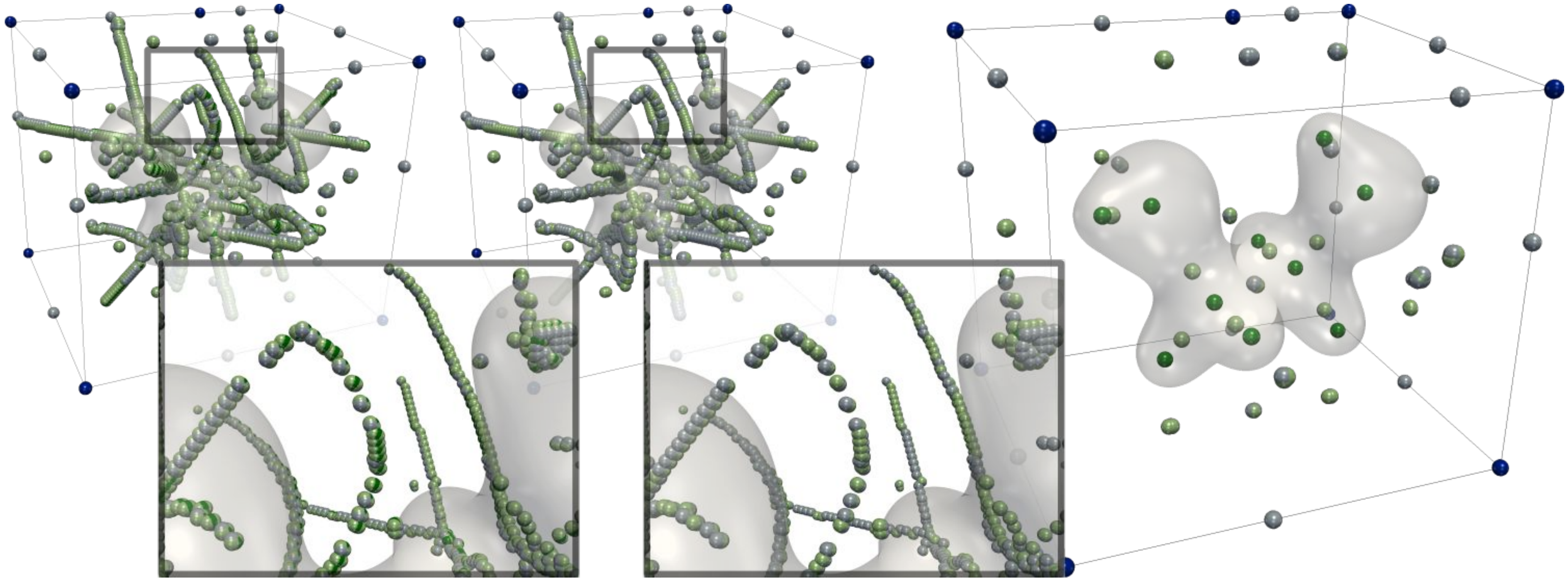
PL matching property



V-path reversal from non-PL critical simplices



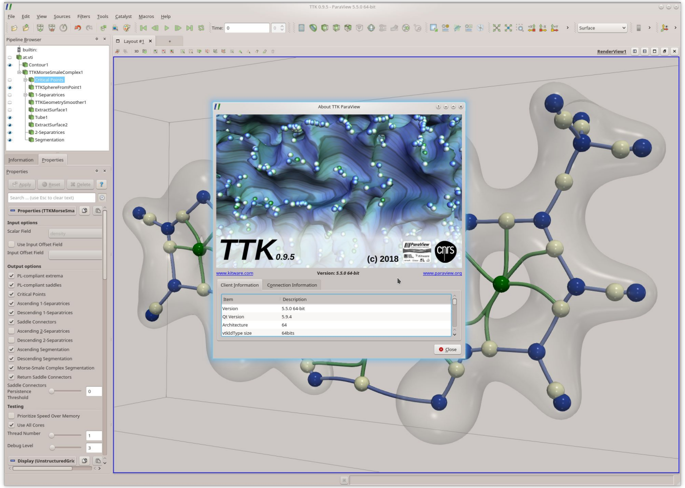
V-path reversal from non-PL critical simplices



The Topology ToolKit

- **Open-source TDA library**
  - ~100k lines in C++, BSD license
  - http://topology-tool-kit.github.io
  - 14k unique visitors, 8.5k Youtube views
  - **Best paper honorable mention IEEE VIS'17**

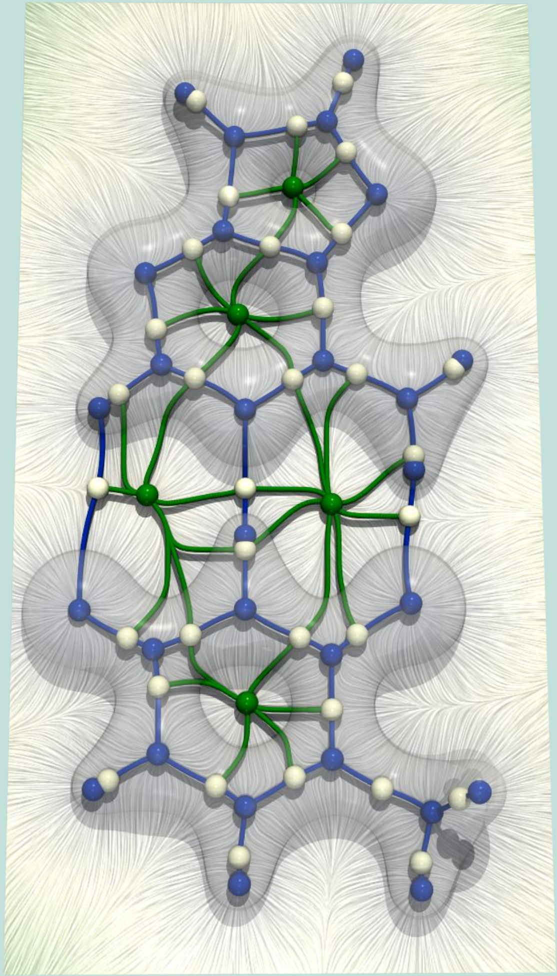
- **Structuring research receptacle**
  - 11 contributing institutions
    - 8 academics, 3 companies
  - Mini-symposia: IEEE VIS'18, TopoInVis'19, Total



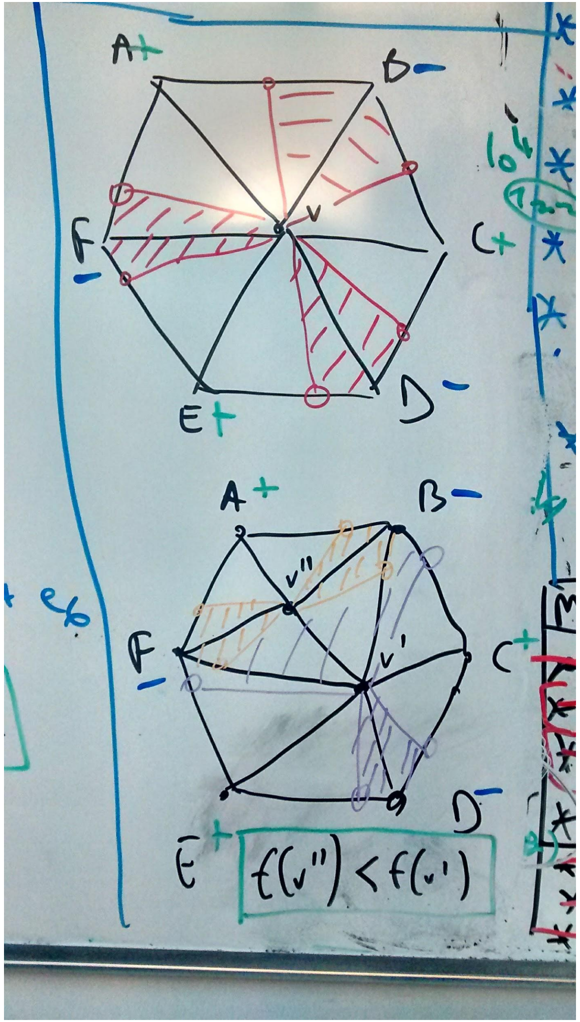
Take-home message

- **Morse-Smale complex**
  - Filament structures
  - Cell-like segmentation
  - Data clustering

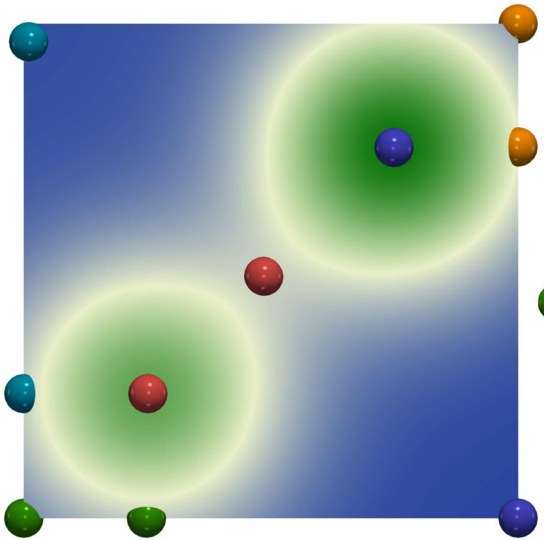
- **Discrete Morse Theory**
  - Simple and elegant
  - Enables robustness in algorithms
  - Possible compliance with PL setting
  - Challenging boundary handling :(

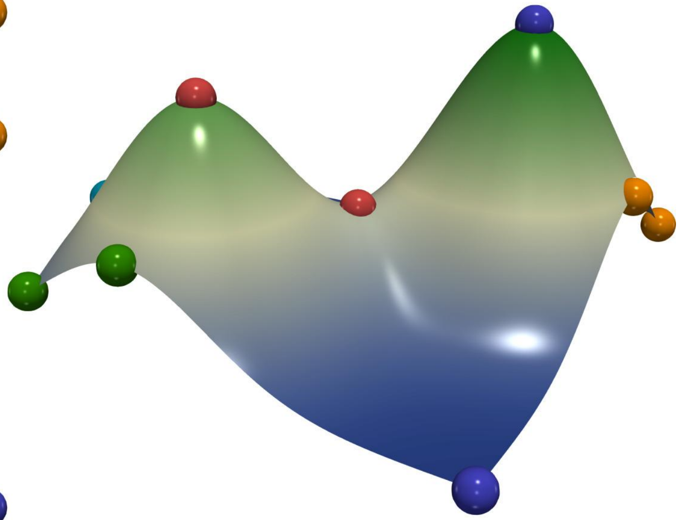


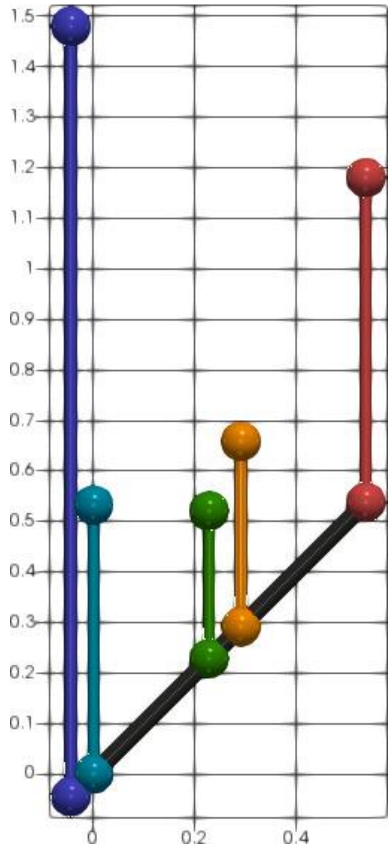
Saddle unfolding



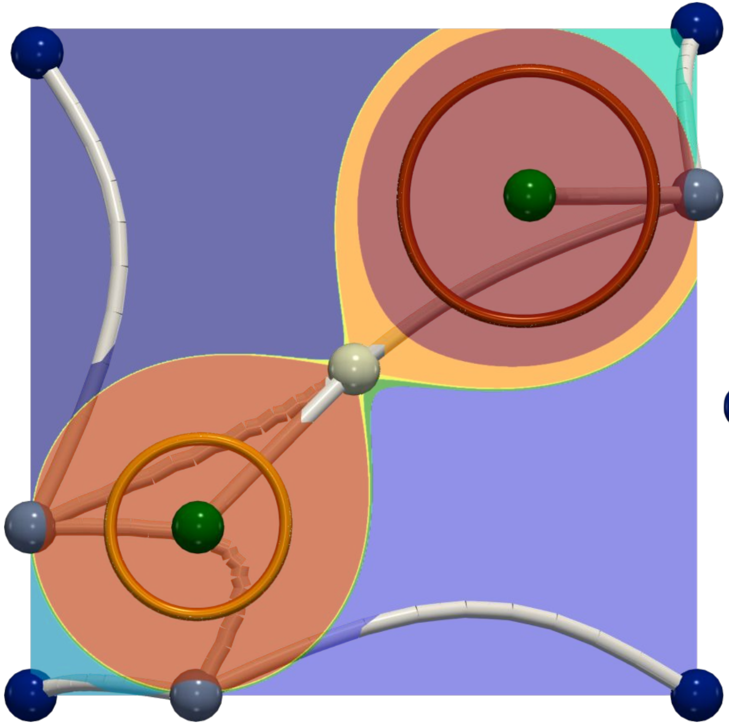
Persistence diagrams

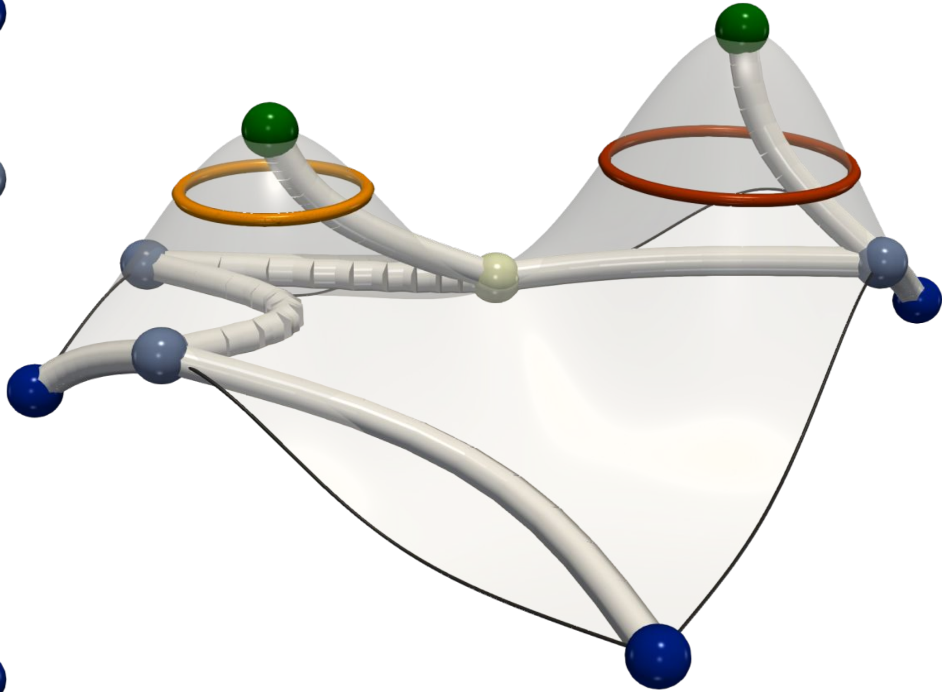






Reeb graphs





Morse-Smale complexes